Free Boundary Problems and Asymptotically Optimal Control for Stochastic Networks.

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based on joint works with X.Liu and S.Saha

IMA Special Workshop
Reflected Brownian Motions, Stochastic Networks, and their Applications.
June 2015

A Problem.

- Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be a filtered probability space.
- Let B be a two dimensional $\{\mathcal{F}_t\}$-Brownian motion with drift θ and covariance Σ.
- Control Problem.

$$\inf_Y \mathbb{E} \int_0^\infty e^{-t} h(X_t) dt$$

Subject to:

$$X_t = B_t + Y_t$$
$$Y_t \geq 0, Y \text{ is RCLL, adapted, non decreasing}$$
$$X_t \geq 0.$$

$$h(x) = (2x_1 - x_2)1_{x_1 \geq x_2} + (2x_2 - x_1)1_{x_2 \geq x_1}.$$

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- Question: Optimal Control?

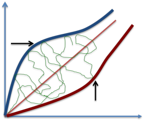
Conjecture.

(See Martins, Shreve and Soner (1996).)

- Solution is a two dimensional reflected Brownian motion in a domain:
- There are functions $\Psi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$ such that Ψ_i are nondecreasing, 1-Lipschitz, $\Psi_i(0) = 0$, $\Psi_i(u) \to \infty$ as $u \to \infty$, and an optimal state process X^* is the unique solution of

$$X_1^*(t) = B_1(t) + \sup_{0 \le s \le t} [B_1(s) - \Psi_1(X_2^*(s))]^-$$

$$X_2^*(t) = B_2(t) + \sup_{0 \le s \le t} [B_2(s) - \Psi_2(X_1^*(s))]^-$$



Free Boundary Problems.

- This is an example of a *Free Boundary Problem*.
- In such singular control problems the goal is to find an optimal control of the following form: There is an open set $\mathcal{O}$ in the state space – the *continuation region*, such that:
  - Starting within $\mathcal{O}$ no control is applied until the boundary of $\mathcal{O}$ is reached.
  - If the initial condition is in $(\bar{\mathcal{O}})^c$ (*action region*), an instantaneous control in a pre-specified direction is applied to bring the state to $\partial\mathcal{O}$.
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- In terms of the associated HJB equation, in $\mathcal{O}$ the value function satisfies a linear elliptic PDE while in $\mathcal{O}^c$ a nonlinear first order PDE is satisfied.
- The boundary $\partial\mathcal{O}$ separating these two regions is referred to as the *free boundary* for the system of PDE and determining this boundary is the *free boundary problem*.

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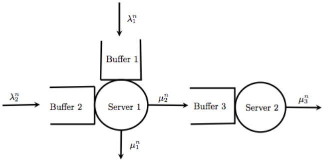
- Arise in the control of SPN.

A SPN Control Problem.

- 3 classes of jobs, 2 Servers.



- Class 1 jobs arrive at rate λ_1 and exit the system after service (at rate μ_1).
- Class 2 jobs arrive at rate λ_2, after processing by server 1 (at rate μ_2) proceed for processing by server 2 (at rate μ_3), becoming class 3 jobs, and then exit the system.

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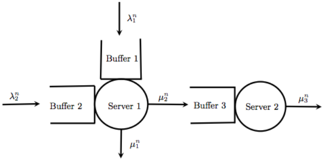

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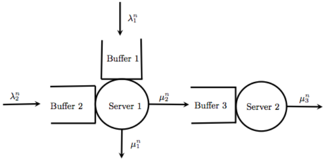

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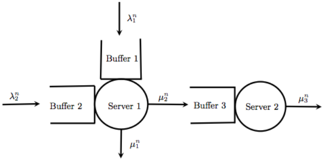

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- Goal: Minimize a expected holding cost discounted over time.
- Exact solutions not available ...so one considers approximate approaches.

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- Interarrival times: For $k = 1, 2$, $\{u_k(i) : i \in \mathbb{N}\}$ are i.i.d. with mean 1 and standard deviation σ_k.

  In the nth network, for $k = 1, 2$ $\{u_k^n(i) : i \in \mathbb{N}\}$ are the interarrival times for Class k customers, where $u_k^n(i) = \frac{1}{\lambda_k^n} u_k(i)$.

- Service times: For $j = 1, 2, 3$, let $\{v_j(i) : i \in \mathbb{N}\}$ a sequence of i.i.d. random variables with mean 1 and standard deviation ς_j.

  In the nth network, for $j = 1, 2, 3$, $\{v_j^n(i) : i \in \mathbb{N}\}$ are the service times for Class j customers, where $v_j^n(i) = \frac{1}{\mu_j^n} v_j(i)$.

- the sequences of inter-arrival times and service times are mutually independent for each $n \in \mathbb{N}$.

Heavy Traffic Condition.

- For $k = 1, 2$ and $j = 1, 2, 3$, there exist $\lambda_k, \mu_j \in (0, \infty)$ such that

$$\lim_{n\to\infty} \lambda_k^n = \lambda_k, \ \lim_{n\to\infty} \mu_j^n = \mu_j.$$

- $\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} = 1$, $\frac{\lambda_2}{\mu_3} = 1$.

- There exist $b_i \in \mathbb{R}$, $i = 1, 2, 3$, such that

$$\lim_{n\to\infty} \sqrt{n}\left(\frac{\lambda_i^n}{\mu_i^n} - \frac{\lambda_i}{\mu_i}\right) = b_i, \ \ i = 1, 2, \ \ \lim_{n\to\infty} \sqrt{n}\left(\frac{\lambda_2^n}{\mu_3^n} - 1\right) = b_3 \, .$$

Control Problem.

- Control Policy: $T^n(t) = (T_1^n(t), T_2^n(t), T_3^n(t))$. Here $T_j^n(t)$ is the amount of time spent on processing class j jobs over $[0, t]$.

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- Let for $k = 1, 2$, $j = 1, 2, 3$,

$$A_k^n(t) = \max\{m \geq 1 : \sum_{i=1}^{m} u_k^n(i) \leq t\}, \; S_j^n(t) = \max\{m \geq 1 : \sum_{i=1}^{m} v_j^n(i) \leq t\}.$$

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- State Process:

$$Q_i^n(t) = A_i^n(t) - S_i^n(T_i^n(t)), \; i = 1, 2,$$
$$Q_3^n(t) = S_2^n(T_2^n(t)) - S_3^n(T_3^n(t)).$$

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- Admissibility: T^n is an admissible control policy if
  - T_j^n is continuous, non-decreasing and $T^n(0) = 0$.
  - $I_1^n(t) = t - T_1^n(t) - T_2^n(t)$, $I_2^n(t) = t - T_3^n(t)$ are non-decreasing processes.
  - $Q^n(t) \geq 0$.
  - T^n is 'non-anticipative'.

Control Problem.

- Cost. Let $\hat{Q}^n(t) = \frac{Q^n(nt)}{\sqrt{n}}$. Let $c = (c_1, c_2, c_3)' > 0$. Cost associated with control $\{T^n\}$:

$$\hat{J}^n(T^n) \doteq \mathbb{E} \int_0^\infty e^{-\gamma t} c \cdot \hat{Q}^n(t) dt.$$

- Goal: Find a sequence $\{T^n\}$ of control policies which is asymptotically optimal, namely it satisfies

$$\lim_{n\to\infty} \hat{J}^n(T^n) = \inf \liminf_{n\to\infty} \hat{J}^n(\tilde{T}^n),$$

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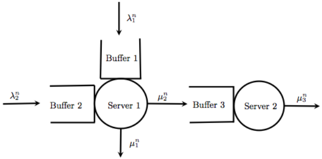

- Parameter Regime: If $c_1\mu_1 \leq c_2\mu_2 - c_3\mu_2$, then priority to Buffer 2 is optimal. Here consider $c_1\mu_1 > c_2\mu_2 - c_3\mu_2$.

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  - The Brownian control problem is to find an $\mathbb{R}^3$-valued $\{\mathcal{F}_t\}$-adapted RCLL stochastic process $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3)'$, which minimizes

$$\mathbb{E}\left(\int_0^\infty e^{-\gamma t} c \cdot \tilde{Q}(t) dt \right),$$

subject to the following conditions. For all $t \geq 0$,

$$0 \leq \tilde{Q}_i(t) \doteq q_i + Z_i(t) + \mu_i \tilde{Y}_i(t), \; i = 1, 2,$$
$$0 \leq \tilde{Q}_3(t) \doteq q_3 + Z_3(t) + \mu_3 \tilde{Y}_3(t) - \mu_2 \tilde{Y}_2(t),$$

and

$$\tilde{I}_1 \doteq \tilde{Y}_1 + \tilde{Y}_2, \; \tilde{I}_2 \doteq \tilde{Y}_3 \text{ are non-decreasing and nonnegative.}$$

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- Roughly,

$$\tilde{Q} \approx \hat{Q}^n, \; \tilde{Y}(t) \approx \sqrt{n}\left(x^* t - \bar{T}^n(t)\right), \; \tilde{I} \approx \hat{I}^n,$$

where $x^* = (\lambda_1/\mu_1, \lambda_2/\mu_2, 1)$ is the 'nominal allocation' and $\bar{T}^n(t) = T(nt)/n$.

Equivalent Workload Formulation

- Workload Process

$$\tilde{W}_1(t) = \frac{\tilde{Q}_1(t)}{\mu_1} + \frac{\tilde{Q}_2(t)}{\mu_2}, \quad \tilde{W}_2(t) = \frac{\tilde{Q}_2(t)}{\mu_3} + \frac{\tilde{Q}_3(t)}{\mu_3}.$$

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- Effective Cost Fix $w_1, w_2 \in [0, \infty)$. Consider the LP:

$$\text{minimize}_q \;\; c_1 q_1 + c_2 q_2 + c_3 q_3, \qquad \text{subject to } \frac{q_1}{\mu_1} + \frac{q_2}{\mu_2} = w_1, \; \frac{q_2}{\mu_3} + \frac{q_3}{\mu_3} = w_2, \; q \geq 0.$$

  - Value of LP:

$$h(w_1, w_2) = \begin{cases} (c_1\mu_1)w_1 + \frac{\mu_3}{\mu_2}(c_2\mu_2 - c_1\mu_1)w_2, & \text{when } \mu_3 w_2 \leq \mu_2 w_1, \\ (c_2\mu_2 - c_3\mu_2)w_1 + (c_3\mu_3)w_2, & \text{when } \mu_3 w_2 \geq \mu_2 w_1, \end{cases}$$

  - Solution of LP:

$$q_1^* = \frac{\mu_1}{\mu_2}(\mu_2 w_1 - \mu_3 w_2), \; q_2^* = \mu_3 w_2, \; q_3^* = 0, \text{if } \mu_3 w_2 \leq \mu_2 w_1,$$

$$q_1^* = 0, \; q_2^* = \mu_2 w_1, \; q_3^* = \mu_3 w_2 - \mu_2 w_1, \; \text{if } \mu_3 w_2 \geq \mu_2 w_1.$$

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- EWF Let $B_1(t) = \frac{Z_1(t)}{\mu_1} + \frac{Z_2(t)}{\mu_2}$, $B_2(t) = \frac{Z_2(t)}{\mu_3} + \frac{Z_3(t)}{\mu_3}$. The EWF of the BCP is to find an $\mathbb{R}_+^2$-valued $\{\mathcal{F}_t\}$-adapted stochastic process $\tilde{I} = (\tilde{I}_1, \tilde{I}_2)'$, which minimizes

$$J(w, \tilde{I}) \doteq \mathbb{E}\left(\int_0^\infty e^{-\gamma t} h(\tilde{W}(t)) dt \right),$$

subject to, for all $t \geq 0$,

$$0 \leq \tilde{W}_1(t) \doteq w_1 + B_1(t) + \tilde{I}_1(t),$$
$$0 \leq \tilde{W}_2(t) \doteq w_2 + B_2(t) + \tilde{I}_2(t),$$
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$$\tilde{Y}_1^*(t) \doteq -\frac{Z_3(t)}{\mu_2} + \tilde{I}_1^*(t) - \frac{\mu_3}{\mu_2}\tilde{I}_2^*(t), \; \tilde{Y}_2^*(t) \doteq \frac{Z_3(t)}{\mu_2} + \frac{\mu_3}{\mu_2}\tilde{I}_2^*(t), \; \tilde{Y}_3^*(t) \doteq \tilde{I}_2^*(t),$$

and when $\mu_3 \tilde{W}_2^*(t) \geq \mu_2 \tilde{W}_1^*(t)$,

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The corresponding optimal queue length:
if $\mu_3 \tilde{W}_2^*(t) < \mu_2 \tilde{W}_1^*(t)$,

$$\tilde{Q}_1^*(t) = \frac{\mu_1}{\mu_2}(\mu_2 \tilde{W}_1^*(t) - \mu_3 \tilde{W}_2^*(t)), \; \tilde{Q}_2^*(t) = \mu_3 \tilde{W}_2^*(t), \; \tilde{Q}_3^*(t) = 0,$$

if $\mu_3 \tilde{W}_2^*(t) \geq \mu_2 \tilde{W}_1^*(t)$,

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Harrison's Program(1988)

- Similar BCP/EWF can be written for a broad family of SPN.
- Harrison proposes the following approach.
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- Most of these have a one dimensional EWF. Last work has a two dimensional EWF whose solution is a RBM in $\mathbb{R}_+^2$ with normal reflections.

Harrison's Program(1988)

- Similar BCP/EWF can be written for a broad family of SPN.
- Harrison proposes the following approach.
  - Solve the BCP.
  - Use the form of the solution to construct a policy sequence for the networks.
  - Rigorously prove asymptotic optimality.
- Carried out in many works: Harrison (1998), Bell-Williams(2001, 2005), Kumar(2000), Stolyar(2004), Mandelbaum and Stolyar(2004), Ata and Kumar (2005), Dai-Lin(2007), B.-Ghosh(2005) ...
- Most of these have a one dimensional EWF. Last work has a two dimensional EWF whose solution is a RBM in $\mathbb{R}^2_+$ with normal reflections.
- For a general family of unitary networks convergence of value functions shown in B.-Ghosh(2006, 2012).

Solving the EWF

- Recall the EWF: Minimize

$$J(w, \tilde{I}) \doteq \mathbb{E}\left(\int_0^\infty e^{-\gamma t} h(\tilde{W}(t)) dt \right),$$

subject to, for all $t \geq 0$,

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- where,

$$h(w_1, w_2) = \begin{cases} (c_1\mu_1)w_1 + \frac{\mu_3}{\mu_2}(c_2\mu_2 - c_1\mu_1)w_2, & \text{when } \mu_3 w_2 \leq \mu_2 w_1, \\ (c_2\mu_2 - c_3\mu_2)w_1 + (c_3\mu_3)w_2, & \text{when } \mu_3 w_2 \geq \mu_2 w_1, \end{cases}$$

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- Parameter Regimes[Martins-Shreve-Soner(1996)]: $c_1\mu_1 - c_2\mu_2 + c_3\mu_2 > 0$
  - Case IIA: $c_2\mu_2 - c_3\mu_2 \geq 0$, $c_2\mu_2 - c_1\mu_1 \geq 0$.
    - Optimal workload process W^* is a RBM in $\mathbb{R}_+^2$ with normal reflections.
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- Since $w_2 \mapsto h(w_1, w_2)$ is nondecreasing, optimal (I_2^*, W_2^*) is characterized as

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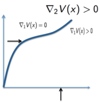
- (B.-Ross(2008)) V is C^1 and $\nabla_2 V(x) > 0$ for all $x_2 > 0$. Also $\nabla_i V \ge 0$.



Case IIB (B.–Ross(2008))

- Free boundary:

$$\Psi(z_2) \doteq \sup\{z_1 \geq 0 : \nabla_1 V(z_1, z_2) = 0\}, \; z_2 \in [0, \infty).$$

- Ψ is nondecreasing, $\Psi(0) = 0$ and $\Psi(u) \to \infty$ as $u \to \infty$.

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- Candidate optimal control:

$$I_1^*(t) \doteq \sup_{0 \leq s \leq t} [w_1 + B_1(s) - \Psi(W_2^*(s))]^-.$$

- Corresponding state process:

$$W_1^*(t) = w_1 + B_1(t) + \sup_{0 \leq s \leq t} [w_1 + B_1(s) - \Psi(W_2^*(s))]^-$$

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Case IIB (B.–Ross(2008))

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- Proof.(Sketch) Basic Idea: Apply Itô's formula...
- Let $G = \{w \in \mathbb{R}^2_+ : w_1 \geq \Psi(w_2)\}$,

$$\partial_1 G = \{w \in G : w_1 = \Psi(w_2)\}, \; \partial_2 G = \{w \in G : w_2 = 0\}.$$

- $\nabla_i V(w) = 0$ for $w \in \partial_i G$ and $\nabla_i V(w) > 0$ for $w \in G^\circ$.
- $\gamma V(w) + AV(w) - h(w) = 0$ for $w \in G^\circ$. Thus V is C^2 in G°.
- $G^\varepsilon \doteq \{w \in \mathbb{R}^2 : d(w, G) < \varepsilon\}$.

$$e = ((2 + 3\frac{\mu_3}{\mu_2}), 2)', \; w(\varepsilon) = w + \varepsilon e \in G^\circ \text{ if } w \in G^\varepsilon.$$

- Let $V^\varepsilon(w) = V(w(\varepsilon))$. V^ε is C^2 on G^ε and

$$|\gamma V^\varepsilon(w) + AV^\varepsilon(w) - h(w)| \leq c\varepsilon, \; w \in G^\varepsilon.$$

Proof of Optimality (Ctd.)

- Applying Itô's formula to V^{ε} with W^*:

$$V^{\varepsilon}(w) = \mathbb{E}e^{-\gamma t}V^{\varepsilon}(W^*(t)) + \mathbb{E}\int_0^t e^{-\gamma s}(\gamma V^{\varepsilon}(W^*(s)) + AV^{\varepsilon}(Z^*(s)))ds$$
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$$V(w) \ge \mathbb{E}e^{-\gamma t}V(W^*(t)) + \mathbb{E}\int_0^t e^{-\gamma s}h(W^*(s))ds$$
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$$= \mathbb{E}e^{-\gamma t}V(W^*(t)) + \mathbb{E}\int_0^t e^{-\gamma s}h(W^*(s))ds.$$

(Since $\nabla_i V(w) = 0$ for $w \in \partial_i G$.)

- Sending $t \to \infty$

$$V(w) \ge \mathbb{E}\int_0^\infty e^{-\gamma s}h(W^*(s))ds = J(w, I^*) \ge V(w).$$

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Constructing an Asymp. Opt. Policy

- Recall: The optimal queue length is:
  if $\tilde{Q}_3^*(t) < \frac{\mu_2}{\mu_1}\tilde{Q}_1^*(t)$

$$\tilde{Q}_1^*(t) = \frac{\mu_1}{\mu_2}(\mu_2 \tilde{W}_1^*(t) - \mu_3 \tilde{W}_2^*(t)),\ \tilde{Q}_2^*(t) = \mu_3 \tilde{W}_2^*(t),\ \tilde{Q}_3^*(t) = 0,$$

  if $\tilde{Q}_3^*(t) \geq \frac{\mu_2}{\mu_1}\tilde{Q}_1^*(t)$,

$$\tilde{Q}_1^*(t) = 0,\ \tilde{Q}_2^*(t) = \mu_2 \tilde{W}_1^*(t),\ \tilde{Q}_3^*(t) = \mu_3 \tilde{W}_2^*(t) - \mu_2 \tilde{W}_1^*(t).$$

- A good policy should:
  - Keep $\hat{Q}_3^n(t)$ close to 0 when $Q_3^n(t) < \frac{\mu_2^n}{\mu_1^n} Q_1^n(t)$.
  - Keep $\hat{Q}_1^n(t)$ close to 0, when $Q_3^n(t) \geq \frac{\mu_2^n}{\mu_1^n} Q_1^n(t)$.
  - $\hat{W}^n(t) \in G$ for all t approximately and no idling when $\hat{W}^n(t) \in G^\circ$.

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- Exponential moment condition. There is a non-empty open neighborhood $\mathcal{N}$ of $0 \in \mathbb{R}$ such that for all $z \in \mathcal{N}$, $k = 1, 2$, $j = 1, 2, 3$

$$\Lambda_{a,k} \doteq \log \mathbb{E}(e^{zu_k(1)/\lambda_k}) < \infty, \; \Lambda_{s,j} \doteq \log \mathbb{E}(e^{zv_j(1)/\mu_j}) < \infty.$$

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- Fix $c, l_0 \in (1, \infty)$. Define

$$L^n \doteq \lfloor l_0 \log n \rfloor, \ C^n \doteq \lfloor cl_0 \log n \rfloor - 1, \ D^n = C^n - L^n + 3.$$

- Control Policy. At time $s \in [0, \infty)$,
  - if $Q_3^n(s) - \frac{\mu_2^n}{\mu_1^n} Q_1^n(s) < L^n$,
    - serve Class 1 customers (provided the queue is non-empty) if either $Q_3^n(s) \geq C^n$ or $Q_2^n(s) = 0$,
    - serve Class 2 customers if $Q_3^n(s) < C^n$ and $Q_2^n(s) > 0$;
  - if $Q_3^n(s) - \frac{\mu_2^n}{\mu_1^n} Q_1^n(s) \geq L^n$,
    - serve Class 1 customers if either $Q_1^n(s) \geq \frac{\mu_1^n}{\mu_2^n} D^n$ or $(Q_2^n(s) = 0$ and $Q_1^n(s) > 0)$,
    - serve Class 2 customers if $Q_1^n(s) < \frac{\mu_1^n}{\mu_2^n} D^n$, $Q_2^n(s) > 0$ and $W_1^n(s) - \sqrt{n}\Psi(W_2^n(s)/\sqrt{n}) \geq 1$,
    - idle Server 1 if $Q_1^n(s) < \frac{\mu_1^n}{\mu_2^n} D^n$, $Q_2^n(s) > 0$ and $W_1^n(s) - \sqrt{n}\Psi(W_2^n(s)/\sqrt{n}) < 1$.

Asymptotic Optimality

- Theorem (B., Liu and Saha (2015)) There exist $c, l_0 \in (1, \infty)$ such that the sequence of scheduling controls $\{T^n\}$ satisfies

$$\lim_{n\to\infty} \hat{J}^n(T^n) = \inf \liminf_{n\to\infty} \hat{J}^n(\tilde{T}^n) = V,$$

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  - With c, l_0 as above, $\lim_{n\to\infty} \hat{J}^n(T^n) = V$.
  - Main step $(\hat{W}^n, \hat{I}^n) \Rightarrow (W^*, I^*)$.

Convergence of Scaled Processes

- With $\hat{I}_1^n(t) = \sqrt{n}(t - \bar{T}_1^n(t) - \bar{T}_2^n(t))$, $\hat{I}_2^n(t) = \sqrt{n}(t - \bar{T}_3^n(t))$

$$\hat{W}_1^n(t) = \frac{\hat{Q}_1^n(t)}{\mu_1^n} + \frac{\hat{Q}_2^n(t)}{\mu_2^n} = w_1^n + \hat{B}_1^n(t) + \hat{I}_1^n(t),$$
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where $\hat{B}^n \Rightarrow B$.

- With $\mathbf{A}(n,t) \doteq \left\{ \hat{Q}_2^n(t) \geq \frac{dl_0 \log n}{\sqrt{n}} \right\}$

$$\hat{W}_2^n(t) = \Gamma\left(w_2^n + \hat{B}_2^n - \frac{\hat{Q}_2^n(\cdot)}{\mu_3^n} 1_{\mathbf{A}(n,\cdot)^c} + \int_0^{\cdot} 1_{\mathbf{A}(n,s)} d\hat{I}_2^n(s) \right)(t) + \frac{\hat{Q}_2^n(t)}{\mu_3^n} 1_{\mathbf{A}(n,t)^c}.$$

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$$\hat{W}_1^n(t) = \frac{\hat{Q}_1^n(t)}{\mu_1^n} + \frac{\hat{Q}_2^n(t)}{\mu_2^n} = w_1^n + \hat{B}_1^n(t) + \hat{I}_1^n(t),$$
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where $\hat{B}^n \Rightarrow B$.

- With $\mathbf{A}(n,t) \doteq \left\{ \hat{Q}_2^n(t) \geq \frac{dl_0 \log n}{\sqrt{n}} \right\}$

$$\hat{W}_2^n(t) = \Gamma\left(w_2^n + \hat{B}_2^n - \frac{\hat{Q}_2^n(\cdot)}{\mu_3^n} 1_{\mathbf{A}(n,\cdot)^c} + \int_0^{\cdot} 1_{\mathbf{A}(n,s)} d\hat{I}_2^n(s) \right)(t) + \frac{\hat{Q}_2^n(t)}{\mu_3^n} 1_{\mathbf{A}(n,t)^c}.$$

- $\int_0^{\cdot} 1_{\mathbf{A}(n,s)} d\hat{I}_2^n(s) \to 0$ and so

$$\hat{W}_2^n \Rightarrow \Gamma(w_2 + B_2) = W_2^*.$$

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- Proof of $\int_0^{\cdot} 1_{\mathbf{A}(n,s)} d\hat{I}_2^n(s) \to 0$ uses exponential moments and large deviation estimates.

Convergence of Scaled Processes

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- Thus

$$\hat{W}_1^n \Rightarrow \Gamma\left(w_1 + B_1^* - \Psi(\hat{W}_2^*(\cdot))\right)(\cdot) + \Psi(\hat{W}_2^*(\cdot))$$
$$= w_1 + B_1^* + \sup_{0\le s\le \cdot}[w_1 + B_1^*(s) - \Psi(\hat{W}_2^*(s))] = W_1^*.$$

Problem: Case IID

- Case IID: $c_2\mu_2 - c_3\mu_2 < 0$, $c_2\mu_2 - c_1\mu_1 < 0$.
- Recall the running cost in the EWF:

$$h(w_1, w_2) = \begin{cases} (c_1\mu_1)w_1 + \frac{\mu_3}{\mu_2}(c_2\mu_2 - c_1\mu_1)w_2, & \text{when } \mu_3 w_2 \le \mu_2 w_1, \\ (c_2\mu_2 - c_3\mu_2)w_1 + (c_3\mu_3)w_2, & \text{when } \mu_3 w_2 \ge \mu_2 w_1, \end{cases}$$

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- Solution of the EWF not known.
- Conjecture. There are functions $\Psi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$ such that Ψ_i are Lipschitz, $\Psi_i(0) = 0$, $\Psi_i(u) \to \infty$ as $u \to \infty$, and an optimal state process Z^* is the unique solution of

$$Z_1^*(t) = B_1(t) + \sup_{0 \le s \le t} [B_1(s) - \Psi_1(Z_2^*(s))]^-$$

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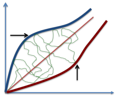

Problem: Free Boundary Characterizations for general SCSC

- Characterize optimal controls for general singular control problems with state constraints.
- 2D with no state constraints (i.e. state space is $\mathbb{R}^2$) studied by Shreve and Soner (1989, 1991).
  - Under suitable smoothness and growth conditions, value function is shown to be C^2.
  - Using this the free boundary is shown to be $C^{2,\alpha}$.
  - An optimal process is then constructed using Lions and Sznitman(1984).
- State Constraints?

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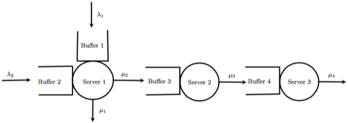
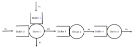
- Multiclass networks require more care (see eg. Dai-Lin(2007)).
- Consider the network below for which Stations 1 and 3 are critical and Station 2 is subcritical. I.e.

$$\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} = 1, \ \frac{\lambda_1}{\mu_3} < 1, \ \frac{\lambda_1}{\mu_4} = 1.$$



Let $J(q^n, T^n) = \mathbb{E} \int_0^\infty e^{-\gamma t} c \cdot \hat{Q}^n(t) dt$, where $\hat{Q}^n$ is the diffusion scaled queue-length process and $q^n = \hat{Q}^n(0) \to q$.

Problem: Control of Bottleneck Subnetworks



- Consider the three dimensional BCP obtained by removing the subcritical node:

$$\text{minimize } \mathbb{E}\left(\int_0^\infty e^{-\gamma t} c \cdot \tilde{Q}(t) dt\right),$$

subject to

$$0 \le \tilde{Q}_i(t) \doteq q_i + Z_i(t) + \mu_i \tilde{Y}_i(t),\ i = 1, 2,$$
$$0 \le \tilde{Q}_4(t) \doteq q_3 + Z_4(t) + \mu_4 \tilde{Y}_3(t) - \mu_2 \tilde{Y}_2(t),$$

and

$$\tilde{I}_1 \doteq \tilde{Y}_1 + \tilde{Y}_2,\ \tilde{I}_3 \doteq \tilde{Y}_3 \text{ are non-decreasing, and } \tilde{I}_i(0) = 0,\ i = 1, 3.$$

- Let $V(q)$ be the value of the BCP.
- Problem. $\lim_{n\to\infty} \inf_{\{T^n\}} J(q^n, T^n) = V(q)$?
- An intermediate step: $\hat{Q}_3^n \to 0$ for all 'reasonable' policies.