

From quantum many body systems to nonlinear dispersive PDE, and back

Nataša Pavlović (University of Texas at Austin)

Based on joint works with
T. Chen, C. Hainzl, R. Seiringer, N. Tzirakis

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Outline

- 1 Interacting bosons and nonlinear Schrödinger equation (NLS)
- 2 From bosons to NLS, via GP
- 3 Going backwards i.e. from NLS to bosons
- 4 Quantum de Finetti as a bridge between the NLS and the GP

Interacting bosons

The mathematical analysis of interacting Bose gases is a hot topic in Math Physics. One of the important research directions is:

- Proof of Bose-Einstein condensation

Bose-Einstein condensation

At very low temperatures dilute Bose gases are characterized by the “macroscopic occupancy of a single one-particle state”.

- **The prediction** in 1920's
Bose, Einstein
- **The first experimental realization** in 1995
Cornell-Wieman et al, Ketterle et al
- **Proof of Bose-Einstein condensation** around 2000
Aizenman-Lieb-Seiringer-Solovej-Yngvason, Lieb-Seiringer, Lieb-Seiringer-Yngvason

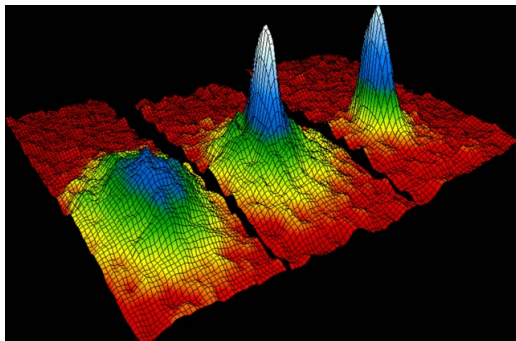


Figure : Velocity distribution data for a gas of rubidium atoms before/just after the appearance of a Bose-Einstein Condensate, and after further evaporation. The photo is a courtesy of Wikipedia.

Nonlinear Schrödinger equation (NLS)

The mathematical analysis of solutions to the nonlinear Schrödinger equation (NLS) has been a hot topic in PDE.

NLS is an example of a **dispersive**¹ equation.

¹Informally, “dispersion” means that different frequencies of the equation propagate at different velocities, i.e. the solution disperses over time.

The Cauchy problem for a nonlinear Schrödinger equation

$$(1.1) \quad iu_t + \Delta u = \mu |u|^{p-1} u$$

$$(1.2) \quad u(x, 0) = u_0(x) \in H^s(\Omega^n), \quad t \in \mathbb{R},$$

where Ω^n is either the space \mathbb{R}^n or the n -dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
The equation (1.1) is called

- defocusing if $\mu = 1$
- focusing if $\mu = -1$.

NLS - basic questions - I

- Local in time well-posedness, LWP** (existence of solutions, their uniqueness and continuous dependence on initial data²)
 - How: usually a fixed point argument.
 - Tools: Strichartz estimates
 - Then (in the '80s, '90s):
 - via **Harmonic Analysis** (e.g. *Kato, Cazenave-Weissler, Kenig-Ponce-Vega*)
 - via **Analytic Number Theory** (e.g. *Bourgain*)
 - via **Probability** (e.g. *Bourgain* a.s. LWP³)
 - Now:
 - via **Probability** (e.g. *Burq-Tzvetkov, Rey-Bellet - Nadmoh - Oh - Staffilani, Nahmod - Staffilani, Bourgain-Bulut*)
 - via **Incidence Theory** (a hot new direction *Bourgain-Demeter*)

²LWP: For any $u_0 \in X$ there exist $T > 0$ and a unique solution u to the IVP in $C([0, T], X)$ that is also stable in the appropriate topology.

³a.s. LWP: There exists $Y \subset X$, with $\mu(Y) = 1$ and such that for any $u_0 \in Y$ there exist $T > 0$ and a unique solution u to the IVP in $C([0, T], X)$ that is also stable in the appropriate topology.

NLS - more on local well-posedness

- (A) **Energy methods:** integrate by parts the IVP to obtain an a priori bound $\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^s} \leq C(T, u_0)$. Then use approximative methods to obtain a sequence for which the bound is valid and take a weak limit.

Bad news: usually too many derivatives are needed.

- (B) **Iterative methods:** by the Duhamel's formula the IVP

$$iu_t + Lu = N(u)$$

is equivalent to the integral equation

$$u(t) = U(t)u_0 + \int_0^t U(t-\tau)N(u(\tau))d\tau,$$

where $U(t)$ is the solution operator associated to the linear problem.

Tools: Strichartz estimates (*Strichartz, Ginibre-Velo, Yajima, Keel-Tao*)

For any admissible pairs (q, r) and (\tilde{q}, \tilde{r}) we have

$$(1.3) \quad \|U(t)u_0\|_{L_t^q L_x^r} \leq C \|u_0\|_{L_x^2}.$$

$$(1.4) \quad \left\| \int_0^t U(t-\tau)N(\tau) d\tau \right\|_{L_t^q L_x^r} \leq C \|N\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Good news: one can treat problems with much less regularity.

Bad news: some smallness is needed (e.g. short times or small data).

NLS - basic questions - II

2 Global in time well-posedness/blow-up

- How: LWP + use of conserved quantities
- Tools: very technical clever constructions in order to access conserved quantities
- Then (in the '00s):
 - via **Harmonic Analysis** (e.g. *Bourgain* and *Colliander-Keel-Staffilani-Takaoka-Tao* induction on energy, *Kenig-Merle* concentration-compactness, *Killip - Visan*)
- Now:
 - via **Probability** (a construction of Gibbs measure e.g. *Burq-Tzvetkov*, *Oh*, *Rey-Bellet - Nadmoh - Oh - Staffilani*, *Bourgain-Bulut*).

Bosons and NLS

What is a connection between:

- interacting bosons
and
- NLS?

Rigorous derivation of the NLS from quantum many body systems

- How: the topic of this talk
- Then (in the late '70s and the '80s):
 - via **Quantum Field Theory** (*Hepp, Ginibre-Velo*)
 - via **Math Physics** (*Spohn*)
- Now:
 - via **Quantum Field Theory** (*Rodnianski-Schlein, Grillakis-Machedon-Margetis, Grillakis-Machedon, X. Chen*)
 - via **Math Physics** (*Fröhlich-Tsai-Yau, Bardos-Golse-Mauser, Erdős-Yau, Adami-Bardos-Golse-Teta, Elgart-Erdős-Schlein-Yau, Erdős-Schlein-Yau*)
 - via **Math Physics + Dispersive PDE** (*Klainerman-Machedon, Kirkpatrick-Schlein-Staffilani, Chen-P., Chen-P.-Tzirakis, Gressman-Sohinger-Staffilani, Sohinger, X. Chen, X. Chen-Holmer, Chen-Hainzl-P.-Seiringer*)

From bosons to NLS following Erdős-Schlein-Yau [2006-07]

Step 1: From N -body Schrödinger to BBGKY hierarchy

The starting point is **a system of N bosons whose dynamics is generated by the Hamiltonian**

$$(2.1) \quad H_N := \sum_{j=1}^N (-\Delta_{x_j}) + \frac{1}{N} \sum_{1 \leq i < j \leq N} V_N(x_i - x_j),$$

on the Hilbert space $\mathcal{H}_N = L^2_{sym}(\mathbb{R}^{dN})$, whose elements $\Psi(x_1, \dots, x_N)$ are fully symmetric with respect to permutations of the arguments x_j .

Here

$$V_N(x) = N^{d\beta} V(N^\beta x),$$

with $0 < \beta \leq 1$.

When $\beta = 1$, the Hamiltonian

$$(2.2) \quad H_N := \sum_{j=1}^N (-\Delta_{x_j}) + \frac{1}{N} \sum_{1 \leq i < j \leq N} V_N(x_i - x_j),$$

is called the Gross-Pitaevskii Hamiltonian.

- We note that physically (2.2) describes a very dilute gas, where **interactions among particles are very rare and strong**.
- This is in contrast to a mean field Hamiltonian, where each particle usually reacts with all other particles via a very weak potential.
- However thanks to the factor $\frac{1}{N}$ in front of the interaction potential, (2.2) can be formally interpreted as a mean field Hamiltonian. In particular, one can still apply to (2.2) similar mathematical methods as in the case of a mean field potential.

Schrödinger equation

The wave function satisfies the Schrödinger equation

$$(2.3) \quad i\partial_t \psi_N = H_N \psi_N,$$

with initial condition $\Psi_{N,0} \in \mathcal{H}_N$.

- Since the Schrödinger equation (2.3) is linear and the Hamiltonian H_N is self-adjoint, global well-posedness of (2.3) is not an issue.

On the N-body Schrödinger equation

Bad news:

- Qualitative and quantitative properties of the solution are hard to extract in physically relevant cases when number of particles N is very large (e.g. it varies from 10^3 for very dilute Bose-Einstein samples, to 10^{30} in stars).

Good news:

- Physicists often care about macroscopic properties of the system, which can be obtained from averaging over a large number of particles.
- Further simplifications are related to obtaining a macroscopic behavior in the limit as $N \rightarrow \infty$, with a hope that the limit will approximate properties observed in the experiments for a very large, but finite N .

To study the limit as $N \rightarrow \infty$, one introduces:

- **the N -particle density matrix**

$$\gamma_N(t, \underline{x}_N; \underline{x}'_N) = \Psi_N(t, \underline{x}_N) \overline{\Psi_N(t, \underline{x}'_N)},$$

- **and its k -particle marginals**

$$\gamma_N^{(k)}(t, \underline{x}_k; \underline{x}'_k) = \int d\underline{x}_{N-k} \gamma_N(t, \underline{x}_k, \underline{x}_{N-k}; \underline{x}'_k, \underline{x}_{N-k}),$$

for $k = 1, \dots, N$.

Here

$$\begin{aligned} \underline{x}_k &= (x_1, \dots, x_k), \\ \underline{x}_{N-k} &= (x_{k+1}, \dots, x_N). \end{aligned}$$

The BBGKY hierarchy is given by

$$(2.4) \quad i\partial_t \gamma_N^{(k)} = -(\Delta_{x_k} - \Delta_{x'_k}) \gamma_N^{(k)} + \frac{1}{N} \sum_{1 \leq i < j \leq k} (V_N(x_i - x_j) - V_N(x'_i - x'_j)) \gamma_N^{(k)}$$

$$(2.5) \quad + \frac{N-k}{N} \sum_{i=1}^k \text{Tr}_{k+1} (V_N(x_i - x_{k+1}) - V_N(x'_i - x'_{k+1})) \gamma_N^{(k+1)}$$

In the limit $N \rightarrow \infty$, the sums weighted by combinatorial factors have the following size:

- In (2.4), $\frac{k^2}{N} \rightarrow 0$ for every fixed k and sufficiently small β .
- In (2.5), $\frac{N-k}{N} \rightarrow 1$ for every fixed k and $V_N(x_i - x_j) \rightarrow b_0 \delta(x_i - x_j)$, with $b_0 = \int dx V(x)$.

Step 2: BBGKY hierarchy \rightarrow GP hierarchy

As $N \rightarrow \infty$, one obtains the infinite GP hierarchy as a weak limit.

$$i\partial_t \gamma_\infty^{(k)} = - \sum_{j=1}^k (\Delta_{x_j} - \Delta_{x'_j}) \gamma_\infty^{(k)} + b_0 \sum_{j=1}^k B_{j;k+1} \gamma_\infty^{(k+1)}$$

where the “**contraction operator**” is given via

$$\begin{aligned} & (B_{j;k+1} \gamma_\infty^{(k+1)})(t, x_1, \dots, x_k; x'_1, \dots, x'_k) \\ &= \gamma_\infty^{(k+1)}(t, x_1, \dots, x_j, \dots, x_k, x_j; x'_1, \dots, x'_k, x_j) \\ & - \gamma_\infty^{(k+1)}(t, x_1, \dots, x_k, x'_j; x'_1, \dots, x'_j, \dots, x'_k, x'_j). \end{aligned}$$

Step 3: Factorized solutions of the GP hierarchy

It is easy to see that

$$\gamma_\infty^{(k)} = |\phi\rangle\langle\phi|^{\otimes k} := \prod_{j=1}^k \phi(t, x_j) \overline{\phi(t, x'_j)}$$

is a solution of the GP if ϕ satisfies the cubic NLS

$$i\partial_t\phi + \Delta_x\phi - b_0|\phi|^2\phi = 0$$

with $\phi_0 \in L^2(\mathbb{R}^d)$.

Step 4: Uniqueness of solutions to the GP hierarchy

While the existence of factorized solutions can be easily obtained, the proof of **uniqueness of solutions** of the GP hierarchy is the most difficult⁴ part in this analysis.

⁴We will describe those difficulties soon.

Summary of the method of ESY

Roughly speaking, the method of *Erdős, Schlein, and Yau* for deriving the cubic NLS justifies the heuristic explained above and it consists of the following two steps:

- (i) **Deriving the GP hierarchy as the limit as $N \rightarrow \infty$ of the BBGKY hierarchy.**
- (ii) **Proving uniqueness of solutions for the GP hierarchy**, which implies that for factorized initial data, the solutions of the GP hierarchy are determined by a cubic NLS. The proof of uniqueness is accomplished by using highly sophisticated **Feynman graphs**.

Why is it difficult to prove uniqueness?

Fix a positive integer r . Let us express the solution $\gamma^{(r)}$ to the GP, with i.d. 0.

$$\begin{aligned} \gamma^{(r)}(t_r, \cdot) &= \int_0^{t_r} e^{i(t_r - t_{r+1})\Delta_{\pm}^{(r)}} B_{r+1}(\gamma^{(r+1)}(t_{r+1})) dt_{r+1} \\ &= \int_0^{t_r} \int_0^{t_{r+1}} e^{i(t_r - t_{r+1})\Delta_{\pm}^{(r)}} B_{r+1} e^{i(t_{r+1} - t_{r+2})\Delta_{\pm}^{(r+1)}} B_{r+2}(\gamma^{(r+2)}(t_{r+2})) dt_{r+1} dt_{r+2} \\ &= \dots \end{aligned}$$

(2.6)

$$= \int_0^{t_r} \dots \int_0^{t_{r+n-1}} J^r(\underline{t}_{r+n}) dt_{r+1} \dots dt_{r+n},$$

where

$$\underline{t}_{r+n} = (t_r, t_{r+1}, \dots, t_{r+n}),$$

$$J^r(\underline{t}_{r+n}) = e^{i(t_r - t_{r+1})\Delta_{\pm}^{(r)}} B_{r+1} \dots e^{i(t_{r+(n-1)} - t_{r+n})\Delta_{\pm}^{(r+(n-1))}} B_{r+n}(\gamma^{(r+n)}(t_{r+n})).$$

Since the interaction term involves the sum, the iterated Duhamel's formula has $r(r+1)\dots(r+n-1)$ terms.

A remark about ESY solutions of the GP hierarchy

- Solutions of the GP hierarchy are studied in “ L^1 -type trace Sobolev” spaces of k -particle marginals

$$\{\gamma^{(k)} \mid \|\gamma^{(k)}\|_{\mathfrak{h}^1} < \infty\}$$

with norms

$$\|\gamma^{(k)}\|_{\mathfrak{h}^\alpha} := \mathrm{Tr}(|\mathbf{S}^{(k,\alpha)}\gamma^{(k)}|),$$

where⁵

$$\mathbf{S}^{(k,\alpha)} := \prod_{j=1}^k \langle \nabla_{x_j} \rangle^\alpha \langle \nabla_{x'_j} \rangle^\alpha.$$

⁵Here we use the standard notation: $\langle y \rangle := \sqrt{1 + y^2}$.

An alternative method for proving uniqueness of GP

Klainerman and Machedon (2008) introduced an alternative method for proving uniqueness in a space of density matrices equipped with the Hilbert-Schmidt type Sobolev norm

$$\|\gamma^{(k)}\|_{H_k^\alpha} := \|\mathcal{S}^{(k,\alpha)}\gamma^{(k)}\|_{L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})}.$$

The method is based on:

- a reformulation of the relevant combinatorics via the “**board game argument**” and
- the use of certain **space-time estimates** of the type:

$$\|B_{j;k+1} e^{it\Delta_\pm^{(k+1)}} \gamma^{(k+1)}\|_{L_t^2 \dot{H}^\alpha(\mathbb{R} \times \mathbb{R}^{dk} \times \mathbb{R}^{dk})} \lesssim \|\gamma^{(k+1)}\|_{\dot{H}^\alpha(\mathbb{R}^{d(k+1)} \times \mathbb{R}^{d(k+1)})}.$$

The method of *Klainerman and Machedon* makes the assumption that the a priori space-time bound

$$(2.7) \quad \|B_{j;k+1}\gamma^{(k+1)}\|_{L_t^1 H_k^1} < C^k,$$

holds, with C independent of k .

Subsequently:

- *Kirkpatrick, Schlein and Staffilani* (2011) were the first to use the KM formulation to derive the cubic NLS in $d = 2$ via proving that the limit of the BBGKY satisfies (2.7).
- *Chen-P* (2011) generalized this to derive the quintic GP in $d = 1, 2$.
- *Xie* (2013) generalized it further to derive a NLS with a general power-type nonlinearity in $d = 1, 2$.
- A derivation of the cubic NLS in $d = 3$ based on the KM combinatorial formulation was settled recently by *Chen-P*; subsequently revisited by *X. Chen, X. Chen-Holmer and T. Chen-Taliaferro*.

Going backwards i.e. from NLS to bosons

Since the GP arises in a derivation of the NLS from quantum many-body system, it is natural to ask:

- Are properties of solutions to NLS generically shared by solutions of the QFT it is derived from?
- Whether methods of nonlinear dispersive PDE can be “lifted” to QFT level?

The Cauchy problem for the GP - joint work with T. Chen

The work of *Klainerman and Machedon* inspired us to study well-posedness for the Cauchy problem for GP hierarchies.

Towards a well-posedness result for the GP

Problem: The equations for $\gamma^{(k)}$ do not close & no fixed point argument.

Solution: Endow the space of sequences

$$\Gamma := (\gamma^{(k)})_{k \in \mathbb{N}}.$$

with a suitable topology.

Revisiting the GP hierarchy

Recall,

$$\Delta_{\pm}^{(k)} = \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k}, \quad \text{with} \quad \Delta_{\underline{x}_k} = \sum_{j=1}^k \Delta_{x_j}.$$

We introduce the notation:

$$\Gamma = (\gamma^{(k)}(t, x_1, \dots, x_k; x'_1, \dots, x'_k))_{k \in \mathbb{N}},$$

$$\widehat{\Delta}_{\pm} \Gamma := (\Delta_{\pm}^{(k)} \gamma^{(k)})_{k \in \mathbb{N}},$$

$$\widehat{B} \Gamma := (B_{k+1} \gamma^{(k+1)})_{k \in \mathbb{N}}.$$

Then, the cubic GP hierarchy can be written as⁶

$$(3.1) \quad i \partial_t \Gamma + \widehat{\Delta}_{\pm} \Gamma = \mu \widehat{B} \Gamma.$$

⁶Moreover, for $\mu = 1$ we refer to the GP hierarchy as defocusing, and for $\mu = -1$ as focusing.

Spaces

Let

$$\mathfrak{G} := \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$$

be the space of sequences of density matrices

$$\Gamma := (\gamma^{(k)})_{k \in \mathbb{N}}.$$

As a crucial ingredient of our arguments, we introduce Banach spaces $\mathcal{H}_{\xi}^{\alpha} = \{ \Gamma \in \mathfrak{G} \mid \| \Gamma \|_{\mathcal{H}_{\xi}^{\alpha}} < \infty \}$ where

$$\| \Gamma \|_{\mathcal{H}_{\xi}^{\alpha}} := \sum_{k \in \mathbb{N}} \xi^k \| \gamma^{(k)} \|_{H^{\alpha}(\mathbb{R}^{dk} \times \mathbb{R}^{dk})}.$$

Properties:

- **Finiteness:** $\| \Gamma \|_{\mathcal{H}_{\xi}^{\alpha}} < C$ implies that $\| \gamma^{(k)} \|_{H^{\alpha}(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} < C \xi^{-k}$.
- **Interpretation:** ξ^{-1} upper bound on typical H^{α} -energy per particle.

Some results for the GP - inspired by the NLS theory

- 1 **Local in time existence** of solutions to GP.
 - *Chen-P* (2010, 2013)
- 2 **Blow-up** of solutions to the focusing GP hierarchies in certain cases.
 - *Chen-P-Tzirakis* (2010)
- 3 **Global existence** of solutions to the GP hierarchy in certain cases.
 - *Chen-P* (2013), *Chen-Taliaferro* (2014)
- 4 **Derivation of the cubic GP hierarchy** in [KM] spaces.
 - *Chen-P* (2013), *X. Chen* (2014), *X. Chen-Holmer* (2014), *Chen-Taliaferro* (2014)
- 5 **Uniqueness of the cubic GP hierarchy** on \mathbb{T}^3 .
 - *Gressman-Sohinger-Staffilani* (2014), *Sohinger* (2014)
- 6 **Uniqueness of the cubic GP hierarchy** on \mathbb{R}^3 revisited.
 - *Chen-Hainzl-P-Seiringer* (2014), *Sohinger* (2014), *Hong-Taliaferro-Xie* (2014)

Local in time existence and uniqueness

(joint work with T. Chen - 2010)

We prove **local in time existence and uniqueness** of solutions to the cubic and quintic GP hierarchy with focusing or defocusing interactions, in a subspace of \mathcal{H}_ξ^α , for $\alpha \in \mathfrak{A}(d, p)$, which satisfy a spacetime bound

$$(3.2) \quad \|\widehat{B}\Gamma\|_{L^1_{t \in I} \mathcal{H}_\xi^\alpha} < \infty,$$

for some $\xi > 0$.

Flavor of the proof:

Note that the GP hierarchy can be formally written as a system of integral equations

$$(3.3) \quad \Gamma(t) = e^{it\hat{\Delta}\pm}\Gamma_0 - i\mu \int_0^t ds e^{i(t-s)\hat{\Delta}\pm} \hat{B}\Gamma(s)$$

$$(3.4) \quad \hat{B}\Gamma(t) = \hat{B}e^{it\hat{\Delta}\pm}\Gamma_0 - i\mu \int_0^t ds \hat{B}e^{i(t-s)\hat{\Delta}\pm} \hat{B}\Gamma(s),$$

where (3.4) is obtained by applying the operator \hat{B} on the linear non-homogeneous equation (3.3).

We prove the local well-posedness result by applying the fixed point argument in the following space:

$$(3.5) \quad \mathfrak{W}_\xi^\alpha(I) := \{ \Gamma \in L_{t \in I}^\infty \mathcal{H}_\xi^\alpha \mid \hat{B}\Gamma \in L_{t \in I}^1 \mathcal{H}_\xi^\alpha \},$$

where $I = [0, T]$.

Dispersive tools at the level of the GP

- 1 Tools at the level of the GP, that are inspired by the NLS techniques, are instrumental in understanding:
 - Well-posedness for the GP hierarchy
 - Well-posedness for quantum many body systems
 - Going from bosons to NLS in Klainerman-Machedon spaces
- 2 But there were still few questions that resisted the efforts to apply newly built tools at the level of the GP, e.g.
 - Long time behavior of the GP hierarchy
 - Uniqueness of the cubic GP on \mathbb{T}^3
 - Uniqueness of the quintic GP on \mathbb{R}^3

Q & A session

- Q** How to address the questions that are “GP tools resistant”?
- A** Use tools at the level of the NLS?
-
- Q** How to use NLS tools when considering the GP?
- A** Apply **the quantum de Finetti theorem**, which roughly says that (relevant) solutions to the GP are given via an average of factorized solutions.

Quantum de Finetti as a bridge between the NLS and the GP

What is quantum De Finetti?

Strong quantum de Finetti theorem

Due to: *Hudson-Moody (1976/77), Stormer (1969), Lewin-Nam-Rougerie (2013)*

Theorem

(Strong Quantum de Finetti theorem) Let \mathcal{H} be any separable Hilbert space and let $\mathcal{H}^k = \bigotimes_{\text{sym}}^k \mathcal{H}$ denote the corresponding bosonic k -particle space. Let Γ denote a collection of admissible bosonic density matrices on \mathcal{H} , i.e.,

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots)$$

with $\gamma^{(k)}$ a non-negative trace class operator on \mathcal{H}^k , and $\gamma^{(k)} = \text{Tr}_{k+1} \gamma^{(k+1)}$, where Tr_{k+1} denotes the partial trace over the $(k+1)$ -th factor. Then, there exists a unique Borel probability measure μ , supported on the unit sphere $S \subset \mathcal{H}$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that

$$(4.1) \quad \gamma^{(k)} = \int d\mu(\phi) (|\phi\rangle\langle\phi|)^{\otimes k}, \quad \forall k \in \mathbb{N}.$$

Weak quantum de Finetti theorem

The limiting hierarchies obtained via weak-* limits from the BBGKY hierarchy of bosonic N -body Schrödinger systems as in *Erdős-Schlein-Yau* do not necessarily satisfy admissibility.

- A weak version of the quantum de Finetti theorem then still applies.
- We use the version that was recently proven by *Lewin-Nam-Rougerie*.

Uniqueness of solutions to the GP via quantum de Finetti theorems

- Until recently, the only available proof of unconditional uniqueness of solutions in $L_{t \in [0, T]}^\infty \mathfrak{S}^1$ to the cubic GP hierarchy in \mathbb{R}^3 was given in the works of Erdős, Schlein, and Yau, who developed an approach based on use of Feynman graphs. A key ingredient in their proof is a powerful combinatorial method that resolves the problem of the factorial growth of number of terms in iterated Duhamel expansions.
- Recently, together with T. Chen, C. Hainzl and R. Seiringer, we obtained a new proof based on quantum de Finetti theorem.

⁷The \mathfrak{S}^1 denotes the trace class Sobolev space defined for the entire sequence $(\gamma^{(k)})_{k \in \mathbb{N}}$:

$$\mathfrak{S}^1 := \left\{ (\gamma^{(k)})_{k \in \mathbb{N}} \mid \text{Tr}(|S^{(k,1)} \gamma^{(k)}|) < M^{2k} \text{ for some constant } M < \infty \right\}.$$

Mild solution to the GP hierarchy

A **mild solution** in the space $L_{t \in [0, T]}^\infty \mathfrak{H}^1$, to the GP hierarchy with initial data $(\gamma^{(k)}(0))_{k \in \mathbb{N}} \in \mathfrak{H}^1$, is a solution of the integral equation

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\lambda \int_0^t U^{(k)}(t-s)B_{k+1}\gamma^{(k+1)}(s)ds \quad , \quad k \in \mathbb{N},$$

satisfying

$$\sup_{t \in [0, T]} \text{Tr}(|S^{(k,1)}\gamma^{(k)}(t)|) < M^{2k}$$

for a finite constant M independent of k .

Here,

$$U^{(k)}(t) := \prod_{\ell=1}^k e^{it(\Delta_{x_\ell} - \Delta_{x'_\ell})}$$

denotes the free k -particle propagator.

Statement of the result

Theorem (Chen-Hainzl-P-Seiringer)

Let $(\gamma^{(k)}(t))_{k \in \mathbb{N}}$ be a mild solution in $L_{t \in [0, T]}^\infty \mathfrak{H}^1$ to the (de)focusing cubic GP hierarchy in \mathbb{R}^3 with initial data $(\gamma^{(k)}(0))_{k \in \mathbb{N}} \in \mathfrak{H}^1$, which is either admissible, or obtained at each t from a weak- $*$ limit.

Then, $(\gamma^{(k)})_{k \in \mathbb{N}}$ is the unique solution for the given initial data.

Moreover, assume that the initial data $(\gamma^{(k)}(0))_{k \in \mathbb{N}} \in \mathfrak{H}^1$ satisfy

$$(4.2) \quad \gamma^{(k)}(0) = \int d\mu(\phi) (|\phi\rangle\langle\phi|)^{\otimes k}, \quad \forall k \in \mathbb{N},$$

where μ is a Borel probability measure supported either on the unit sphere or on the unit ball in $L^2(\mathbb{R}^3)$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one. Then,

$$(4.3) \quad \gamma^{(k)}(t) = \int d\mu(\phi) (|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k}, \quad \forall k \in \mathbb{N},$$

where $S_t : \phi \mapsto \phi_t$ is the flow map of the cubic (de)focusing NLS.

Key tools that we use:

- 1 **The boardgame combinatorial organization** as presented by *Klainerman and Machedon* (KM)
- 2 **The quantum de Finetti theorem** allows one to avoid using the condition that was assumed in the work of KM.

Setup of the proof

Assume that we have two positive semidefinite solutions $(\gamma_j^{(k)}(t))_{k \in \mathbb{N}} \in L^\infty_{t \in [0, T]} \mathfrak{H}^1$ satisfying the same initial data,

$$(\gamma_1^{(k)}(0))_{k \in \mathbb{N}} = (\gamma_2^{(k)}(0))_{k \in \mathbb{N}} \in \mathfrak{H}^1.$$

Then,

$$(4.4) \quad \gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t) \quad , \quad k \in \mathbb{N},$$

is a solution to the GP hierarchy with initial data $\gamma^{(k)}(0) = 0 \quad \forall k \in \mathbb{N}$, and it suffices to prove that

$$\gamma^{(k)}(t) = 0$$

for all $k \in \mathbb{N}$, and for all $t \in [0, T)$.

Remarks:

- From de Finetti theorems, we have

$$\begin{aligned} \gamma_j^{(k)}(t) &= \int d\mu_t^{(j)}(\phi) (|\phi\rangle\langle\phi|)^{\otimes k}, \quad j = 1, 2, \\ (4.5) \quad \gamma^{(k)}(t) &= \int d\tilde{\mu}_t(\phi) (|\phi\rangle\langle\phi|)^{\otimes k}, \end{aligned}$$

where $\tilde{\mu}_t := \mu_t^{(1)} - \mu_t^{(2)}$ is the difference of two probability measures on the unit ball in $L^2(\mathbb{R}^3)$.

- From the assumptions of Theorem 2, we have that

$$(4.6) \quad \sup_{t \in [0, T)} \text{Tr}(|S^{(k,1)} \gamma_i^{(k)}(t)|) < M^{2k}, \quad k \in \mathbb{N}, \quad i = 1, 2,$$

for some finite constant M , which is equivalent to

$$(4.7) \quad \int d\mu_t^{(j)}(\phi) \|\phi\|_{H^1}^{2k} < M^{2k}, \quad j = 1, 2,$$

for all $k \in \mathbb{N}$.

Representation of solution using KM and de Finetti

KM implies that we can represent $\gamma^{(k)}(t)$ in upper echelon form:

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{N}_{k,r}} \int_{D(\sigma,t)} dt_1 \cdots dt_r U^{(k)}(t - t_1) B_{\sigma^{(k+1)}, k+1} U^{(k+1)}(t_1 - t_2) \cdots \\ \cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{\sigma^{(k+r)}, k+r} \gamma^{(k+r)}(t_r)$$

Now using the quantum de Finetti theorem, we obtain:

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{N}_{k,r}} \int_{D(\sigma,t)} dt_1, \dots, dt_r \int d\tilde{\mu}_r(\phi) J^k(\sigma; t, t_1, \dots, t_r),$$

where

$$J^k(\sigma; t, t_1, \dots, t_r; \underline{x}_k; \underline{x}'_k) = \left(U^{(k)}(t - t_1) B_{\sigma^{(k+1)}, k+1} U^{(k+1)}(t_1 - t_2) \cdots \right. \\ \left. \cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{\sigma^{(k+r)}, k+r} (|\phi\rangle\langle\phi|)^{\otimes(k+r)} \right) (\underline{x}_k; \underline{x}'_k).$$

Roadmap of the proof

- 1 recognize that a certain product structure gets preserved from right to left (via recursively introducing kernels that account for contractions performed by B operators)
- 2 get an estimate on integrals in upper echelon form via recursively performing **Strichartz estimates (at the level of the Schrödinger equation)** from left to right

Recent related works

- **Existence of scattering states for the GP via quantum de Finetti**
Chen-Hainzl-P-Seiringer (2014)
- **Uniqueness of solutions to the cubic GP in low regularity spaces**
Hong-Taliaferro-Xie (2014)
- **Uniqueness of solutions to the quintic GP on \mathbb{R}^3**
Hong-Taliaferro-Xie (2014)
- **Uniqueness of solutions to the cubic GP on \mathbb{T}^d**
Sohinger (2014), Herr-Sohinger (2014)
- **Uniqueness of solutions to the infinite hierarchy that appears in a connection to the Chern-Simons-Schrödinger system**
X. Chen-Smith (2014)

Back and forth from many body systems to nonlinear equations

Other examples:

- “From Newton to Boltzmann: hard spheres and short-range potentials”
Gallagher - Saint-Raymond - Texier, 2012
- “Kac’s Program in Kinetic Theory”
Mischler - Mouhot, 2011