

# Homogenization of a Transmission Problem

F. Cakoni <sup>1</sup>   B. Guzina <sup>2</sup>   S. Moskow <sup>3</sup>

<sup>1</sup>University of Delaware

<sup>2</sup>University of Minnesota

<sup>3</sup>Drexel University

Institute for Mathematics and its Applications  
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- Bounded domain usually Dirichlet boundary conditions
- Scatterers in free space obey transmission boundary conditions
- This matters; boundary layers play a large role

# Model of a periodic scatterer

Scattering of time harmonic incident field  $u^i$  by periodic inhomogeneity  $D$ .  
Scattered field given by  $u^s$ , total field  $u = u^s + u^i$ .

$$\nabla \cdot a(x/\epsilon) \nabla u + k^2 n(x/\epsilon) u = 0 \quad \text{in } D \quad (1)$$

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^d \setminus \bar{D} \quad (2)$$

$$(u^s + u^i) = u \quad \text{on } \partial D \quad (3)$$

$$\nabla(u^s + u^i) \cdot \nu = a(x/\epsilon) \nabla u \cdot \nu \quad \text{on } \partial D \quad (4)$$

$$\lim_{r \rightarrow \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0 \quad (5)$$

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- Assume  $a(y)$  is positive definite symmetric matrix valued function
- Assume  $n(x/\epsilon)$ ,  $a(x/\epsilon) \in C^\infty(D)$  positive

# Model of a periodic scatterer

The above scattering problem for inhomogeneous media  $D$  with periodically varying coefficients, can be formulated as the following transmission problem for  $u_\epsilon := u$  in  $D$  and  $u_\epsilon := u^s$  in  $\mathbb{R}^d \setminus \overline{D}$

$$\begin{aligned} \nabla \cdot a(x/\epsilon) \nabla u_\epsilon + k^2 n(x/\epsilon) u_\epsilon &= 0 && \text{in } D \\ \Delta u_\epsilon + k^2 u_\epsilon &= 0 && \text{in } \mathbb{R}^d \setminus \overline{D} \\ u_\epsilon^+ - u_\epsilon^- &= f && \text{on } \partial D \\ (\nabla u_\epsilon \cdot \nu)^+ - (a(x/\epsilon) \nabla u_\epsilon \cdot \nu)^- &= g && \text{on } \partial D \end{aligned} \tag{6}$$

where  $u_\epsilon$  satisfies the Sommerfeld radiation condition at infinity. Here  $f := u^i$  and  $g := \nu \cdot \nabla u^i$  on  $\partial D$ .

- Well known homogenization theory
- Bensoussan, Lions, Papanicolaou '78
- Tartar, Sanchez-Palencia

## Homogenized problem (expected)

$$\begin{aligned}\nabla \cdot A \nabla u_0 + k^2 \bar{n} u_0 &= 0 && \text{in } D \\ \Delta u_0 + k^2 u_0 &= 0 && \text{in } \mathbb{R}^d \setminus \bar{D} \\ u_0^+ - u_0^- &= f && \text{on } \partial D \\ (\nabla u_0 \cdot \nu)^+ - (A \nabla u_0 \cdot \nu)^- &= g && \text{on } \partial D\end{aligned}\tag{7}$$

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# Homogenized problem (expected)

$\bar{n}$  denotes the unit cell average of  $n$ , i.e.

$$\bar{n} = \int_Y n(y) dy,$$

and  $A$  is the constant-valued homogenized matrix

$$A_{ij} = \int_Y \left( a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) dy, \quad (8)$$

Here  $\chi^j(y)$  are the so-called cell functions which represent the  $Y$ -periodic solutions to

$$\frac{\partial}{\partial y_i} \left( a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) = 0. \quad (9)$$

The additive constant for  $\chi^j$  is always chosen so that

$$\int_Y \chi^j dy = 0.$$

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- Write the equation for  $u_\epsilon$  inside of  $D$  as a first-order system

$$\begin{aligned} a(x/\epsilon)\nabla u_\epsilon - v_\epsilon &= 0 \\ \nabla \cdot v_\epsilon + k^2 n(x/\epsilon)u_\epsilon &= 0 \end{aligned} \tag{10}$$

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$$\begin{aligned} a(x/\epsilon)\nabla u_\epsilon - v_\epsilon &= 0 \\ \nabla \cdot v_\epsilon + k^2 n(x/\epsilon)u_\epsilon &= 0 \end{aligned} \quad (10)$$

- Write the ansatz

$$\begin{aligned} u_\epsilon &= u_0(x, x/\epsilon) + \epsilon u^{(1)}(x, x/\epsilon) + \epsilon^2 u^{(2)}(x, x/\epsilon) + \dots \\ v_\epsilon &= v_0(x, x/\epsilon) + \epsilon v^{(1)}(x, x/\epsilon) + \epsilon^2 v^{(2)}(x, x/\epsilon) + \dots \end{aligned} \quad (11)$$

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- Plug into equations using

$$\nabla = \nabla_x + \frac{1}{\epsilon} \nabla_y,$$

and equate like powers of  $\epsilon$ .

This yields

$$a(y)\nabla_y u_0 = 0 \quad (12)$$

$$\nabla_y \cdot v_0 = 0 \quad (13)$$

$$a(y)\nabla_y u^{(1)} + a(y)\nabla_x u_0 - v_0 = 0 \quad (14)$$

$$\nabla_y \cdot v^{(1)} + \nabla_x \cdot v_0 + k^2 n(y)u_0 = 0 \quad (15)$$

## Corrections (first order)

- Equations (13) and (14) yield the equation (9) for  $\chi^j$  and

$$u^{(1)} = -\chi^j(y) \frac{\partial u_0}{\partial x_j}, \quad (16)$$

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- Taking the cell average in (15) yields the homogenized PDE in the interior of  $D$  (7).
- Furthermore

$$\begin{aligned} (v_0(x, y))_i &= (a(y) \nabla_x u_0(x) + a(y) \nabla_y u^{(1)}(x, y))_i \\ &= \left( a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^k}{\partial y_j}(y) \right) \frac{\partial u_0}{\partial x_j}, \end{aligned} \quad (17)$$

and

$$\bar{v}_0 = \int_Y v_0 \, dy = A \nabla u_0.$$



# Corrections (first order and second order)

We go further to find

$$u^{(2)} = \chi^{ij}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j} + k^2 \beta(y) u_0 \quad (18)$$

$$v^{(1)} = -a \chi^j \nabla_x \frac{\partial u_0}{\partial x_j} + a \nabla_y \chi^{ij} \frac{\partial u_0}{\partial x_i \partial x_j} + k^2 a \nabla_y \beta u_0 \quad (19)$$

where  $\chi^{ij}$  are defined by

$$\nabla_y \cdot (a \nabla_y \chi^{ij}) = b_{ij}(y) - \bar{b}_{ij} \quad (20)$$

with

$$b_{ij}(y) = -a_{ij} + a_{ik} \frac{\partial \chi^j}{\partial y_k} + \frac{\partial}{\partial y_k} (a_{ki} \chi^j), \quad (21)$$

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and where we also have a cell function from the lower order term

$$\nabla_y \cdot (a \nabla_y \beta(y)) = \bar{n} - n(y), \quad (22)$$

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- This is further disturbed by the the presence of  $v^{(1)}$ .
- To compensate need a boundary corrector.

# First order boundary correction

The first order boundary corrector function is

$$\begin{aligned}\nabla \cdot a(x/\epsilon)\nabla\theta_\epsilon + k^2n(x/\epsilon)\theta_\epsilon &= 0 \quad \text{in } D \\ \Delta\theta_\epsilon + k^2\theta_\epsilon &= 0 \quad \text{in } \mathbb{R}^d \setminus \bar{D} \\ \theta_\epsilon^+ - \theta_\epsilon^- &= u^{(1)} \quad \text{on } \partial D \\ (\nabla\theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\theta_\epsilon \cdot \nu)^- &= \left( \frac{v_0 - \bar{v}_0}{\epsilon} + v^{(1)} \right) \cdot \nu \quad \text{on } \partial D\end{aligned}\tag{23}$$

with Sommerfeld radiation conditions at infinity.



## Theorem

Let  $u_\epsilon$  be the solution to (6),  $u_0$  the solution to (7), and the bulk and boundary corrections  $u^{(1)}$  and  $\theta_\epsilon$  given by (16) and (23) respectively. Then for any ball  $B_R$  of radius  $R > 0$  which contains  $D$ ,

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon \theta_\epsilon)\|_{L^2(B_R)} \leq C_R \epsilon^2 \|u_0\|_{H^4(D)}$$

and

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon \theta_\epsilon)\|_{H^1(D)} + \|u_\epsilon - (u_0 + \epsilon \theta_\epsilon)\|_{H^1(B_R \setminus D)} \leq C_R \epsilon^{3/2} \|u_0\|_{H^4(D)}$$

where the constant  $C_R$  is independent of  $\epsilon$  and  $u_0$ .

- Note that the above result shows (as was done for Dirichlet and Neumann problems  $n = 0$  Santosa, Vogelius '93 Vogelius, M. '97) that the only bulk effect at first order is  $u^{(1)} = -\chi^j(y) \frac{\partial u_0}{\partial x_j}$ .

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- We can show this with the formal asymptotics directly- assume  $u^{(1)} = -\chi^j(y) \frac{\partial u_0}{\partial x_j} + \hat{u}(x)$

- One finds that  $\hat{u}$  solves

$$\begin{aligned} \nabla \cdot A \nabla \hat{u} + k^2 \bar{n} \hat{u} = \\ - \left( - \overline{a_{ki} \chi^j} + \overline{a_{kl} \frac{\partial \chi^{ij}}{\partial y_l}} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} - k^2 \left( \overline{a_{ki} \frac{\partial \beta}{\partial y_i}} - \overline{n \chi^k} \right) \frac{\partial u_0}{\partial x_k}. \end{aligned}$$

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- The source term can be shown to sum to zero by integration by parts.
- So there is no mean field at order  $\epsilon$ .

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- They used Bloch decomposition to derive dispersion equations for periodic media in time domain
- They showed dispersion will appear for large times, numerically demonstrate effects at  $t \sim 1/\epsilon^2$

- Here we find that if we set

$$u^{(2)} = \chi^{ij}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j} + k^2 \beta(y) u_0 + \hat{u}^{(2)}(x) \quad (24)$$

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- We get

$$A : \nabla \nabla \hat{u}^{(2)} + k^2 \bar{n} \hat{u}^{(2)} = - \left( \mathcal{A} : \nabla \nabla \nabla \nabla u_0 + k^2 \mathcal{N} : \nabla \nabla u_0 \right), \quad (25)$$

where  $\mathcal{A}$  and  $\mathcal{N}$  are respectively fourth- and second-order constant tensors and

$$\mathcal{A} : \nabla \nabla \nabla \nabla u_0 = \mathcal{A}_{ijkl} \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l},$$

exhibiting dispersive effects.

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- However, the presence of  $\theta_\epsilon$  or something close to it is *necessary* to obtain a higher order approximation.
- We must understand its behavior to go past first order.

# Boundary corrector for Dirichlet problems

- Presented in BLP
- Behavior first analyzed in Santosa Vogelius '93 and further Vogelius, M '97. Found limit as  $\epsilon \rightarrow 0$  not necessarily unique.
- Tails analyzed in Allaire, Amar '99



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- If boundary is flat at a rational angle with respect to the periodic structure, precise value of  $\epsilon$  dictates what portion of the medium the boundary "sees".
- Gérard-Varet, Masmoudi '12 show that boundary layer limit exists and is unique for smooth domains with no flat parts.

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- Not simply the average and depends on the normal
- For polygons limiting boundary value can be found as the limit at infinity of a function
- Periodic half strip for rational normals Vogelius, M '97
- Half space for irrational normals Gérard-Varet, Masmoudi '12

# An example of a boundary layer limit

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- Use some techniques from above works but requires new analysis due to transmission condition
- Consider one side of the square at a time
- Consider highest order part of data  $u^{(1)}$  and  $\frac{v_0 - \bar{v}_0}{\epsilon}$  and the boundary cell average of  $v^{(1)}$
- oscillatory part of  $v^{(1)}$  will not contribute to the limit at this order

# An example of a boundary layer limit

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$$\Delta \theta_\epsilon + k^2 \theta_\epsilon = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}$$

$$\theta_\epsilon^+ - \theta_\epsilon^- = 1_{\{x_1=1\}} \chi^j(x/\epsilon) \frac{\partial u_0}{\partial x_j} \quad \text{on } \partial D$$

$$(\nabla \theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon) \nabla \theta_\epsilon \cdot \nu)^- = 1_{\{x_1=1\}} \left( \epsilon^{-1} g_j(x/\epsilon) \frac{\partial u_0}{\partial x_j} + \overline{v^{(1)}}^\partial \right) \quad \text{on } \partial D$$

together with the Sommerfeld radiation condition (5) at infinity, where

$$g_1(x/\epsilon) = a_{11}(x/\epsilon) - a_{1k}(x/\epsilon) \frac{\partial \chi^1}{\partial y_k}(x/\epsilon) - A_{11}. \quad (27)$$

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- It is for this reason that one can expect different limits of the boundary layer function for different sequences of  $\epsilon$  going to zero.
- Suppose that the fractional part of  $1/\epsilon_k$  is constant, i.e.

$$\frac{1}{\epsilon_k} - \lfloor \frac{1}{\epsilon_k} \rfloor = \delta,$$

# An example of a boundary layer limit

- Set the oscillatory boundary functions to their restrictions:

$$\chi^1(y_2) = \chi^1(\delta, y_2); \quad g_1(y_2) = g_1(\delta, y_2). \quad (28)$$

- Scale up boundary cell to strip,

$$G = \{-\infty < y_1 < \infty; y_2 \in [0, 1]\}$$

with its two halves

$$G^+ = \{y_1 > 0; y_2 \in [0, 1]\}$$

and

$$G^- = \{y_1 < 0; y_2 \in [0, 1]\}.$$

# An example of a boundary layer limit

Let  $\hat{w}(y_1, y_2)$  solve

$$\begin{aligned} \nabla_y \cdot a(y_1 + \delta, y_2) \nabla \hat{w} &= 0 \quad \text{in } G^- & (29) \\ \Delta_y \hat{w} &= 0 \quad \text{in } G^+ \\ \hat{w}(0, y_2)^+ - \hat{w}(0, y_2)^- &= \chi^1(y_2) \\ \partial_{y_1} \hat{w}(0, y_2)^+ - a_{1i}(\delta, y_2) \partial_{y_i} \hat{w}(0, y_2)^- &= g_1(y_2) \\ \hat{w} [0, 1] &\text{-- periodic in } y_2 \end{aligned}$$

There exists  $\gamma > 0$  such that  $e^{\gamma y_1} \nabla \hat{w} \in L^2(G^+)$   
and  $e^{-\gamma y_1} \nabla \hat{w} \in L^2(G^-)$ .



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$$d^+ = \lim_{y_1 \rightarrow \infty} \hat{w} \quad \text{and} \quad d^- = \lim_{y_1 \rightarrow -\infty} \hat{w}.$$

- We can now define our limiting boundary value

$$\chi_1^* = d^+ - d^- \tag{30}$$

# An example of a boundary layer limit

Then, we can define  $w(y_1, y_2)$  similarly

$$\begin{aligned} \nabla_y \cdot a(y_1 + \delta, y_2) \nabla w &= 0 \quad \text{in } G^- & (31) \\ \Delta_y w &= 0 \quad \text{in } G^+ \\ w(0, y_2)^+ - w(0, y_2)^- &= \chi^1(y_2) - \chi_1^* \\ \partial_{y_1} w(0, y_2)^+ - a_{1i}(\delta, y_2) \partial_{y_i} w(0, y_2)^- &= g_1(y_2) \\ w &[0, 1] - \text{periodic in } y_2 \end{aligned}$$

There exists  $\gamma > 0$  such that  $e^{\gamma y_1} \nabla w \in L^2(G^+)$   
and  $e^{-\gamma y_1} \nabla w \in L^2(G^-)$ .

Now  $w$  itself also decays to zero exponentially as  $|y_1| \rightarrow \infty$ .

# An example of a boundary layer limit

## Theorem

Let  $D = (0, 1) \times (0, 1)$  be the unit square and let  $\epsilon_k$  be a sequence approaching zero such that  $\frac{1}{\epsilon_k} - \lfloor \frac{1}{\epsilon_k} \rfloor = \delta$  for all  $k$ . Then if  $\theta_{\epsilon_k}$  solves (26) for  $\epsilon = \epsilon_k$ , we have that  $\theta_{\epsilon_k} \rightarrow \theta^*$  strongly in  $L^2_{loc}(\mathbb{R}^2)$  where  $\theta^*$  solves

$$\nabla \cdot A \nabla \theta^* + k^2 \bar{n} \theta^* = 0 \quad \text{in } D$$

$$\Delta \theta^* + k^2 \theta^* = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}$$

$$(\theta^*)^+ - (\theta^*)^- = 1_{\{x_1=1\}} \chi_j^* \frac{\partial u_0}{\partial x_j} \quad \text{on } \partial D$$

$$(\nabla \theta^* \cdot \nu)^+ - (A \nabla \theta^* \cdot \nu)^- = 1_{\{x_1=1\}} \left( \overline{a_{12} w_j}^{\partial} \frac{\partial^2 u_0}{\partial x_j \partial x_2} + \overline{v(1)}^{\partial} \right) \quad \text{on } \partial D$$

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- We characterize these effects for a square
- Boundary layer limits difficult to characterize explicitly in general
- If  $a$  is constant (periodicity only in lower order term  $n$ ), for smooth domains with no flat parts, boundary layer limit is zero at first order.