Virtual Element Spaces

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Papers on VEMs from our group


- F. Brezzi, R.S. Falk, L.D. Marini: Basic principles of mixed Virtual Element Method, M2AN 48 (2014), 1227-1240


Typical Functional Spaces

Here are the functional spaces most commonly used in variational formulations of PDE problems

$L^2(\Omega)$ (ex. pressures, densities)
$H(\text{div}; \Omega)$ (ex. fluxes, $D$, $B$)
$H(\text{curl}; \Omega)$ (ex. vector potentials, $E$, $H$)
$H(\text{grad}; \Omega)$ ($H^1$) (ex. displacements, velocities)
$H(\mathbb{D}^2; \Omega)$ ($H^2$) (ex. in K-L plates, Cahn-Hilliard)
Continuity requirements

For a **piecewise smooth** vector valued function, at the common boundary between two elements,

**in order to belong to**

- $L^2(\Omega)$
- $H(\text{div}; \Omega)$
- $H(\text{curl}; \Omega)$
- $H(\text{grad}; \Omega)$
- $H(D^2; \Omega)$

**you need to match**

- nothing
- normal component
- tangential components
- all the components
  - $w$, $w_x$, $w_y$

Note that *the freedom you gain by relaxing the continuity properties can be used to satisfy other properties*
Difficulties with FEM’s: distorted elements

Distorted quads can degenerate in many ways:

- YES
- NO
- NO
- NO

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More difficulties: FE approximations of $H^2(\Omega)$

There are relatively few $C^1$ Finite Elements on the market. Here are some:

- HCT
- reduced HCT
- Argyris
- Bell
Cod liver oil
(Olio di fegato di merluzzo, Huile de foie de morue
Aceite de hígado de bacalao, Dorschlebertran)
A flavor of VEM’s

For a decomposition in general sub-polygons, FEM’s face considerable difficulties. With VEM, instead, you can take a decomposition like

having four elements with 8, 12, 14, and 41 nodes, respectively! Can we work in 3D as well?
A flavor of VEM’s

WE CAN !! These are three possible 3D elements
Polygonal and Polyhedral elements

There is a wide literature on Polygonal and Polyhedral Elements

- **Rational Polynomials** (Wachspress, 1975, 2010)
- **Voronoi tassellations** (Sibson, 1980; Hiyoshi-Sugihara, 1999; Sukumar et al., 2001)
- **Mean Value Coordinates** (Floater, 2003)
- **Metric Coordinates** (Malsch-Lin-Dasgupta, 2005)
- **Maximum Entropy** (Arroyo-Ortiz, 2006; Hormann-Sukumar, 2008)
- **Harmonic Coordinates** (Joshi et al., 2007; Martin et al., 2008; Bishop 2013)
Why Polygonal/Polyhedral Elements

There are several types of problems where Polygonal and Polyhedral elements are used:

- Crack propagation and Fractured materials (e.g. T. Belytschko, N. Sukumar)
- Topology Optimization (e.g. O. Sigmund, G.H. Paulino)
- Computer Graphics (e.g. M.S. Floater)
- Fluid-Structure Interaction (e.g. W.A. Wall)
- Complex Micro structures (e.g. N. Moes)
- Two-phase flows (e.g. J. Chessa)
The ”interface” big squares are treated as polygons with 11 edges.
Possible Inclusions

512 polygons, 2849 vertices
Note that the pink element is a polygon with 9 edges, while the blue element is a polygon (not simply connected) with 13 edges. We are exact on linears...
For reasons of ”glastnost”, we take as exact solution

\[ w = x(x - 0.3)^3(2 - y)^2 \sin(2\pi x) \sin(2\pi y) + \sin(10xy) \]
Robustness of the method

\[ \max |u - u_h| = 0.074424 \]

Mesh of 512 (16 \times 16 \times 2) elements. Max-Err = 0.074
Finer grids

max \vert u - u_h \vert = 0.019380

Mesh of 2048 \((32 \times 32 \times 2)\) elements. Max-Err=0.019
Solution on the finer grid

Mesh of 8192 \((64 \times 64 \times 2)\) elements. Max-Err=0.005

Note the \(O(h^2)\) convergence in \(L^\infty\).
The next steps? (by M.C. Escher)
The next steps? (by M.C. Escher)
The first step: a pegasus-shaped polygon with 82 edges.
The second step: local numbering of the 82 nodes.
The third step: a mesh of $2 \times 2$ pegasus.
A mesh of $20 \times 20$ pegasus.
Solution on a $20 \times 20$-pegasus mesh. Max-Err $= 0.077$
Going **totally** berserk

A mesh of $40 \times 40$ pegasus.
Going **totally** berserk

\[ \max |u-u_h| = 0.026436 \]

Solution on a $40 \times 40$-pegasus mesh. Max-Err$=0.026$
The main features of VEM

As for other methods on polyhedral elements

- the trial and test functions inside each element are rather complicated (e.g. solutions of suitable PDE’s or systems of PDE’s).

Contrary to other methods on polyhedral elements,

- they do not require the approximate evaluation of trial and test functions at the integration points.
- In most cases they satisfy the patch test exactly (up to the computer accuracy).
- We have now a full family of spaces.
In every element, to *define* the generic (scalar or vector valued) element $v$ of our VEM space:

- You start from the **boundary** d.o.f. and use a 1D edge-by-edge reconstruction
- Then you define $v$ **inside** as the solution of a (system of) PDE’s, typically with a polynomial right-hand side.
- The construction is such that all **polynomials** of a certain degree belong to the local space. In general the local space also contains some additional elements.

Let us see some examples.
Nodal 2D elements

We take, for every integer $k \geq 1$

$$V_h^E = \{ v | v|_e \in \mathbb{P}_k(e) \forall \text{ edge } e \text{ and } \Delta v \in \mathbb{P}_{k-2}(E) \}$$

It is easy to see that the local space will contain all $\mathbb{P}_k$. As degrees of freedom we take:

- the values of $v$ at the vertices,
- the moments $\int_e v \ p_{k-2} \, \text{d}e$ on each edge,
- the moments $\int_E v \ p_{k-2} \, \text{d}E$ inside.

It is easy to see that these d.o.f. are unisolvent.
A fantastic trick (sometimes called *The Three Card Monte trick*), often allows the *exact* computation of the moments of order $k - 1$ and $k$ of every $v \in V_h^E$.

This is very useful for dealing with the 3D case.
The *Three Card Monte Trick* is hard to believe

“*I can’t believe it.*”
~Luke

“*That is why you fail.*”
~Yoda
Example: Degrees of freedom of nodal VEM’s in 2D

\[ \begin{array}{ccc}
  k=1 & & k=2 \\
  k=3 & & k=4 \\
\end{array} \]
More general geometries $k = 1$
More general geometries $k = 2$
Approximations of $H^1(\Omega)$ in 3D

For a given integer $k \geq 1$, and for every element $E$, we set

$$V_h^E = \left\{ v \in H^1(E) \mid v_e \in P_k(e) \quad \forall \text{ edge } e, \right.$$  

$$v_f \in V_h^f \quad \forall \text{ face } f, \quad \text{and } \Delta v \in P_{k-2}(E) \right\}$$

with the degrees of freedom:
- values of $v$ at the vertices,
- moments $\int_e v p_{k-2}(e)$ on each edge $e$,
- moments $\int_f v p_{k-2}(f)$ on each face $f$, and
- moments $\int_E v p_{k-2}(E)$ on $E$.

Ex: for $k = 3$ the number of degrees of freedom would be: the number of vertices, plus $2 \times$ the number of edges, plus $3 \times$ the number of faces, plus 4. On a cube this makes $8 + 24 + 18 + 4 = 54$ against 64 for $Q_3$. 
The spaces $G_k$, $G_k^\perp$, $R_k$, and $R_k^\perp$

In the sequel it will be convenient to introduce the following notation

- $G_k := \text{grad}(P_{k+1})$
- $R_k := \text{rot}(P_{k+1})$ (in 2 dimensions)
- $R_k := \text{curl}(P^3_{k+1})$ (in 3 dimensions)

Moreover, for every vector valued polynomial space $P_k(E) \subset P^d_k(E)$ we denote

- $P_k^\perp(E) := \{ q \in P^d(E) \text{ s.t. } (q, p)_{0,E} = 0 \forall p \in P_k(E) \}$
VEM approximations of $H(\text{div}; \Omega)$ in 2d and in 3d

In each element $E$, and for each integer $k$, we define

$$B_k(\partial E) := \{ g \mid g|_e \in \mathbb{P}_k \forall \text{ edge } e \in \partial E \} \text{ in 2d}$$

$$B_k(\partial E) := \{ g \mid g|_f \in \mathbb{P}_k \forall \text{ face } f \in \partial E \} \text{ in 3d}.$$ 

Then we define, in 2 dimensions:

$$V_k(E) = \{ \tau \mid \tau \cdot n \in B_k(\partial E), \nabla \text{div} \tau \in G_{k-2}, \text{rot} \tau \in \mathbb{P}_{k-1} \}$$

and in 3 dimensions

$$V_k(E) = \{ \tau \mid \tau \cdot n \in B_k(\partial E), \nabla \text{div} \tau \in G_{k-2}, \text{curl} \tau \in \mathbb{R}_{k-1} \}.$$
Variants of VEMs in $H(\text{div}; \Omega)$

For $k$, $r$ and $s$ integer, we define, in 2 dimensions:

$$V_{(k,r,s)}(E) = \{ \tau | \tau \cdot n \in B_k(\partial E), \nabla \text{div} \tau \in G_{r-1}, \text{rot} \tau \in P_s \}$$

and in 3 dimensions

$$V_{(k,r,s)}(E) = \{ \tau | \tau \cdot n \in B_k(\partial E), \nabla \text{div} \tau \in G_{r-1}, \text{curl} \tau \in R_s \}.$$ 

In general we might say that

$$V_k \equiv V_{(k,k-1,k-1)} \simeq BDM_k \quad \text{and} \quad V_{(k,k,k-1)} \simeq RT_k$$

On a triangle: $V_{(0,0,-1)} = RT_0$. We point out that $\forall k \geq 0$

$$(\mathbb{P}_k)^d \subset V_{(k,k-1,k-1)} \quad \text{and} \quad \nabla(P_{k+1}) \subset V_{(k,k-1,-1)}$$
Degrees of freedom in $V_k \equiv V_{(k,k-1,k-1)}(E)$ in 2d

In $V_k \equiv V_{(k,k-1,k-1)}$ we can take the following d.o.f.

- $\int_e \boldsymbol{\tau} \cdot \mathbf{n} \, q_k \, \text{d}e \quad \forall q_k \in \mathbb{P}_k(e) \quad \forall \text{ edge } e$
- $\int_E \boldsymbol{\tau} \cdot \text{grad} q_{k-1} \, \text{d}E \quad \forall q_{k-1} \in \mathbb{P}_{k-1}$
- $\int_E \boldsymbol{\tau} \cdot \mathbf{g}^\perp_k \, \text{d}E \quad \forall \mathbf{g}^\perp_k \in \mathcal{G}^\perp_k$

with natural variants for other spaces of the type $V_{(k,s,r)}$. 

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Degrees of freedom for $V_{(0,0,-1)}$
Degrees of freedom for $V_{(1,1,0)}$
Degrees of freedom in $V_k \equiv V_{(k,k-1,k-1)}(E)$ in 3d

In $V_k \equiv V_{(k,k-1,k-1)}$ we can take the following d.o.f.

1. $\int_f \tau \cdot n \, q_k \, df \quad \forall q_k \in \mathbb{P}_k(f) \ \forall \text{face } f$
2. $\int_E \tau \cdot \text{grad} q_{k-1} \, dE \quad \forall q_{k-1} \in \mathbb{P}_{k-1}$
3. $\int_E \tau \cdot g_k^\perp \, dE \quad \forall g_k^\perp \in G_k^\perp$

with natural variants for other spaces of the type $V_{(k,s,r)}$. 
VEM approximations of $H(\text{rot}; \Omega)$ in 2d

In each element $E$, and for each integer $k$, we recall

$$B_k(\partial E) := \{ g \mid g|_e \in \mathbb{P}_k \ \forall \text{ edge } e \in \partial E \} \text{ in } 2d$$

Then we set

$$V_k(E) = \{ \varphi \mid \varphi \cdot t \in B_k(\partial E), \text{div} \varphi \in \mathbb{P}_{k-1}, \text{rotrot} \varphi \in \mathcal{R}_{k-2} \}$$

and for integers $k$, $r$, and $s$

$$V_{(k,r,s)}(E) = \{ \varphi \mid \varphi \cdot t \in B_k(\partial E), \text{div} \varphi \in \mathbb{P}_r, \text{rotrot} \varphi \in \mathcal{R}_{s-1} \}$$
In $V_k \equiv V_{(k,k-1,k-1)}(E)$ in $2d$ we can take the following d.o.f.

- $\int_e \varphi \cdot \mathbf{t} q_k \mathrm{d}e$ \quad $\forall q_k \in \mathbb{P}_k(e) \ \forall$ edge $e$

- $\int_E \varphi \cdot \text{rot} q_{k-1} \mathrm{d}E$ \quad $\forall q_{k-1} \in \mathbb{P}_{k-1}$

- $\int_E \varphi \cdot \mathbf{r}_k^\perp \mathrm{d}E$ \quad $\forall \mathbf{r}_k^\perp \in \mathcal{R}_k^\perp$

with natural variants for other spaces of the type $V_{(k,r,s)}$.
Degrees of freedom for $V_{(0,-1,0)}$
Degrees of freedom for $V_{(1,0,1)}$
In each element $E$, and for each integer $k$, we set

$$\mathcal{B}_k(\partial E) := \{ \varphi | \varphi|_f \in V_k(f) \forall \text{ face } f \in \partial E \text{ and } \varphi \cdot \mathbf{t}_e \text{ is single valued at each edge } e \in \partial E \}$$

Then we set

$$V_k(E) = \{ \varphi | \text{ such that } \varphi \cdot \mathbf{t} \in \mathcal{B}_k(\partial E), \text{ div}\varphi \in P_{k-1}, \text{ curl}\text{curl}\varphi \in \mathcal{R}_{k-2} \}$$
Degrees of freedom in $V_k(E) \equiv V_{(k,k-1,k-1)}$ in 3d

- for every edge $e$ \( \int_e \varphi \cdot t q_k \, de \quad \forall q_k \in \mathbb{P}_k(e) \)
- for every face $f$ \( \int_f \varphi \cdot \text{rot} q_{k-1} \, df \quad \forall q_{k-1} \in \mathbb{P}_{k-1}(f) \)
  \[ \int_f \varphi \cdot r_{k}^{\perp} \, df \quad \forall r_{k}^{\perp} \in \mathcal{R}^{\perp}_k(f) \]
- and inside $E$ \( \int_E \varphi \cdot \text{curl} q_{k-1} \, dE \quad \forall q_{k-1} \in (\mathbb{P}_{k-1}(E))^3 \)
  \[ \int_E \varphi \cdot r_{k}^{\perp} \, dE \quad \forall r_{k}^{\perp} \in \mathcal{R}^{\perp}_k(E) \]
VEM approximations of $L^2(\Omega)$

In each element $E$, and for each integer $k$, we set

$$V_k(E) := \mathbb{P}_k(E),$$

and then obviously

$$V_k(\Omega) = \{ v | \text{ such that } v|_E \in \mathbb{P}_k(E), \forall E \in \mathcal{T}_h \}$$
 Degrees of freedom in $V_k(E)$

As degrees of freedom in $V_k(E)$ we can obviously choose

$$\int_E \nu \, q_k \, dE$$

$$\forall q_k \in (\mathbb{P}_k(E))^3$$

$k = 0$

$k = 1$
VEM approximations of $H^2(\Omega)$

For $r$, $s$, $k$, with

- $r \geq k$,
- $s \geq k - 1$

we set

$$V_h := \{ \mathbf{v} \in V : \mathbf{v} \in \mathbb{P}_r(e), \mathbf{v}_n \in \mathbb{P}_s(e) \ \forall \text{ edge } e, \text{ and } \Delta^2 \mathbf{v} \in \mathbb{P}_{k-4}(E) \ \forall \text{ element } E \}$$

It is clear that for every element $E$ the restriction $V_h^E$ of $V_h$ to $E$ contains all the polynomials of degree $\leq k$. 
Degrees of freedom for K-L plate elements

We had:

\[ V_h := \{ \mathbf{v} \in V : \mathbf{v} \in P_r(e), \mathbf{v}_n \in P_s(e) \}
\forall \text{ edge } e \text{ and } \Delta^2 \mathbf{v} \in P_{k-4}(E) \forall \text{ element } E \} \]

In each \( E \) the functions in \( V_h^E \) are identified by

- their value and the value of their derivatives on \( \partial E \),
- (for \( k > 3 \)) the moments up to the order \( k - 4 \) in \( E \)

Hence we have to worry only for the boundary degrees of freedom.
Example: \( r = 3, \ s = 1 \)

On each vertex we assign \( v, \ v_x, \ v_y \).
Example: $r = 4, s = 3$ (Pac-Plate)

On each vertex we assign $v, v_x, v_y$. On each midpoint we assign $v, v_n, v_{nt}$.
Manipulating VEM’s

When we deal with VEM, we cannot manipulate them as we please. As we don’t want to use approximate solutions of the PDE problems in each element, we have to use only the degrees of freedom and all the information that you can deduce exactly from the degrees of freedom.

In a sense, is like doing Robotic Surgery
Example of manipulation

For instance if you know a function $v$ on $\partial E$ and its mean value in $E$ you can compute

$$\int_E \nabla v \cdot q_1 dE = \int_{\partial E} v \ q_1 \cdot n ds - \int_E v \ \text{div} q_1 dE$$

for every vector valued polynomial $q_1 \in (\mathbb{P}_1)^2$. 
A very useful property

We observe that the classical differential operators $\text{grad}$, $\text{curl}$, and $\text{div}$ send these VEM spaces one into the other (up to the obvious adjustments for the polynomial degrees). Indeed:

\[
\text{grad}(VEM, \text{nodal}) \subseteq VEM, \text{edge} \\
\text{curl}(VEM, \text{edge}) \subseteq VEM, \text{face} \\
\text{div}(VEM, \text{face}) \subseteq VEM, \text{volume}
\]

\[
\mathbb{R} \xrightarrow{i} V^\text{ver}_k(\Omega) \xrightarrow{\text{grad}} V^\text{edg}_{k-1}(\Omega) \xrightarrow{\text{curl}} V^\text{fac}_{k-2}(\Omega) \xrightarrow{\text{div}} V^\text{vol}_{k-3}(\Omega) \xrightarrow{o} 0
\]
The crucial feature common to all these choices is the possibility to construct (starting from the degrees of freedom, and without solving approximate problems in the element) a symmetric bilinear form \([u, v]_h\) such that, on each element \(E\), we have

\[
[p_k, v]^E_h = \int_E p_k \cdot v \, dE \quad \forall p_k \in (P_k(E))^d, \forall v \text{ in the VEM space}
\]

and \(\exists \alpha^* \geq \alpha_* > 0\) independent of \(h\) such that

\[
\alpha_* \|v\|_{L^2(E)}^2 \leq [v, v]^E_h \leq \alpha^* \|v\|_{L^2(E)}^2, \quad \forall v \text{ in the VEM space}
\]
The crucial feature - 2

In other words: In each VEM space (nodal, edge, face, volume) we have a corresponding inner product

\[
\begin{bmatrix} \cdot, \cdot \end{bmatrix}_{VEM,\text{nodal}}, \begin{bmatrix} \cdot, \cdot \end{bmatrix}_{VEM,\text{edge}}, \begin{bmatrix} \cdot, \cdot \end{bmatrix}_{VEM,\text{face}}, \begin{bmatrix} \cdot, \cdot \end{bmatrix}_{VEM,\text{volume}}
\]

that scales properly, and reproduces exactly the \( L^2 \) inner product when at least one of the two entries is a polynomial of degree \( \leq k \).

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General idea on the construction of Scalar Products

- First note that you can always integrate a polynomial
  \[
  \int_E x^3 \, dE = \int_{\partial E} \frac{x^4}{4} n_x \, ds.
  \]

- You construct \( \Pi_k : V^E \rightarrow (P_k(E))^d \) defined by
  \[
  \int_E (v - \Pi_k v) \cdot p_k \, dE = 0 \quad \forall p_k,
  \]

- and then set, for all \( u \) and \( v \) in \( V^E \)
  \[
  [u, v]_E := \int_E (\Pi_k u \cdot \Pi_k v) \, dE + S(u - \Pi_k u, v - \Pi_k v)
  \]

where the stabilizing bilinear form \( S \) is for instance the measure of \( E \) times the Euclidean inner product in \( \mathbb{R}^N \).
Strong formulation of Darcy’s law

- \( p = \text{pressure} \)
- \( u = \text{velocities} \ (\text{volumetric flow per unit area}) \)
- \( f = \text{source} \)
- \( K = \text{material-depending (full) tensor} \)
- \( u = -K \nabla p \) (Constitutive Equation)
- \( \text{div } u = f \) (Conservation Equation)

\[- \text{div} (K \nabla p) = f \quad \text{in } \Omega,\]
\[p = 0 \quad \text{on } \partial \Omega, \quad \text{for simplicity.}\]
The variational formulation of Darcy problem is:

\[ \text{find } p \in H^1_0(\Omega) \text{ such that} \]
\[ \int_{\Omega} K \nabla p \cdot \nabla q \, dx = \int_{\Omega} f q \, dx \quad \forall q \in H^1_0(\Omega). \]

and as VEM approximate problem we can take:

\[ \text{find } p_h \in VEM,\text{nodal} \text{ such that:} \]
\[ [K \nabla p_h, \nabla q_h]_{VEM,\text{edge}} = [f, q_h]_{VEM,\text{nodal}} \quad \forall q_h \in VEM,\text{nodal} \]

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Darcy problem, in *mixed form*, is instead:

*find* \( p \in L^2(\Omega) \) *and* \( u \in H(\text{div}; \Omega) \) *such that:*

\[
\int_{\Omega} K^{-1} u \cdot v \, dV = \int_{\Omega} p \, \text{div} v \, dV \quad \forall v \in H(\text{div}; \Omega)
\]

*and*

\[
\int_{\Omega} \text{div} u \, q \, dV = \int_{\Omega} f \, q \, dV \quad \forall q \in L^2(\Omega).
\]
The approximate mixed formulation can be written as:

\[
\text{find } p_h \in VEM, \text{ volume and } u_h \in VEM, \text{ face such that }
\]

\[
[K^{-1}u_h, v_h]_{VEM, \text{face}} = [p_h, \text{div} v_h]_{VEM, \text{volume}} \quad \forall v_h \in VEM, \text{face}
\]

and

\[
[\text{div} u_h, q_h]_{VEM, \text{volume}} = [f, q_h]_{VEM, \text{volume}} \quad \forall q_h \in VEM, \text{volume}.
\]
Strong formulation of Magnetostatic problem

- \( \mathbf{j} \) = divergence free current density
- \( \mu \) = magnetic permeability
- \( \mathbf{u} \) = vector potential with the gauge \( \text{div} \, \mathbf{u} = 0 \)
- \( \mathbf{B} = \text{curl} \, \mathbf{u} \) = magnetic induction
- \( \mathbf{H} = \mu^{-1} \mathbf{B} = \mu^{-1} \text{curl} \, \mathbf{u} \) = magnetic field
- \( \text{curl} \, \mathbf{H} = \mathbf{j} \)

The classical magnetostatic equations become now

\[
\text{curl} \, \mu^{-1} \text{curl} \, \mathbf{u} = \mathbf{j} \quad \text{and} \quad \text{div} \, \mathbf{u} = 0 \quad \text{in} \ \Omega,
\]

\[
\mathbf{u} \times \mathbf{n} = 0 \quad \text{on} \ \partial\Omega.
\]
Variational formulation of the magnetostatic problem

The variational formulation of the magnetostatic problem (setting $B = \mu H = \text{curl } u$) is:

\[
\begin{cases}
\text{Find } u \in H_0(\text{curl, } \Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that:} \\
(\mu^{-1} \text{curl } u, \text{curl } v) - (\nabla p, v) = (j, v) \quad \forall \, v \in H_0(\text{curl}; \Omega) \\
(u, \nabla q) = 0 \quad \forall \, q \in H_0^1(\Omega),
\end{cases}
\]

and the VEM approximation can be chosen as:

\[
\begin{cases}
\text{Find } u_h \in \text{VEM,edges} \text{ and } p_h \in \text{VEM,nodal} \text{ such that:} \\
[\mu^{-1} \text{curl } u_h, \text{curl } v_h]_{\text{VEM,face}} - [\text{grad } p_h, v_h]_{\text{VEM,edge}} \\
= [j, v_h]_{\text{VEM,edge}} \quad \forall \, v_h \in \text{VEM, edge}, \\
[u, \text{grad } q_h]_{\text{VEM,edge}} = 0 \quad \forall \, q_h \in \text{VEM, nodal}.
\end{cases}
\]
Numerical results for Darcy problem with "BDM-like" VEM

Mesh of squares 4x4, 8x8, ..., 64x64
Exact solution p = sin(2x)cos(3y)
Numerical results - Squares

\[ L^2 \text{--error} \]

\[ \log h \]

\[ \| p - p_{h} \|_{0} \]

\[ \| u - h_{h} \|_{0} \]

\[ \text{BDM1} \]

\[ \text{VEM1} \]

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VEM

IMA October 24-th, 2014
Numerical results - Voronoi Meshes

Voronoi polygons 88, ..., 7921
Exact solution $p = \sin(2x)\cos(3y)$
Numerical results-Voronoi

\[ \| p - p_h \|_{L^2} \]

\[ \| \sigma - \Pi^a \sigma_h \|_{L^2} \]
Mesh of distorted quads: 10x10; 20x20; 40x40
Exact solution: \( p = \sin(2x) \cos(3y) \)
Numerical results–Distorted Quads

$||p - p_h||$ in $L^2$

$||\sigma - \Pi^a \sigma_h||$ in $L^2$
Numerical results—Winged horses meshes

Mesh of horses: 4x4; 8x8; 10x10; 16x16

Exact solution: $p = \sin(2x) \cos(3y)$
Numerical results–Winged horses

\[ \|p - p_h\|_{L^2} \]

\[ \|\sigma - \Pi^\theta \sigma_h\|_{L^2} \]

Franco Brezzi (vv)
Conclusions

- Virtual Elements are a new method, and a lot of work is needed to assess their *pros* and *cons*.
- Their major interest is on polygonal and polyhedral elements, but their use on distorted quads, hexa, and the like, is also quite promising.
- For triangles and tetrahedra the interest seems to be concentrated in higher order continuity (e.g. plates).
- The use of VEM mixed methods seems to be quite interesting, in particular for their connections with other methods for polygonal/polyhedral elements.