Central-Upwind Schemes for Shallow Water Models

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Shallow Water Equations

Shallow water equations are derived by depth-averaging the Navier-Stokes equations in the case when the horizontal length scale is much greater than the vertical length scale, e.g. [Gerbeau, Perthame; 2001].

Shallow water equations are widely used to model water flows in

- Rivers
- Channels
- Lakes
- Coastal areas

Shallow water equations are also used in oceanography and atmospheric sciences to model a variety of waves and phenomena including such natural disasters as tropical cyclones, tsunamis and floods caused by rainfalls, storm surges, dam breaks, etc.

Shallow water models are extremely important in hydraulic engineering and coastal management.
$w = B + h$

$h(x,t)$

$B(x)$
1-D Saint-Venant System

\[
\begin{aligned}
  h_t + q_x &= 0 \\
  q_t + \left( hu^2 + \frac{g}{2}h^2 \right)_x &= -ghB_x
\end{aligned}
\]

This is a system of hyperbolic balance laws

\[
U_t + F(U, B)_x = S(U, B), \quad U := (h, q)
\]

- \(h\): depth
- \(u\): velocity
- \(q := hu\): discharge
- \(B\): bottom topography
- \(g\): gravitational constant
Finite-Volume Methods

1-D System:

\[ U_t + F(U)_x = 0 \]

\[ \bar{U}_j(t) \approx \frac{1}{\Delta x} \int_{C_j} U(x,t) \, dx : \text{cell averages over} \ C_j := (x_j - \frac{1}{2}, x_j + \frac{1}{2}) \]

This solution is approximated by a piecewise polynomial (conservative, high-order accurate, non-oscillatory) reconstruction:

\[ \tilde{U}(x) = P_j(x) \quad \text{for} \ x \in C_j \]

Second-order schemes employ piecewise linear reconstructions:

\[ \tilde{U}(x) = \bar{U}_j + (U_x)_j(x - x_j) \quad \text{for} \ x \in C_j \]
For example,

\[ (U_x)_j = \minmod \left( \theta \frac{U_j - U_{j-1}}{\Delta x}, \frac{U_{j+1} - U_{j-1}}{2\Delta x}, \theta \frac{U_{j+1} - U_j}{\Delta x} \right) \]

\[ \theta \in [1, 2] \]

where the \textit{minmod} function is defined as:

\[
\text{minmod}(z_1, z_2, \ldots) := \begin{cases} 
\min_j \{z_j\}, & \text{if } z_j > 0 \ \forall j, \\
\max_j \{z_j\}, & \text{if } z_j < 0 \ \forall j, \\
0, & \text{otherwise.}
\end{cases}
\]

The reconstructed point values at cell interfaces are:

\[
U_{j+\frac{1}{2}}^- := p_j(x_{j+\frac{1}{2}}) = U_j + \frac{\Delta x}{2} (U_x)_j
\]

\[
U_{j+\frac{1}{2}}^+ := p_{j+1}(x_{j+\frac{1}{2}}) = U_{j+1} - \frac{\Delta x}{2} (U_x)_{j+1}
\]
The discontinuities appearing at the reconstruction step at the interface points \( \{ x_{j+\frac{1}{2}} \} \) propagate at finite speeds estimated by:

\[
\begin{align*}
a_{j+\frac{1}{2}}^+ &:= \max \left\{ \lambda_N \left( A(U_{j+\frac{1}{2}}^-) \right), \lambda_N \left( A(U_{j+\frac{1}{2}}^+) \right), 0 \right\} \\
a_{j+\frac{1}{2}}^- &:= \min \left\{ \lambda_1 \left( A(U_{j+\frac{1}{2}}^-) \right), \lambda_1 \left( A(U_{j+\frac{1}{2}}^+) \right), 0 \right\}
\end{align*}
\]

\( \lambda_1 < \lambda_2 < \ldots < \lambda_N \): \( N \) eigenvalues of the Jacobian \( A(U) := \frac{\partial F}{\partial \mathbf{U}} \)
Central-Upwind Schemes

Godunov-type central schemes with a built-in upwind nature

[Kurganov, Tadmor; 2000]

[Kurganov, Petrova; 2000, 2001]

[Kurganov, Noelle, Petrova; 2001]

[Kurganov, Lin; 2007]
1-D Semi-Discrete Central-Upwind Scheme

\[
\frac{d}{dt} U_j(t) = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x}
\]

The central-upwind numerical flux is:

\[
H_{j+\frac{1}{2}} = \frac{a^+_j \mathbf{F}(U^-_{j+\frac{1}{2}}) - a^-_j \mathbf{F}(U^+_{j+\frac{1}{2}})}{a^+_j - a^-_j} + \left( a^+_j \frac{a^-_j}{a^+_j - a^-_j} \right) \frac{U^+_{j+\frac{1}{2}} - U^-_{j+\frac{1}{2}}}{a^+_j - a^-_j} - d_{j+\frac{1}{2}}
\]

The built-in “anti-diffusion” term is:

\[
d_{j+\frac{1}{2}} = \text{minmod} \left( \frac{U^+_{j+\frac{1}{2}} - U^*_{j+\frac{1}{2}}}{a^+_j - a^-_j}, \frac{U^*_{j+\frac{1}{2}} - U^-_{j+\frac{1}{2}}}{a^+_j - a^-_j} \right)
\]

The intermediate values \(U^*_{j+\frac{1}{2}}\) are:

\[
U^*_{j+\frac{1}{2}} = \frac{a^+_j U^+_{j+\frac{1}{2}} - a^-_j U^-_{j+\frac{1}{2}} - \left\{ \mathbf{F}(U^+_{j+\frac{1}{2}}) - \mathbf{F}(U^-_{j+\frac{1}{2}}) \right\}}{a^+_j - a^-_j}
\]
Remarks

1. $d_{j+\frac{1}{2}} \equiv 0$ corresponds to the central-upwind scheme from [Kurganov, Noelle, Petrova; 2001]

2. For the system of balance laws

$$U_t + F(U)_x = S$$

the central-upwind scheme is:

$$\frac{d}{dt} \bar{U}_j(t) = - \frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x} + \bar{S}_j(t)$$

where

$$\bar{S}_j(t) \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} S(x, t) \, dx$$
2-D Semi-Discrete Central-Upwind Scheme

Rectangular Grid

[Kurganov, Petrova; 2001]

[Kurganov, Noelle, Petrova; 2001]

[Kurganov, Tadmor; 2002]

[Kurganov, Lin; 2007]

Triangular Grid

[Kurganov, Petrova; 2005]
Saint-Venant System — Numerical Challenges

\[
\begin{aligned}
    h_t + q_x &= 0 \\
    q_t + \left( hu^2 + \frac{g}{2}h^2 \right)_x &= -ghB_x
\end{aligned}
\]

- Steady-state solutions:
  \[ q = \text{Const}, \quad \frac{u^2}{2} + g(h + B) = \text{Const} \]

- “Lake at rest” steady-state solutions:
  \[ u = 0, \quad h + B = \text{Const} \]

- Dry \((h = 0)\) or near dry \((h \sim 0)\) states
• **Well-Balanced Schemes** (preserving “lake at rest” steady states)

[LeVeque; 1998]: incorporating the source term into the Riemann solver
[Shi Jin; 2001]: well-balanced source term averaging
[Perthame, Simeoni; 2001]: kinetic scheme
[Kurganov, Levy; 2002]: central-upwind scheme
[Gallouët, Hérard, Seguin; 2003]: Roe-type scheme
[Audusse, Bouchut, Bristeau, Klein, Perthame; 2004]: hydrostatic reconstruction
[Russo; 2005]: staggered central scheme
[Xing, Shu; 2005, 2006]: WENO schemes
[Noelle, Pankratz, Puppo, Natvig; 2006]: high-order schemes
[Lukáčová-Medvidová, Noelle, Kraft; 2007]: FVEG scheme
[Berthon, Marche; 2008]: relaxation schemes
[Fjordholm, Mishra, Tadmor; 2008, 2011]: energy stable schemes

Abgrall, Audusse, Bristeau, Castro, Chertock, Dawson, Donat, Epshteyn, George, Karni, Klingenberg, Mohammadian, Parés, Ricchiuto, ...
• **Well-Balanced Schemes** (preserving moving steady states)

[Noelle, Shu, Xing; 2007, 2009, 2011]: WENO schemes
[Russo, Khe; 2009, 2010]: staggered central schemes
[Xing; 2014]: discontinuous Galerkin method

• **Positivity Preserving Schemes**

[Perthame, Simeoni; 2001]: kinetic scheme
[Audusse, Bouchut, Bristeau, Klein, Perthame; 2004]: hydrostatic reconstruction
[Kurgnov, Petrova; 2007]: central-upwind scheme with continuous piecewise linear bottom reconstruction
[Berthon, Marche; 2008]: relaxation schemes
[Bollermann, Noelle, Lukáčová-Medvid’ová; 2011]: special time-quadrature for the fluxes
[Bollermann, Chen, Kurganov, Noelle; 2013]: well-balanced reconstruction of wet/dry fronts
Central-upwind schemes is simple, highly accurate and efficient tool that allows one to quite easily incorporate techniques required to numerically solve more realistic (and more complicated) Saint-Venant based shallow water models.

Additional terms: friction, viscosity, Coriolis forces, temperature variations, third-order dispersive terms

Robustness of central-upwind schemes made them attractive to practitioners. For example:

Central-upwind schemes have been recently implemented at the National Center of Computational Hydroscience and Engineering at the University of Mississippi, where they became the core part of the GIS-based decision support systems for flood mitigation and management. These systems are currently being used by several US federal agencies.
Well-Balanced Central-Upwind Scheme

[Kurganov, Levy; 2002]

\( w = h + B \): water surface

\[
\begin{align*}
    h_t + q_x &= 0 \\
    q_t + (hu^2 + \frac{g}{2}h^2)_x &= -ghB_x
\end{align*}
\]

\( \Downarrow \quad (h, q) \rightarrow (w, q) \)

\[
\begin{align*}
    w_t + q_x &= 0 \\
    q_t + \left( \frac{q^2}{w-B} + \frac{g}{2}(w-B)^2 \right)_x &= -g(w-B)B_x
\end{align*}
\]

“Lake at rest” steady states: \( u \equiv 0, \quad w \equiv \text{Const} \)
At the “lake at rest” steady state: \( q = 0, \ w = \text{Const} \)

\[ \implies \text{the flux is} \quad F = (q, \frac{q^2}{w-B} + \frac{g}{2}(w-B)^2)^T = (0, \frac{g}{2}(w-B)^2)^T \]

\[ \implies \text{the second component of the numerical flux is} \]

\[ H^{(2)}_{j+\frac{1}{2}} = \frac{g}{2} \left( w - B(x_{j+\frac{1}{2}}) \right)^2, \quad H^{(2)}_{j-\frac{1}{2}} = \frac{g}{2} \left( w - B(x_{j-\frac{1}{2}}) \right)^2 \]

\[ \implies \frac{d}{dt} \bar{q}_j(t) = -\frac{H^{(2)}_{j+\frac{1}{2}}(t) - H^{(2)}_{j-\frac{1}{2}}(t)}{\Delta x} + \bar{S}^{(2)}_j(t) \]

\[ = g \cdot \frac{B(x_{j+\frac{1}{2}}) - B(x_{j-\frac{1}{2}})}{\Delta x} \left( w - B(x_{j+\frac{1}{2}}) \right) + (w - B(x_{j-\frac{1}{2}})) \frac{2}{2} + \bar{S}^{(2)}_j(t) \]

\[ \implies \text{The well-balanced quadrature is} \]

\[ \bar{S}^{(2)}_j(t) = -g \cdot \frac{B(x_{j+\frac{1}{2}}) - B(x_{j-\frac{1}{2}})}{\Delta x} \cdot \left( \bar{w}_j - \frac{B(x_{j+\frac{1}{2}}) + B(x_{j-\frac{1}{2}})}{2} \right) \]
Well-Balanced Positivity Preserving Central-Upwind Scheme

[Kurganov, Petrova; 2007]

**Step 1:** Piecewise linear reconstruction of the bottom
Step 2: Positivity preserving reconstruction of $w$
\[ h_{j+\frac{1}{2}}^\pm = w_{j+\frac{1}{2}}^\pm - B_{j+\frac{1}{2}} \]
\[ B_{j-1/2} = w_{j-1/2}^+ \]

\[ B_{j} \]

\[ B_{j+1/2} \]

\[ x_{j-1/2} \]

\[ x_{j} \]

\[ x_{j+1/2} \]
Step 3: **Desingularization** \( (u \neq \frac{q}{h} \text{ for small } h) \)

– Simplest:

\[
    u = \begin{cases} 
        \frac{q}{h}, & \text{if } h \geq \varepsilon \\
        0, & \text{if } h < \varepsilon 
    \end{cases}
\]

– More sophisticated (smoother transition for small \( h \)):

\[
    u = \frac{2h q}{h^2 + \max(h^2, \varepsilon)} \quad \text{or} \quad u = \frac{\sqrt{2} h q}{\sqrt{h^4 + \max(h^4, \varepsilon)}}
\]

**Remark:** For consistency, one has to recompute the discharge:

\[
    q = h \cdot u
\]
Positivity Preserving Property

If an SSP ODE solver is used, then

$$\bar{h}_{\frac{j}{2}}^{n+1} = \alpha_{\frac{j-1}{2}}h_{\frac{j-1}{2}} + \alpha_{\frac{j-1}{2}}h_{\frac{j-1}{2}} + \alpha_{\frac{j+1}{2}}h_{\frac{j+1}{2}} + \alpha_{\frac{j+1}{2}}h_{\frac{j+1}{2}}$$

where the coefficients $\alpha_{\frac{j}{2}}^\pm > 0$ provided an appropriate CFL condition is satisfied:

- **1-D**: CFL number is $1/2$
- **2-D Cartesian mesh**: CFL number is $1/4$
- **2-D triangular mesh**: CFL number is $1/3$

**Remark**: For high-order SSP methods, adaptive timestep control has to be implemented.
Example — Small Perturbation of a Steady State

The bottom topography is:

\[
B(x) = \begin{cases} 
10(x - 0.3), & 0.3 \leq x \leq 0.4 \\
1 - 0.0025 \sin^2(25\pi(x - 0.4)), & 0.4 \leq x \leq 0.6 \\
-10(x - 0.7), & 0.6 \leq x \leq 0.7 \\
0, & \text{otherwise}
\end{cases}
\]

The initial data are:

\[
(w(x, 0), u(x, 0)) = \begin{cases} 
(1 + \varepsilon, 0), & 0.1 < x < 0.2 \\
(1, 0), & \text{otherwise}
\end{cases}
\]

\[
\varepsilon = 10^{-3}
\]
\Delta x = 1/200

NEW SCHEME
OLD SCHEME

\Delta x = 1/800
Example — Dry State and Discontinuous Bottom

\[ B(x) = \begin{cases} 
2, & x \leq 0.5 \\
0.1, & x > 0.5 
\end{cases} \]

\[ (w(x, 0), u(x, 0)) = \begin{cases} 
(2.222, -1), & x \leq 0.5 \\
(0.8246, -1.6359), & x > 0.5 
\end{cases} \]
Example — ShW with Friction and Discontinuous Bottom

\[
\begin{align*}
\begin{cases}
  h_t + q_x &= 0 \\
  q_t + \left( hu^2 + \frac{g}{2} h^2 \right)_x &= -ghB_x - \kappa(h)u,
\end{cases}
\end{align*}
\]

\[
\kappa(h) = \frac{0.001}{1 + 10h}
\]

\[
B(x) = \begin{cases}
  1, & x < 0 \\
  \cos^2(\pi x), & 0 \leq x \leq 0.4 \\
  \cos^2(\pi x) + 0.25(\cos(10\pi(x - 0.5)) + 1), & 0.4 \leq x \leq 0.5 \\
  0.5 \cos^4(\pi x) + 0.25(\cos(10\pi(x - 0.5)) + 1), & 0.5 \leq x \leq 0.6 \\
  0.5 \cos^4(\pi x), & 0.5 \leq x < 1 \\
  0.25 \sin(2\pi(x - 1)), & 1 < x \leq 1.5 \\
  0, & x > 1.5.
\end{cases}
\]

\[
(w(x, 0), u(x, 0)) = \begin{cases}
  (1.4, 0), & x < 0 \\
  (B(x), 0), & x > 0
\end{cases}
\]

(Dam break)
Central-Upwind Schemes for the 2-D Saint-Venant System

**Cartesian Grid:** [Kurganov, Levy, 2002], [Kurganov, Petrova; 2007]

**Triangular Grid:** [Bryson, Epshteyn, Kurganov, Petrova; 2011]

**Polygonal Cell-Vertex Mesh:**
[Beljadid, Mohammadian, Kurganov; preprint]
Example — Small Perturbation over an Exponential Hump

The bottom consists of an elliptical shaped hump:

\[ B(x, y) = 0.8 \exp(-5(x - 0.9)^2 - 50(y - 0.5)^2) \]

Initially, the water is at rest and its surface is flat everywhere except for 0.05 < x < 0.15:

\[
    w(x, y, 0) = \begin{cases} 
        1 + \varepsilon, & 0.05 < x < 0.15, \\
        1, & \text{otherwise,}
    \end{cases}
    \quad u(x, y, 0) \equiv v(x, y, 0) \equiv 0
\]

We take \( \varepsilon = 10^{-4} \)
Example — Small Perturbation over Submerged Flat Plateau

\[ B(x, y) = \begin{cases} 
1 - 2\varepsilon, & r \leq 0.1, \\
10(1 - 2\varepsilon)(0.2 - r), & 0.1 \leq r \leq 0.2, \\
0, & \text{otherwise,}
\end{cases} \]

\[ w(x, y, 0) = \begin{cases} 
1 + \varepsilon, & 0.1 < x < 0.2, \\
1, & \text{otherwise,}
\end{cases} \]

\[ u(x, y, 0) \equiv v(x, y, 0) \equiv 0 \]

We take \( \varepsilon = 0.01 \)
Example — Small Perturbation Bending around an Island

\[ B(x, y) = \begin{cases} 
1.1, & r \leq 0.1, \\
11(0.2 - r), & 0.1 < r < 0.2, \\
0, & \text{otherwise,}
\end{cases} \]

\[ r := \sqrt{x^2 + y^2} \]

\[ w(x, y, 0) = \begin{cases} 
1 + \varepsilon, & -0.4 < x < -0.3, \\
\max(1, B(x, y)), & \text{otherwise,}
\end{cases} \]

\[ u(x, y, 0) \equiv v(x, y, 0) \equiv 0 \]

We take \( \varepsilon = 0.01 \)
Shallow Water System with Friction Terms

[Chertock, Cui, Kurganov, Wu; preprint]

Friction from [Gerbeau, Perthame; 2001]

\[
\begin{cases}
    h_t + q_x = 0 \\
    q_t + \left( hu^2 + \frac{g}{2}h^2 \right)_x = -ghB_x - \frac{0.001}{1 + 10h}u,
\end{cases}
\]

Classical Manning friction [Manning; XIX century]

\[
\begin{cases}
    h_t + q_x = R \\
    q_t + \left( hu^2 + \frac{g}{2}h^2 \right)_x = -ghB_x - g\frac{n^2}{h^{1/3}}|u|u
\end{cases}
\]

\[n: \text{Manning coefficient}\]

\[R = R(x,t): \text{rain source}\]
Steady-State Solutions

Nontrivial steady states (for $R \equiv 0$)

\[ q \equiv \text{Const}, \quad h \equiv \text{Const}, \quad B_x \equiv \text{Const} \]

correspond to the situation when the water flows over a slanted infinitely long surface with the constant slope.

Assume that $B_x \equiv -C$, where $C > 0$ and denote $q \equiv q_0$, then

\[ h \equiv h_n = \left( \frac{n^2 q_0^2}{C} \right)^{3/10}, \quad q \equiv q_0, \quad B_x \equiv -C, \]

where $h_n$ is the so-called normal depth.

This steady state is stable in the super-critical case when

\[ h_n < h_c := \left( \frac{q_0}{g} \right)^{1/3} \]
The bottom setting in numerical examples:

Left: 1-D steady state

Right: a case of urban draining with obstacles like houses
As before, we:

- replace the bottom with its piecewise linear approximation
- correct reconstruction of $w = h + B$ to ensure that $w > B$
- regularize computed velocities to avoid division by $h \sim 0$
- use well-balanced quadrature for the geometric source term

A straightforward midpoint discretization of the friction term leads to the well-balanced positivity preserving central-upwind scheme
Modified Positivity Correction

Instead of

\[ B_{j-1/2} = w_{j-1/2}^+ \]

\[ B_{j+1/2} = w_{j+1/2}^- \]

\[ B_j = w_j \]

\[ B_{j-1/2} = w_{j-1/2} \]

\[ B_{j+1/2} = w_{j+1/2}^- \]

\[ w_{j-1/2}^+ \]

\[ w_{j+1/2}^- \]

\[ x_{j-1/2} \]

\[ x_j \]

\[ x_{j+1/2} \]
we do
Example — Small Perturbation of a Steady Flow Over a Slanted Surface ($R \equiv 0$)

\[ h(x, 0) = h_n + \begin{cases} 
0.2h_n, & 1 \leq x \leq 1.25, \\
0, & \text{otherwise,} 
\end{cases} \]

\[ q(x, 0) \equiv q_0 \]
Example — Rainfall-Runoff Over An Urban Area

We consider a rainfall-runoff situation, which occurs over a 2-D surface containing houses as outlined in
The setting corresponds to the laboratory experiment reported in [Cea, Garrido, Puertas; 2010].

The surface structure is shown in

The precise data was provided by Dr. Luis Cea
The experiment was built to mimic an urban area within the laboratory simulator of size $2m \times 2.5m$. To model urban buildings, several blocks were placed onto the surface according to three different geometries:
Notice that across the walls of the houses the bottom topography is discontinuous and thus the bilinear interpolant $\tilde{B}$ has very sharp gradients there
We set the almost dry initial conditions:

\[ h(x, y, 0) \equiv 10^{-8}, \quad u(x, y, 0) \equiv v(x, y, 0) \equiv 0 \]

The rain of a constant intensity starts falling at time \( t = 0 \) and stops at \( t = T_s \):

\[
R(x, y, t) = \begin{cases}
\frac{1}{12000}, & 0 \leq t \leq T_s \\
0, & \text{otherwise}
\end{cases}
\]

We take \( T_s = 20, 40 \) or \( 60 \)

At the lower part of boundary, the total outlet discharge is recorded in the laboratory experiments at different time moments and then it is compared with the computed values of

\[
\sum_{j=1}^{N_x} \left( H_{j,1/2}^y \right)^{(1)}
\]
Example — Rainfall Runoff Over An Urban Area (REVISED)

First, we remove the houses from the computational domain which becomes a punctured rectangle. Each of the holes is depicted in

The house walls become the internal boundary, which is numerically treated using a solid wall ghost cell technique.
Second, we need to **redistribute the rain water** falling onto the roof so that it is placed inside the modified computational domain. In the laboratory experiment, the water falling on the house blocks streams down from the long (lower) edges and finally joins the surface water flow:
In reality, the **gutter system** is commonly used and the rain water streams down from the rain pipes typically located at the house corners:

In both cases, the building-roof rainfall is uniformly distributed on the shaded cells near and outside the building edges.
The modified rain source can then be written as follows:

\[
\hat{R}(x, y, t) = \begin{cases} 
\frac{1}{12000} \left(1 + \frac{A_h}{A_s}\right), & \text{in the shaded cells} \\
\frac{1}{12000}, & \text{otherwise}
\end{cases}
\]

which is as before switched on only for \( t \in [0, T_s] \)

\( A_h \): the area of the house

\( A_s \): the area of the shaded region near that house
Water height snapshots for $T_s = 40$
Shallow Water System with Coriolis Forces

[Chertock, Dudzinski, Kurganov, Lukáčová-Medviďová; preprint]

\[
\begin{aligned}
& h_t + (hu)_x + (hv)_y = 0 \\
& (hu)_t + \left( hu^2 + \frac{g}{2} h^2 \right)_x + (huv)_y = -ghB_x + fhv \\
& (hv)_t + (huv)_x + \left( hv^2 + \frac{g}{2} h^2 \right)_y = -ghB_y - fhu
\end{aligned}
\]

\( f \): Coriolis parameter
Steady-State Solutions

\[
\begin{align*}
    h_t + (hu)_x + (hv)_y &= 0 \\
    (hu)_t + \left(hu^2 + \frac{g}{2}h^2\right)_x + (huv)_y &= -ghB_x + fhv \\
    (hv)_t + (huv)_x + \left(hv^2 + \frac{g}{2}h^2\right)_y &= -ghB_y - fhu
\end{align*}
\]

- “Lake at rest”: \( u \equiv 0, \; v \equiv 0, \; h + B \equiv \text{Const} \)
- Geostrophic equilibria ("jets in the rotational frame"): \( u \equiv 0, \; v_y \equiv 0, \; h_y \equiv 0, \; B_y \equiv 0, \; K \equiv \text{Const} \)
  \( v \equiv 0, \; u_x \equiv 0, \; h_x \equiv 0, \; B_x \equiv 0, \; L \equiv \text{Const} \)

Here,

\[ K := g(h + B - V) \quad \text{and} \quad L := g(h + B + U) \]

are the potential energies defined through the primitives of the Coriolis force \((U, V)_T\):

\[ V_x := \frac{f}{g}v \quad \text{and} \quad U_y := \frac{f}{g}u \]
1-D Shallow Water System with Coriolis Forces

\[
\begin{aligned}
& h_t + (h u)_x = 0 \\
& (h u)_t + \left( h u^2 + \frac{g}{2} h^2 \right)_x = -g h B_x + f h v \\
& (h v)_t + (h u v)_x = -f h u
\end{aligned}
\]

- 1-D geostrophic equilibria ("jets in the rotational frame"):

\[
\begin{aligned}
& u \equiv 0, \quad K \equiv \text{Const} \\
& K := g (h + B - V), \quad V_x := \frac{f}{g} v
\end{aligned}
\]
Special Piecewise Linear Reconstruction

Way to design a well-balanced scheme (KEY IDEA):
Reconstruct equilibrium variables!

1-D case: $u, v, K$

2-D case: $u, v, K, L$
1-D Reconstruction

First, compute the point values of the velocities at $x_j$:

$$u_j = \frac{(hu)_j}{\bar{h}_j}, \quad v_j = \frac{(hv)_j}{\bar{h}_j}$$

The values $V(x_{j+\frac{1}{2}})$ are obtained using the midpoint quadrature:

$$V_{j+\frac{1}{2}} = \frac{f}{g} \sum_{m=1}^{j} v_m \Delta x$$

The corresponding values $V(x_j)$ are

$$V_j = \frac{1}{2} \left( V_{j-\frac{1}{2}} + V_{j+\frac{1}{2}} \right)$$

Thus, $K(x_j)$ are computed as

$$K_j = g(\bar{h}_j + B_j - V_j)$$
Equipped with \( \{u_j\} \), \( \{v_j\} \) and \( \{K_j\} \), we construct piecewise linear approximants \( \tilde{u} \), \( \tilde{v} \) and \( \tilde{K} \):

\[
\begin{align*}
\tilde{u}(x) &= u_j + (u_x)_j(x - x_j), \quad x \in C_j \\
\tilde{v}(x) &= v_j + (v_x)_j(x - x_j), \quad x \in C_j \\
\tilde{K}(x) &= K_j + (K_x)_j(x - x_j), \quad x \in C_j
\end{align*}
\]

The corresponding reconstruction \( \tilde{h} \) then reads:

\[
\tilde{h}(x) = \frac{\tilde{K}(x)}{g} + \frac{1}{\Delta x} \left[ (V_{j-\frac{1}{2}} - B_{j-\frac{1}{2}})(x_{j+\frac{1}{2}} - x) + (V_{j+\frac{1}{2}} - B_{j+\frac{1}{2}})(x - x_{j-\frac{1}{2}}) \right]
\]

(Recall that \( K := g(h + B - V) \Rightarrow h = \frac{K}{g} + V - B \))
Example — 2-D Geostrophic Adjustment Simulation

\[ B \equiv 0, \; f = 1, \; g = 1 \]

\[ h(x, y, 0) = 1 + \frac{1}{4} \left[ 1 - \tanh \left( 10 \left( \sqrt{2.5x^2 + 0.4y^2} - 1 \right) \right) \right] \]

\[ u(x, y, 0) \equiv 0, \quad v(x, y, 0) \equiv 0 \]

The initial perturbation generates two circular shock waves propagating outwards with a clockwise rotating elevation staying behind the shocks.

As time evolves, the solution converges to a nontrivial geostrophic steady state, which is accurately captured by the well-balanced central-upwind scheme.
Two-Layer Shallow Water Equations

\[
\begin{align*}
(h_1)_t + (q_1)_x &= 0 \\
(q_1)_t + \left( h_1 u_1^2 + \frac{g}{2} h_1^2 \right)_x &= -gh_1 B_x - gh_1 (h_2)_x \\
(h_2)_t + (q_2)_x &= 0 \\
(q_2)_t + \left( h_2 u_2^2 + \frac{g}{2} h_2^2 \right)_x &= -gh_2 B_x - gh_2 (\hat{h}_1)_x
\end{align*}
\]

\[u_1, u_2: \text{ velocities}\]

\[q_1 := h_1 u_1, \; q_2 := h_2 u_2: \text{ discharges}\]

\[g: \text{ gravitational constant}\]

\[r := \frac{\rho_1}{\rho_2}: \text{ constant density ratio}\]

\[\hat{h}_1 := rh_1.\]
\[
\begin{align*}
(h_1)_t + (q_1)_x &= 0 \\
(q_1)_t + \left( h_1 u_1^2 + \frac{g}{2} h_1^2 \right)_x &= -gh_1 B_x - gh_1 (h_2)_x \\
(h_2)_t + (q_2)_x &= 0 \\
(q_2)_t + \left( h_2 u_2^2 + \frac{g}{2} h_2^2 \right)_x &= -gh_2 B_x - gh_2 (\hat{h}_1)_x
\end{align*}
\]

**Difficulties:**

- Nonlinear hyperbolic system \(\implies\) shock waves
- Rarefaction waves
- Contact waves (if \(B\) is discontinuous)

- Geometric source terms (nonflat bottom topography)

- Nonconservative products (interlayer exchange)
\[\begin{align*}
(h_1)_t + (q_1)_x &= 0 \\
(q_1)_t + \left( h_1 u_1^2 + \frac{g}{2} h_1^2 \right)_x &= -g h_1 B_x - g h_1 (h_2)_x \\
(h_2)_t + (q_2)_x &= 0 \\
(q_2)_t + \left( h_2 u_2^2 + \frac{g}{2} h_2^2 \right)_x &= -g h_2 B_x - g h_2 (\hat{h}_1)_x
\end{align*}\]

\[\downarrow\]

\[\begin{align*}
(h_1, q_1, h_2, q_2) &\rightarrow (h_1, q_1, w := h_2 + B, q_2)
\end{align*}\]

\[\begin{align*}
(h_1)_t + (q_1)_x &= 0 \\
(q_1)_t + \left( \frac{q_1^2}{h_1} + g \varepsilon h_1 \right)_x &= g \varepsilon (h_1)_x \\
w_t + (q_2)_x &= 0 \\
(q_2)_t + \left( \frac{q_2^2}{w - B} + \frac{g}{2} w^2 - \frac{g}{2} r h_1^2 - g B \hat{\varepsilon} \right)_x &= -g \hat{\varepsilon} B x - g \varepsilon (\hat{h}_1)_x
\end{align*}\]

\[\varepsilon = h_1 + h_2 + B = h_1 + w, \quad \hat{\varepsilon} := \hat{h}_1 + w\]
The new system is preferable for numerical computations thanks to the following two reasons:

(i) the "lake at rest" steady state is given by \( U \equiv (C_1, 0, C_2, 0)^T \) with \( C_1, C_2 \) constants

(ii) the coefficients in the nonconservative products \( g\varepsilon(h_1)_x \) and \( g\varepsilon(\hat{h}_1)_x \) are proportional to \( \varepsilon \), which vanishes at "lake at rest" steady states, and, what is even more important, in most oceanographic applications remains very small.

Note that when \( r \sim 1 \), the variable \( \hat{\varepsilon} \) is also expected to be small in typical oceanographic applications.
As in the single-layer case, we:

- replace the bottom with its piecewise linear approximation
- correct reconstruction of $w$ to ensure that $w > B$
- regularize computed velocities to avoid division by $h_i \sim 0$
- use well-balanced quadrature for the geometric source term

The e-values are given implicitly by

$$(\lambda^2 - 2u_1 \lambda + u_1^2 - gh_1)(\lambda^2 - 2u_2 \lambda + u_2^2 - gh_2) = g^2 \hat{h}_1 h_2$$

We either:

- use the first-order approximation of the e-values (when $r \sim 1$ and $u_1 \sim u_2$)
- use their upper/lower bounds to enforce stability (if the system is still in the hyperbolic regime)
Example — Interface Propagation

\[(h_1, q_1, h_2, q_2)(x, 0) = \begin{cases} 
(0.50, 1.250, 0.50, 1.250), & x < 0.3 \\
(0.45, 1.125, 0.55, 1.375), & x > 0.3 
\end{cases}\]

\[B \equiv -1, \ r = 0.98\]
\[(h_1, q_1, h_2, q_2)(x, 0) = \begin{cases} (1.8, 0, 0.2, 0), & x < 0, \\ (0.2, 0, 1.8, 0), & x > 0, \end{cases} \quad B \equiv -2, \quad r = 0.98\]
Example — Lock Exchange Problem

\[(h_1, q_1, h_2, q_2)(x, 0) = \begin{cases} 
(-B(x), 0, 0, 0), & x < 0 \\
(0, 0, -B(x), 0), & x > 0 
\end{cases}\]

\[B(x) = e^{-x^2} - 2, \quad r = 0.98\]

The computational domain is \([-3, 3]\) and the boundary conditions are \(q_1 = -q_2\) at each end of the interval.
Example — Internal Dam Break

\[(h_1, q_1, h_2, q_2)(x, 0) = \begin{cases} 
(1.95, 0, -1.95 - B(x), 0), & x < 0 \\
(0.05, 0, -0.05 - B(x), 0), & x > 0 
\end{cases}\]

\[B(x) = 0.5e^{-x^2} - 2.5, \quad r = 0.998\]
Example — 2-D Interface Propagation

\[
(h_1, q_1, p_1, h_2, q_2, p_2)(x, y, 0) = \begin{cases} 
(0.50, 1.250, 1.250, 0.50, 1.250, 1.250), & \text{if } (x, y) \in \Omega \\
(0.45, 1.125, 1.125, 0.55, 1.375, 1.375), & \text{otherwise}
\end{cases}
\]

where

\[
\Omega = \{x < -0.5, y < 0\} \cup \{(x + 0.5)^2 + (y + 0.5)^2 < 0.25\} \\
\cup \{x < 0, y < -0.5\}.
\]

The initial location of the interface is shown by the dashed line.

\[B(x, y) \equiv -1\]
$\varepsilon$

$h_1$

$h_1$ along the diagonal $y=x$

$\Delta x = \Delta y = 1/200$
$\Delta x = \Delta y = 1/400$
$\Delta x = \Delta y = 1/800$
Savage-Hutter Type Model of Submarine Landslides and Generated Tsunami Waves

[Fernández-Nieto, Bouchut, Bresch, Castro, Mangene; 2008]
\[
\begin{align*}
(h_1)_t + (q_1 \cos \theta)_x &= 0 \\
(q_1)_t + \left( h_1 u_1^2 \cos \theta + \frac{g}{2} h_1^2 \cos^3 \theta \right)_x &= -gh_1 \cos \theta B_x \\
&\quad - \frac{g}{2} h_1^2 \cos^2 \theta (\cos \theta)_x - gh_1 \cos \theta \left( h_2 \cos^2 \theta \right)_x \\
(h_2)_t + (q_2 \cos \theta)_x &= 0 \\
(q_2)_t + \left( h_2 u_2^2 \cos \theta + \frac{g}{2} h_2^2 \cos^3 \theta \right)_x &= -gh_2 \cos \theta B_x \\
&\quad - \frac{g}{2} h_2^2 \cos^2 \theta (\cos \theta)_x - rgh_2 \cos \theta \left( h_1 \cos^2 \theta \right)_x + \frac{T}{\cos \theta}
\end{align*}
\]

$B(x)$: non-erodible bottom

$\theta(x)$: the angle of $B(x)$ from horizontal

$T = T(h_2, u_2; \delta_0)$: the Coulomb friction term that describes the friction between the lower layer and the bottom

$\delta_0$: angle of repose (parameter of the fluid-granular mixture)
“Lake-at-rest” steady-state solution (flat water surface in the absence of a lower-layer):

\[ h_1 \cos^2 \theta + B \equiv \text{const}, \quad h_2 \equiv 0, \quad q_1 \equiv q_2 \equiv 0 \]
\[ \downarrow \quad (h_1, h_2) \rightarrow (w_i := h_i \cos^2 \theta, \varepsilon := w_1 + w_2 + B) \]

\[
\begin{align*}
\varepsilon_t + \left( (q_1 + q_2) \cos^3 \theta \right)_x &= 2(q_1 + q_2) \cos^2 \theta (\cos \theta)_x \\
(q_1)_t + \left( \frac{q_1^2 \cos^3 \theta}{\varepsilon - (w_2 + B)} + \frac{g \varepsilon^2 - 2\varepsilon(w_2 + B)}{2 \cos \theta} \right)_x &= -\frac{g}{\cos \theta} \varepsilon(w_2 + B)_x \\
(w_2)_t + \left( q_2 \cos^3 \theta \right)_x &= 2q_2 \cos^2 \theta (\cos \theta)_x \\
(q_2)_t + \left( \frac{q_2^2 \cos^3 \theta}{w_2} + \frac{g(1 - r)w_2^2 + 2rw_2\varepsilon}{2 \cos \theta} \right)_x &= -\frac{(1 - r)g}{\cos \theta} w_2 B_x \\
&\quad + \frac{rg}{\cos \theta} \varepsilon(w_2)_x - \frac{g(1 - r)w_2^2 + 2r\varepsilon w_2}{2 \cos^2 \theta} (\cos \theta)_x + \frac{T}{\cos \theta}
\end{align*}
\]
\((\varepsilon, q_1, w_2, q_2)^T\): equilibrium variables

“Lake-at-rest” steady-state solution:

\[\varepsilon \equiv \text{const}, \quad w_2 \equiv 0, \quad q_1 \equiv q_2 \equiv 0\]
As for the two-layer shallow water equations, we:

- replace the bottom with its piecewise linear approximation
- correct reconstruction of $w$ to ensure that $w > B$
- regularize computed velocities to avoid division by $w_i \sim 0$
- use well-balanced quadrature for the geometric source term
- use upper/lower bounds of e-values to enforce stability
Coulomb Friction

To include the friction term, we apply an operator splitting

We first obtain \((q_2^*)_j\) and then update \(q_2\) as follows:

\[
(q_2)_j(t + \Delta t) = \begin{cases} 
(q_2^*)_j + T_j \Delta t, & \text{if } |(q_2^*)_j| > \sigma_j^* \Delta t / \cos \theta_j, \\
0, & \text{otherwise,}
\end{cases}
\]

with

\[
:= -\text{sgn}\{(q_2^*)_j\} \left[ \sigma^*_j + \frac{(\hat{w}_2)_j^{+\frac{1}{2}} + (\hat{w}_2)_j^{-\frac{1}{2}}}{2} \cdot (\hat{u}_2)_j^2 \cdot \frac{\sin \theta_j^{+\frac{1}{2}} - \sin \theta_j^{-\frac{1}{2}}}{\Delta x} \cdot \tan \delta_0 \right],
\]

where

\[
\sigma^*_j := g(1 - r) \frac{(\hat{w}_2)_j^{+\frac{1}{2}} + (\hat{w}_2)_j^{-\frac{1}{2}}}{2} (\cos \theta_j)^2 \tan \delta_0
\]

\[
(\hat{u}_2)_j := \frac{\sqrt{2(\hat{w}_2^*)_j(q_2^*)_j}(\cos \theta_j)^2}{\sqrt{((\hat{w}_2^*)_j^4 + \max (((\hat{w}_2^*)_j^4, \beta))}}
\]

\[
(\hat{w}_2)_j^{+\frac{1}{2}} := \frac{1}{2} \left( \frac{(\hat{w}_2)_j}{(\cos \theta_j)^2} + \frac{(\hat{w}_2)_j^{+1}}{(\cos \theta_j^{+1})^2} \right)
\]
Remark

Desingularization is critical to the success of the friction procedure since the locations where \((w_2)\) is small is exactly where one would expect the friction cause the lower layer to stop.

Without desingularization, the velocities \((\hat{u}_2)\) at these locations may become very large, and as a result the momentum may never fall below the critical threshold \(\sigma^* \Delta t/ \cos \theta\).
Example — Small Perturbation of Steady State

\[(\varepsilon, q_1, w_2, q_2)(x, 0) = \begin{cases} 
(0, 0, \eta_2(x), 0), & \text{if } 0.9 < x < 0.95 \\
(\eta_1, 0, 0, 0), & \text{if } 2 < x < 2.1 \\
(0, 0, 0, 0), & \text{otherwise}
\end{cases}\]

\[B(x) = \begin{cases} 
0.05(\cos(5\pi(x - 1)) + 1) - 2, & \text{if } 0.8 < x < 1.2 \\
-2, & \text{otherwise}
\end{cases}\]

\[r = 0.2, \quad g = 1, \quad \delta_0 = 25^\circ\]
Case I

\[ \eta_1 = 10^{-3}, \quad \eta_2(x) \equiv 0 \]
Reformulated System, Well-Balanced
Original System, not Well-Balanced

$t=1.4$

$t=3$
Case II

\[ \eta_1 = 0, \quad \eta_2(x) = -1.9 - B(x) \]

\[ \begin{array}{cccc}
\eta_1 & 0.8 & 1.2 & 2 \\
\eta_2 & 0.1 & 2 & 2
\end{array} \]
Reformulated System, Well-Balanced
Original System, not Well-Balanced
Example — Surface Wave Generation

\[(\varepsilon, q_1, w_2, q_2)(x, 0) = \begin{cases} 
(0, 0, \cos(\arctan(0.2)), 0), & \text{if } 7 < x < 8 \\
(0, 0, 0, 0), & \text{otherwise}
\end{cases} \]

\[B(x) = 0.2x - 2.7\]

\[r = 0.2, \quad g = 9.81, \quad \delta_0 = 25^\circ\]
Zoom of the interface between the upper and lower layers
Example — Large-Scale Wave Generation

\[(\varepsilon, q_1, w_2, q_2)(x, 0) = (0, 0, \max\{-B(x) - 1.8 \cdot 10^{-4}(x - 10000)^2, 0\}, 0)\]

\[B(x) = \begin{cases} 
-10 - 490 e^{-6.1429 (1-x/10000)}, & x \leq 10000 \\
-2500 + 2000 e^{-1.5050 (x/10000 - 1)}, & x > 10000 
\end{cases}\]

\[r = 0.2, \quad g = 9.81, \quad \delta_0 = 12^\circ\]
Example — Tsunami-Like Wave Generation

\[(\varepsilon, q_1, w_2, q_2)(x, 0) = \begin{cases} 
(0, 0, \max \{-1 - B(x), 0\}, 0), & \text{if } 0.5 < x < 3.5 \\
(0, 0, 0, 0), & \text{otherwise}
\end{cases}\]

\[B(x) = -\frac{1}{4} - \frac{1}{2} \left[1 - (2 - x) - \text{sgn}(2 - x) \left(1 - (|2 - x|^c + 1)^{1/c}\right)\right]\]

\[r = 0.2, \quad g = 9.81, \quad \delta_0 = 35^\circ\]
Water Surface ($\varepsilon$)

$t=10$

$t=20$

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