Mapping of Temperatures from Coarser to Finer Grid using Temporal Derivatives

Team 1

Mentor: Ravindra Akarapu, *Corning Incorporated*

Ilona Ambartsumyan, *University of Pittsburgh*
Cuiyu He, *Purdue University*
Eldar Khattatov, *University of Pittsburgh*
Sewoong Kim, *Pohang University of Science and Technology*
Lidia Mrad, *Purdue University*

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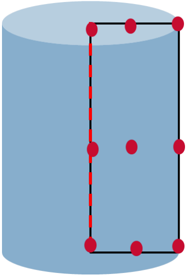
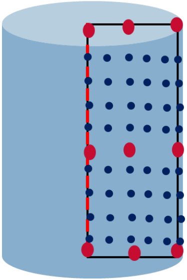
Overview

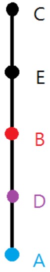
Problem: Improving the spatial resolution of thermocouple data

Objective: Mapping temperature from coarser grid to finer grid

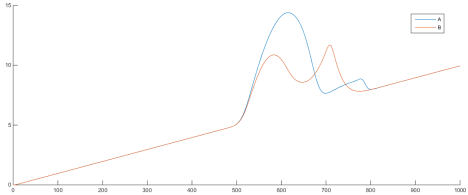




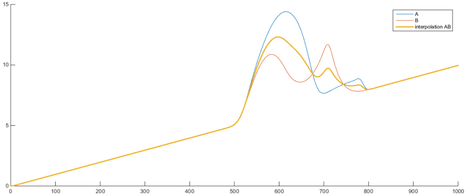
Point Locations at which measurements are taken:



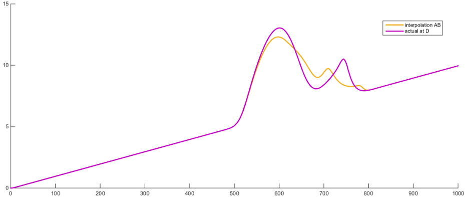
Measured temperature at A and B as time evolves:



Bilinear interpolation of A and B to get temperature at D:



Difference between actual measured temperature and interpolation at D:



Two Approaches

First Approach: Method of Fundamental Solutions

- Method
- Difficulties
- Initial Results and Plan

Second Approach: Conjugate Gradient Method

- Method
- Initial Results
- Plan

Mathematical model

Inverse Heat Conduction Problem (IHCP)

$$\begin{aligned}
\frac{\partial T(x,t)}{\partial t} - \triangle_x T(x,t) &= G(x,t) \quad \text{in } \Omega \times [0,T] & (1)\\
T(x,t) &= g(x,t) \quad \text{on } \Gamma_1 \times [0,T] & (2)\\
\frac{\partial T(x,t)}{\partial n} &= 0 \quad \text{on } \Gamma_2 \times [0,T] & (3)\\
T(x,0) &= 0 \quad \text{on } \Omega & (4)\\
T(x_i,t) &= g_i(t) \quad t \in [0,T], i = 1,2,\cdots,M. & (5)
\end{aligned}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \neq \emptyset$.

- G: heat source function
- x_i: interior points where sensors are placed

If G and g are known: *direct* heat conduction problem
Our case: $G(x,t)$ unknown; $g_i(t)$ over-determined data

Fundamental Solutions Approach

Method of Fundamental Solutions and Radial Basis Functions

Method: Initial Attempt

Laplace Transformation: $\bar{T}(x,y,s) = \int_0^\infty e^{-st} T(x,y,\tau)\, d\tau$

$$\frac{\partial T(x,y,t)}{\partial t} - \triangle T(x,y,t) = G(x,y,t)$$

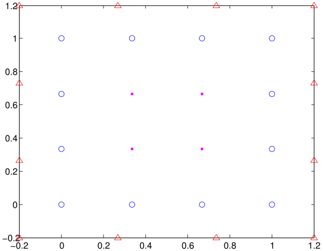
$$\downarrow \qquad \text{Laplace Transform}$$

$$\triangle \bar{T}(x,y,s) - s\bar{T}(x,y,s) = -T_0(x,y) - \bar{G}(x,y,s)$$

Non-homogeneous Modified Helmhotlz Equation, unknown RHS $f(x,y,s)$

$$\bar{T} = \bar{T}^{(h)} + \bar{T}^{(p)}$$

Mierzwiczak & Kolodziej, *Application of the Method of Fundamental Solutions with the Laplace Transformation for the Inverse Transient Heat Source Problem*



Homogeneous solution: $\bar{T}^{(h)}(x, y, s) = \sum_{j=1}^{NS} W_j K_0(r_j \sqrt{s})$

- $r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$, source points $\triangle$, fictitious boundary
- K_0: modified Bessel function of 2nd kind and zero order
- Unknowns: W_j's

Non-homogeneous solution: Interpolate RHS by means of Radial Basis Functions (RBFs) to get a particular solution of the form

$$\bar{T}^{(p)}(x, y, s) = \sum_{m=1}^{M} \alpha_m \hat{\psi}(r_m, s) + \sum_{k=1}^{K} \beta_k \widetilde{\psi_k}(x, y, s)$$

- (x_m, y_m), interpolation points inside the region (dots)
- $\hat{\psi}(r_m, s)$: Radial Basis Function , $\widetilde{\psi_k}$: monomial
- Unknowns: α_m's and β_k's

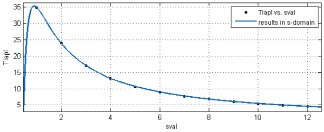
Apply Laplace Transform to boundary conditions and to the extra condition to complete the linear system:

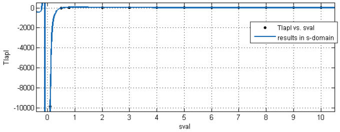
$$A\omega = b$$

- $A : (NB + M + K) \times (NS + M + K)$ matrix
- ω : unknown coefficients
- b : prescribed boundary, additional and interpolation conditions

Difficulties

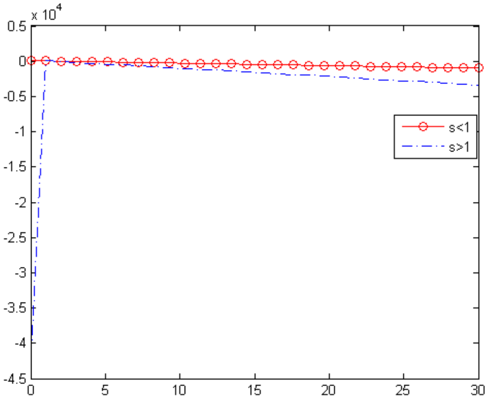
- Numerical Laplace Transform:
  Data of RHS from time-domain to s-domain
- Numerical Inverse Laplace Transform:
  Solution from s-domain back to time-domain (What values of s to take?)





Large s vs. Small s

$$T(x,y,t) = t\frac{(x-6)^3+(y-6)^3}{6}$$



Error in time-domain

Second Attempt

Mierzwiczak & Kolodziej, *Application of the Method of Fundamental Solutions and Radial Basis Functions for Inverse Transient Heat Source Problem*

Key: Overcome Numerical Laplace Transform difficulties by discristizing in time

$$\frac{\partial T(X,Y,F_n)}{\partial F} \approx \frac{T(X,Y,F_{n+1}) - T(X,Y,F_n)}{\tau}$$

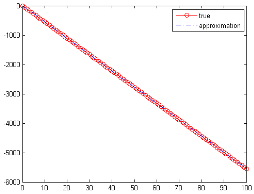
$$\Delta T(X,Y,F_n) \approx \theta.\Delta T(X,Y,F_{n+1}) + (1-\theta)\Delta T(X,Y,F_n)$$

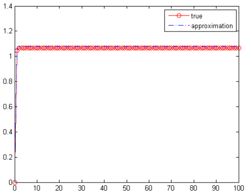
F: Fourier number
Let $T(X,Y,F_n) = w_n(X,Y)$. Get Non-Homogeneous Helmholtz again:

$$\Delta w_{n+1} - \frac{1}{\theta.\tau} w_{n+1} = \frac{1}{\theta.\tau} w_n - \frac{(1-\theta)}{\theta}\Delta w_n - \frac{1}{\theta} Q_n$$

Initial Results

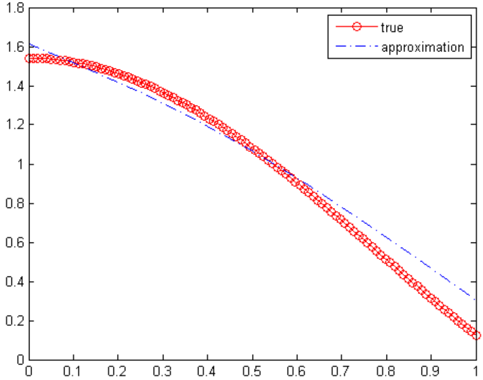
* MODIFY code: iterate and update RHS, $\theta = 1$





$$T_1(x,y,t) = t\frac{(x-6)^3+(y-6)^3}{6} \qquad T_2(x,y,t) = (1-e^{-4t})(\cos(2x)+\cos(2y))$$

Time Distribution for one spacial point$(0.5, 0.5)$



$$T_2(x, y, t) = (1 - e^{-4t})(\cos(2x) + \cos(2y))$$

Spacial Distribution 10 time intervals, $y = 0.5$, 4 interpolation points

Plan

Plan:

- Determine the number of interpolation points to be used to get more accurate results
- Try real data!
- A code to handle varying conductivity?

Advantages of Method:

⋆ Attack 2-D right away

⋆ Mesh-free

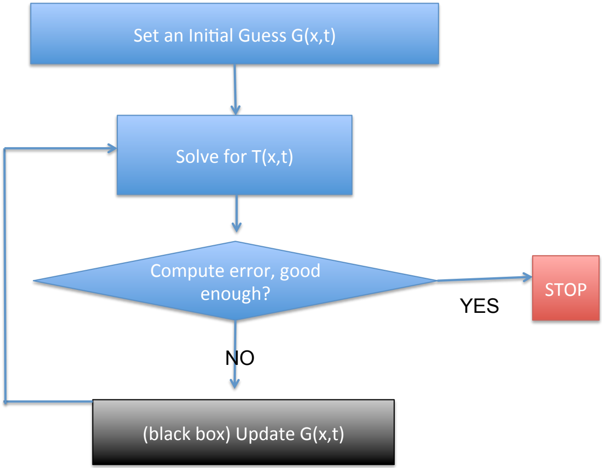
⋆ Avoid numerical integration of singular fundamental solution

⋆ Get value of temperature and value of strength of heat source at ANY point

Conjugate Gradient Approach

Conjugate Gradient Method with Adjoint Problems for Function Estimation.

Big Picture


Set an Initial Guess G(x,t)
Solve for T(x,t)
Compute error, good enough?
YES
STOP
NO
(black box) Update G(x,t)

Direct Problem

$$\begin{cases} \dfrac{\partial T(x,t)}{\partial t} - \triangle_x T(x,t) = G(x,t) \\ T(x,t) = g(x,t) \qquad \text{on} \Gamma_1 \\ \dfrac{\partial T(x,t)}{\partial n} = 0 \qquad \text{on} \Gamma_2 \\ T(x,0) = 0 \end{cases} \tag{6}$$

Minimizing Objective Functional

- In order to estimate $G(z, t)$, we approach by minimizing the following objective functional:

$$S(G(x,t)) = \sum_{m=1}^{M} \int_0^T (Z_m(t) - T(x_m, t))^2 \, dt$$
$$+ \int_\Omega \int_0^T \lambda(x,t) \left(\frac{\partial T(x,t)}{\partial t} - \triangle_x T(x,t) - G(x,t) \right) dtdx$$

where M is the number of sensors and $\lambda(x, t)$ is the Lagrange multiplier.

Adjoint Problem

$$\begin{cases} \frac{\partial \lambda(x,t)}{\partial t} - \Delta\lambda(x,t) = 2\sum_{m=1}^{M}(T(x,t) - Y_m(t))\delta(x - x_m) \\ \lambda(x,t) = 0 \qquad \text{on}\Gamma_1 \\ \frac{\partial \lambda(x,t)}{\partial n} = 0 \qquad \text{on}\Gamma_2 \\ \lambda(x,0) = 0. \end{cases} \tag{7}$$

Solving this problem for $\lambda(x,t)$. Then we can derive

$$\nabla S[G(x,t)] = \lambda(x,t).$$

Sensitivity problem: how sensitive the objective function to $G(x, t) is$.

By replacing the direct problem (6), T by $T + \triangle T$, G by $G + \triangle G$, and subtracting the resulting expressions the original problem, we have

$$\begin{cases} \dfrac{\partial \triangle T(x,t)}{\partial t} - \triangle_x T(x,t) = \triangle G(x,t) \\ \triangle T(x,t) = 0 \qquad\qquad\qquad \text{on} \Gamma_1 \\ \dfrac{\partial \triangle T(x,t)}{\partial n} = 0 \qquad\qquad\qquad \text{on} \Gamma_2 \\ \triangle T(x,0) = 0. \end{cases} \tag{8}$$

Iterative Procedure

$$G^{k+1}(x,t) = G^k(x,t) - \beta^k d^k(x,t),$$

where

$$d^k(x,t) = \nabla S(G^k) + \gamma^k d^{k-1}(x,t),$$

$$\gamma^k = \frac{\int\int \nabla S(G^k)\cdot(\nabla S(G^k) - \nabla S(G^{k-1}))dxdt}{\int\int \nabla S(G^{k-1})^2 dxdt},$$

and

$$\beta^k = \frac{\sum_{m=1}^{M}\left(\int_0^T T^k(x_m,t) - Z_m(t)\right)\triangle T^k(x_m,t,d^k)dt}{\sum_{m=1}^{M}\int_0^T \triangle T^k(x_m,t,d^k)^2 dt}.$$

Initial Results

- We have successfully tested the direct solver.
- We have done most of the coding in 1D and we are now debugging.

Plan

- Of course, get the current code work for 1D :).
- Then for 2D.
- Since we know some information about the source function, how can we make the maximal use of this information instead of starting blindly?

Until the Final Presentation ...