Rapid convergence to quasi-stationary states in the 2D Navier-Stokes equation

Margaret Beck
Heriot-Watt University and Boston University

joint work with C. Eugene Wayne (Boston University)

IMA, Sept 24, 2012
Observed dynamics

2D incompressible Navier-Stokes on the torus with small viscosity:

- Vorticity evolves from small scale to large scale structures
- Localized vortices persist and organize the dynamics
- Separation of time scales
  - Rapid convergence to localized vortices
  - Slow motion and merger of vortices
2D Navier-Stokes: decaying turbulence

Some questions:
- How to characterize the quasi-stationary states? [Y, M, C 2003]
- What causes the separation in time scales? [This talk]

Determine quasi-stationary states via statistical mechanics:
- Stationary solutions of inviscid Euler equations seem to play a role
- Such states with maximum entropy are good candidates

[Yin, Montgomery, Clercx 2003]
Quasi-stationary states

Yin, Montgomery, Clercx 2003:

- Euler: formal calculations and numerical analysis determined these states
- Navier-Stokes: dynamic calculations confirmed predictions ($\nu = 1/5000$)
Related work for stochastically forced Navier-Stokes equation

Statistical equilibrium consists of bars and dipoles [Bouchet, Simonnet 09]:

- Square torus: dipole dominates
- Asymmetric (rectangular) torus: bar dominates

Figures produced by Gabriel Lord (Heriot-Watt)
1D Burgers equation; figure is for similarity variables:

\[ u_t = \nu u_{xx} - uu_x \]

\[ 0 < \nu \ll 1 \]

Zero viscosity:
N-waves stable

Nonzero viscosity:
Diffusion waves stable

Results from [Kim & Tzavaras 01]:
- Observed numerically
- Explained formally using asymptotic expansions
Related work for Burgers equation

Burgers Equation:

\[ u_t = \nu u_{xx} - uu_x, \quad x \in \mathbb{R}, \quad t > 0, \quad u \in \mathbb{R} \]
\[ u(x, 0) = u_0(x), \quad 0 < \mu \ll 1 \]

Scaling variables - deal with continuous spectrum:

\[ u(x, t) = \frac{1}{\sqrt{t + 1}} w \left( \frac{x}{\sqrt{t + 1}}, \log(t + 1) \right) \]
\[ \xi = \frac{x}{\sqrt{t + 1}}, \quad \tau = \log(t + 1) \]

Scaled Burgers equation:

\[ w_\tau = \nu w_{\xi \xi} + \frac{1}{2} \xi w_\xi + \frac{1}{2} w - w w_\xi \]
\[ \mathcal{L}_\nu w = \partial_\xi^2 w + \frac{1}{2} \partial_\xi (\xi w) \]
Related work for Burgers equation

\[ w_\tau = \mathcal{L}_\nu w - ww_\xi \]

In the space

\[ L^2(m) := \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + \xi^2)^m w^2(\xi) d\xi < \infty \right\} \]

the spectrum of \( \mathcal{L} \) is [Gallay & Wayne 02]

\[ \sigma(\mathcal{L}) = \left\{ -\frac{n}{2} : n \in \mathbb{N} \right\} \cup \left\{ \lambda \in \mathbb{C} : \text{Re}\lambda \leq \frac{1 - 2m}{4} \right\} \]

Cole-Hopf still applies:

\[ W(\xi, \tau) = w(\xi, \tau)e^{-\frac{1}{2\nu} \int_{-\infty}^{\xi} w(y, \tau) dy} \Rightarrow W_\tau = \mathcal{L}_\nu W. \]
Related work for Burgers equation

[Beck, Wayne 09]

Reason for timescales:
- Spectrum independent of $\nu$
- Large coefficients in eigenfunction expansion: $w(\tau) = c_0\phi_0 + c_1\phi_1e^{-\frac{1}{2}\tau} \ldots$
- Due to pseudospectrum or Cole-Hopf?
Related results in reaction-diffusion equations

Metastability in gradient systems:

\[ u_t = \epsilon^2 u_{xx} - u(u^2 - 1), \quad x \in (0, 1) \]

Eg: [Carr & Pego 89], [Fusco & Hale 89], [Chen 04], [Otto & Reznikoff 07]

- Stable states: \( u \equiv \pm 1 \)
- Metastable states: step functions connecting \( \pm 1 \) numerous times

Different mechanisms:

- Gradient: utilize energy functional

\[
E[u](t) = \int_0^1 \left[ \frac{\epsilon^2}{2} u_x^2 + \frac{1}{4} (u^2 - 1)^2 \right] dx
\]

- Burgers: spectrum independent of viscosity.
- Navier-Stokes: “spectrum” depends strongly on viscosity.
- Timescale differences
2D Navier-Stokes on the torus

\[ \partial_t u = \nu \Delta u - (u \cdot \nabla)u - \nabla p, \quad \nabla \cdot u = 0, \quad (x, y) \in \mathbb{T}^2 \]

Assume viscosity is small

\[ 0 < \nu \ll 1, \quad \text{physical range} = \mathcal{O}(10^{-3}). \]

Vorticity formulation: \( \omega = \nabla \times u \)

\[ \partial_t \omega = \nu \Delta \omega - u \cdot \nabla \omega, \quad \int_{\mathbb{T}^2} \omega = 0, \quad u = \left( \frac{-\partial_y \Delta^{-1} \omega}{\partial_x \Delta^{-1} \omega} \right). \]

Decay of energy due to diffusion

\[ \frac{d}{dt} \frac{1}{2} \int_{\mathbb{T}^2} \omega^2(x, y) \, dx \, dy = -\nu \int_{\mathbb{T}^2} |\nabla \omega(x, y)|^2 \, dx \, dy \leq -\nu \int_{\mathbb{T}^2} \omega^2(x, y) \, dx \, dy \]

is very slow

\[ \|\omega(t)\|_{L^2} = \mathcal{O}(e^{-\nu t}). \]
Explicit families of metastable states

\[ \omega^{\text{bar}}(x, y, t) = e^{-\nu t} \cos(x), \quad \omega^{\text{dipole}}(x, y, t) = e^{-\nu t} [\cos(x) + \cos(y)] \]

These solutions:

- Are quasi-stationary if \( 0 < \nu \ll 1 \).
- Match observations of [Yin et al 03] and [Bouchet and Simonnet 09].
- Are stationary solutions of the Euler equations when \( \nu = 0 \).
- Should attract (some) nearby solutions faster than \( O(e^{-\nu t}) \).
- Are part of an infinite family:

\[ \omega^{\text{slow}}(x, y, t) = e^{-\nu m^2 t} [a_1 \cos(mx) + a_2 \cos(my) + a_3 \sin(mx) + a_4 \sin(my)] \]
Linearization about a bar state

\[ \partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \quad \mathbf{u} = \begin{pmatrix} -\partial_y \Delta^{-1} \omega \\ \partial_x \Delta^{-1} \omega \end{pmatrix}. \]

Ansatz: \( \omega = \omega^{\text{bar}} + \nu \)

\[ \partial_t \nu = \nu \Delta \nu - e^{-\nu t}[\sin x \partial_y (1 + \Delta^{-1})] \nu - \mathbf{u}^{\nu} \cdot \nabla \nu. \]

Approximate linearization similar to advection of passive scalar by a shear flow:

\[ \partial_t \nu = \nu \Delta \nu - \sin x \partial_y \nu \]

- Asymptotic of eigenvalues in [Vanneste, Byatt-Smith 07]: \( O(e^{-\sqrt{\nu} t}) \)
- Compute spectrum (Eigtool; Fourier approximation, \((k, l) = (k, 1)):\n
\begin{align*}
\text{Approximate operator} & \quad \nu = 0.001 & \text{Full operator}
\end{align*}
What causes the fast decay?

\[ u_t = Lu \]

Villani, 2009, considers operators of the form

\[ L = A^*A + B, \quad B^* = -B \]

- **\( AB = BA \):** antisymmetry of \( B \) implies \( \|e^{Bt}u\| = \|u\| \), and so

\[ \|e^{Lt}\| = \|e^{A^*At}e^{Bt}\| = \|e^{A^*At}\|, \]

so \( B \) cannot increase the decay rate of the semigroup.

- **\( AB \neq BA \):** rapid decay possible via hypoceoricvity

Define commutator \( C = [A, B] = AB - BA \) and an inner product

\[ ((u, u)) = (u, u) + \alpha(Au, Au) - 2\beta Re(Au, Cu) + \gamma(Cu, Cu) \]

Careful choice of \( \alpha, \beta, \) and \( \gamma \) can show faster than expected decay.
Back to our problem...

\[ \partial_t v = \nu \Delta v - e^{-\nu t} [\sin x \partial_y (1 + \Delta^{-1})] v =: \mathcal{L}(t) v \]

**Slow modes:** Cannot expect rapid decay on all of \( L^2 \)

\[ \lambda v_{\text{slow}} = \partial_t v_{\text{slow}} = \mathcal{L}(t) v_{\text{slow}}, \quad \lambda = \mathcal{O}(\nu) \]

\[ v_{\text{slow}} \in \{ e^{-\nu m^2 t + imx}, e^{-\nu t \pm iy} : m \in \mathbb{Z}_0 \}. \]

Like an infinite-dimensional eigenspace – need to “project” off it.

**Intuitively:**
- Expect something like a center manifold with slow decay \( \mathcal{O}(e^{-\nu t}) \)
- and something like a stable manifold with rapid decay \( \mathcal{O}(e^{-\sqrt{\nu} t}) \)
- Use hypocoercivity to get rapid decay rate in stable manifold.
- But operator is time-dependent.
- Can’t use spectral projections to obtain manifolds.

**Invariant subspaces:**
- Need an alternative way to construct them
- Should be related to movement of energy between Fourier modes.
Construct invariant subspaces

\[ v(x, y) = \sum_{k,l \in \mathbb{Z}, (k,l) \neq (0,0)} \hat{v}(k, l) e^{i(kx+ly)} \]

Goal: don’t excite the slow modes

\[ \{ e^{-\nu m^2 t + imx}, e^{-\nu t \pm iy} \} \Rightarrow (k, l) \in \{(0, \pm 1), (m, 0)\} \]

In Fourier space, \( \nu_t = \nu \Delta v - e^{-\nu t} \partial_y \sin x(1 + \Delta^{-1})v \) becomes

\[
\partial_t \hat{v}(k, l) = -\nu (k^2 + l^2) \hat{v}(k, l)
- \frac{l}{2} e^{-\nu t} \left[ \left(1 - \frac{1}{(k-1)^2 + l^2}\right) \hat{v}(k-1, l) - \left(1 - \frac{1}{(k+1)^2 + l^2}\right) \hat{v}(k+1, l) \right]
\]

Try \( \mathcal{M}_x = \{ v \in L^2(\mathbb{T}^2) : \hat{v}(m, 0) = 0, \ m \in \mathbb{Z} \} \)

\[ \partial_t \hat{v}(m, 0) = -\nu m^2 \hat{v}(m, 0) \quad \text{invariant} \]

Try: \( \tilde{\mathcal{M}}_y = \{ v \in L^2(\mathbb{T}^2) : \hat{v}(0, \pm 1) = 0 \} \)

\[ \partial_t \hat{v}(0, \pm 1) = -\nu \hat{v}(0, \pm 1) + \frac{1}{4} e^{-\nu t} \left[ \hat{v}(-1, \pm 1) - \hat{v}(1, \pm 1) \right] \quad \text{not invariant} \]
Construct invariant subspaces

Recall: we don’t want to excite the modes $e^{\pm imx}$ and $e^{\pm iy}$

- x-modes: $\mathcal{M}_x = \{ w \in L^2(\mathbb{T}^2) : \hat{w}(m, 0) = 0 \}$
- y-modes: Formal calculations with Fourier equation lead to...

Define

$$ p_j^\pm := \hat{w}(2j, \pm 1) + \hat{w}(-2j, \pm 1), \quad q_j^\pm := \hat{w}(2j + 1, \pm 1) - \hat{w}(-2j - 1, \pm 1) $$

One can show:

$$ \begin{pmatrix} p_j^\pm \\ q_j^\pm \end{pmatrix} = A^\pm(t) \begin{pmatrix} p_j^\pm \\ q_j^\pm \end{pmatrix} $$

Proposition A solution of $w_t = \mathcal{L}(t)$ satisfies $\hat{w}(0, \pm 1)(t) = 0$ for all $t \geq 0$ if and only if $w(0) \in \mathcal{M}_y$, where

$$ \mathcal{M}_y = \{ w \in L^2 : p_j^\pm = q_j^\pm = 0 \ \forall j \}. $$

Recall: In [YCM '03], only special initial data converge rapidly to bar states.
Rapid decay in this subspace

From now on, we only work in \( M_x \cap M_y \).

\[
\partial_t \nu = \nu \Delta \nu - e^{-\nu t} [\sin x \partial_y (1 + \Delta^{-1})] \nu.
\]

Since there is no \( y \)-dependence in the bar state: \( \nu(x, y) = \sum_{l \in \mathbb{Z}} \hat{\nu}_l(x) e^{ily} \)

\[
\partial_t \hat{\nu}_l = \nu \Delta_l \hat{\nu}_l - ile^{-\nu t} [\sin x (1 + \Delta_l^{-1})] \hat{\nu}_l, \quad \Delta_l = \partial_x^2 - l^2.
\]

Recall: want \( L = A^* A + B \), with \( B^* = -B \)

- \( A = \partial_x \), \( A^* = -\partial_x \), so that \( \nu \partial_x^2 = -\nu A^* A \)
- But the second term is not anti-symmetric! Change variables...

Motivated by Wilkinson’s book “The algebraic eigenvalue problem”:

\[
u := \sqrt{1 + \Delta_l^{-1}} \hat{\nu}_l
\]

\[
1 + \Delta_l^{-1} = 1 - \frac{1}{k^2 + l^2} \quad \Leftrightarrow \quad |l| + |k| > 1.
\]

Invertible transformation in our subspace.
Transformed equation

\[ \partial_t u = \nu \Delta_l u - \nu t e^{-\nu t} \left[ \sqrt{1 + \Delta_l^{-1}} \sin x \sqrt{1 + \Delta_l^{-1}} \right] u. \]

We have

- \( A := \partial_x \)
- \( B := -\nu t e^{-\nu t} \left[ \sqrt{1 + \Delta_l^{-1}} \sin x \sqrt{1 + \Delta_l^{-1}} \right] \), \( B^* = -B \)
- \( C := [\partial_x, B] = -\nu t e^{-\nu t} \left[ \sqrt{1 + \Delta_l^{-1}} \cos x \sqrt{1 + \Delta_l^{-1}} \right] \), \( C^* = -C \).

Problem: \([B, C] \neq 0\); will lead to difficult terms in Villani’s framework.
Partial solution: first consider only the approximate equation

\[ \partial_t u = \nu \Delta_l u - \nu t e^{-\nu t} \sin x u := \mathcal{L}_{\text{approx}}(t)u. \]

- \( A := \partial_x \)
- \( B := -\nu t e^{-\nu t} \sin x \), \( B^* = -B \)
- \( C := [\partial_x, B] = -\nu t e^{-\nu t} \cos x \), \( C^* = -C \).
- \([B, C] = 0\)
Why is this new inner product useful?

Motivated by work of Gallagher, Gallay, and Nier 2009, we rescale time:

\[ \partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu} Bu. \]

Define, for \((u, u) = \|u\|_{L^2}^2, \alpha, \beta, \gamma > 0,\)

\[ \Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta \text{Re}(\partial_x u, Cu) + \gamma(Cu, Cu) \]

If \(\beta^2 < \alpha\gamma/4,\) Young’s inequality implies

\[ \|u\|^2 + \frac{\alpha}{2} \|u_x\|^2 + \frac{\gamma}{2} \|Cu\|^2 < \Phi < \|u\|^2 + \frac{3\alpha}{2} \|u_x\|^2 + \frac{3\gamma}{2} \|Cu\|^2. \]

Therefore, by controlling the dynamics of \(\Phi,\) we can control the above norm.

Strategy:

- Compute \(d\Phi/dt\)
- Chose \(\alpha, \beta, \gamma\) to obtain a decay estimate
- Show this implies rapid convergence of solutions to the bar states
Main result

Function space: $C = C(l) = -ie^{-\nu t} \cos x$

$$X = \left\{ u : \hat{u}_0 = 0, \sum_{l \neq 0} \left[ \| \hat{u}_l \|^2 + \sqrt{\frac{\nu}{|l|}} \| \partial_x \hat{u}_l \|^2 + \frac{1}{\sqrt{\nu}} |l|^{3/2} \| C(l) \hat{u}_l \|^2 \right] < \infty \right\}$$

**Theorem** Pick $T \in [0, 1/\nu]$. There exist constants $K$ and $M$, $O(1)$ with respect to $\nu$, such that the following holds. If $\nu$ is sufficiently small, then the solution to $u_t = \mathcal{L}_{\text{approx}}(t)u$ with initial condition $u^0 \in X$ satisfies

$$\| u(t) \|^2_X \leq K e^{-M \sqrt{\nu} t} \| u^0 \|^2_X$$

for all $t \in [0, T]$.

Implies rapid decay of solutions:
- Decay $e^{-M \sqrt{\nu} t}$ much faster than the viscous time scale $e^{-\nu t}$
- If $T = 1/\nu$, then

$$e^{-M \sqrt{\nu} T} = e^{-\frac{M}{\sqrt{\nu}}} \ll 1, \quad e^{-\nu T} = e^{-1}$$
Proof of Theorem

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu}Bu$$

$$\Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta\text{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

Differentiate:

$$\frac{d}{dt} \Phi(t) = [(u_t, u) + (u, u_t)] + \alpha[(\partial_x u_t, \partial_x u) + (\partial_x u, \partial_x u_t)]$$

$$-2\beta\text{Re}[(\partial_x u_t, Cu) + (\partial_x u, Cu_t)] + \gamma[(Cu_t, Cu) + (Cu, Cu_t)]$$

$$+ \gamma[(C_t u, Cu) + (Cu, C_t u)].$$

The first term gives

$$(u_t, u) + (u, u_t) = ((-l^2 + \partial_x^2 + \frac{1}{\nu}B)u, u) + (u, (-l^2 + \partial_x^2 + \frac{1}{\nu}B)u)$$

$$= -2l^2\|u\|^2 - 2\|u_x\|^2 + \frac{1}{\nu}[(Bu, u) + (u, Bu)]$$

$$= 0$$

by the anti-symmetry of $B$. 
Proof of Theorem

\[ \partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu}Bu \]

\[ \Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta \text{Re}(\partial_x u, Cu) + \gamma(Cu, Cu) \]

The \( \alpha \) term gives

\[
(\partial_x u_t, \partial_x u) + (\partial_x u, \partial_x u_t) = (\partial_x(-l^2 + \partial_x^2 + \frac{1}{\nu}B)u, u_x) \\
+ (u_x, \partial_x(-l^2 + \partial_x^2 + \frac{1}{\nu}B)u) \\
= -2l^2\|u_x\|^2 - 2\|u_{xx}\|^2 \\
+ \frac{1}{\nu}[(\partial_x(Bu), u_x) + (u_x, \partial_x(Bu))] \\
\]

We can bound

\[
[(\partial_x(Bu), u_x) + (u_x, \partial_x(Bu))] = (Bu_x, u_x) + (\overset{\text{c}}{[\partial_x, B]}u, u_x) \\
+ (u_x, Bu_x) + (u_x, [\partial_x, B]u) \\
= 2\text{Re}(u_x, Cu) \\
\leq 2\|u_x\|\|Cu\|. 
\]
Proof of Theorem

\[ \partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu} Bu \]

\[ \Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta \text{Re}(\partial_x u, Cu) + \gamma(Cu, Cu) \]

The \( \beta \) term gives

\[ (\partial_x u_t, Cu) + (\partial_x u, Cu_t) = -2l^2 \text{Re}(\partial_x u, Cu) + [(u_{xxx}, Cu) + (u_x, Cu_{xx})] + \frac{1}{\nu} [(\partial_x(Bu), Cu) + (u_x, C(Bu))] \]

One can show

\[ (\partial_x(Bu), Cu) + (u_x, C(Bu)) = \|Cu\|^2 + (u_x, [C, B]u) = \|Cu\|^2 \]

**Important term:** \(-\frac{2\beta}{\nu}\|Cu\|^2\)

The \( \gamma \) and \( C_t \) terms are similar.
Proof of Theorem

Collecting these estimates, we have shown

\[
\frac{d}{dt} \Phi(t) \leq -2l^2\|u\|^2 - [2 + 2\alpha l^2]\|u_x\|^2 - 2\alpha\|u_{xx}\|^2 \\
+ \left(\frac{2\alpha}{\nu} + 2\beta(2l^2 + 1 + \nu)\right)\|u_x\||C_u|| + 4\beta\|u_{xx}\||C_u_x| \\
- \left((2l^2 + 2)\gamma + \frac{2\beta}{\nu} - 2\gamma\nu\right)\|C_u\|^2 - 2\gamma\|C_{u_x}\|^2 + 2\gamma\|B_u\|^2.
\]

We now use the fact that \(2ab \leq a^2 + b^2\) and scale the parameters as

\[
\alpha = \sqrt{\nu}\alpha_0, \quad \beta = \beta_0, \quad \gamma = \frac{1}{\sqrt{\nu}}\gamma_0
\]

With appropriate conditions on \(\alpha_0, \beta_0, \gamma_0\), this gives

\[
\frac{d}{dt} \Phi(t) \leq -2\|u\|^2 + 2\gamma_0\|B_u\|^2 - \frac{1}{4}\|u_x\|^2 - \frac{3\beta_0}{2\nu}\|C_u\|^2
\]

**Goal:** Show \(\Phi' \leq -(M/\sqrt{\nu})\Phi\)
Proof of Theorem

\[ \|u\|^2 + \frac{\alpha_0 \sqrt{\nu}}{2} \|u_x\|^2 + \frac{\gamma_0}{2 \sqrt{\nu}} \|Cu\|^2 < \Phi < \|u\|^2 + \frac{3 \alpha_0 \sqrt{\nu}}{2} \|u_x\|^2 + \frac{3 \gamma_0}{2 \sqrt{\nu}} \|Cu\|^2 \]

\[ \frac{d}{dt} \Phi(t) \leq -2 \|u\|^2 + 2 \frac{\gamma_0}{\sqrt{\nu}} \|Bu\|^2 - \frac{1}{4} \|u_x\|^2 - \frac{3 \beta_0}{2 \nu} \|Cu\|^2 \]

Proposition If \(|l| > 1\), then there exists a constant \(M_0\) such that, for all \(0 < t < T\),

\[ \frac{1}{8} \|u_x\|^2 + \frac{\beta_0}{2 \nu} \|Cu\|^2 \geq \frac{M_0 |l| \sqrt{\beta_0}}{\sqrt{\nu}} \|u\|^2 . \]

Proof: Follows like a similar result in [Gallagher, Gallay, & Nier '09]. Essentially due to connection with harmonic oscillator:

\[ H = a \partial_{xx} + bx^2 \Rightarrow (Hu, u)_{L^2(\mathbb{R})} \geq \sqrt{ab} (u, u)_{L^2(\mathbb{R})} \]

Need to be careful about the role of \(|l|\). Also, \(M_0 = \mathcal{O}(e^{-\nu t})\). \(\square\)

This implies (after choosing \(\alpha_0, \beta_0, \gamma_0\))

\[ \Phi'(t) \leq - \frac{M}{\sqrt{\nu}} \Phi(t) \]
Summary and future directions

We have shown:
- Rapid decay for approximate operator: $O(e^{-\sqrt{\nu}t}) \ll O(e^{-\nu t})$
- Proof based on Villani’s treatment of $L = A^*A + B$, $[A, B] \neq 0$.

To extend to full linear operator:
- Existence of invariant subspaces (and projections) for full operator.
- Use transformation $u = \sqrt{1 + \Delta^{-1}} v$ to make $B$ antisymmetric.

Nonlinear equation; metastability of bar states:
- Use projection operators
- Use estimates similar to invariant manifold existence proofs

Dipoles:
\[
\omega^d(x, y, t) = e^{-\nu t}[\cos(x) + \cos(y)],
\]
- Much of the proof could be similar
- Need to understand slow modes and invariant subspaces