Symmetry of components for semilinear elliptic systems

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Dynamical Systems in Studies of Partial Differential Equations

Minneapolis, September 27, 2012

Motivation

$$-\Delta u + \omega u = u^p \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{1}$$

where

$$n \geq 3, \quad \omega \in \mathbb{R}, \quad p > 1, \quad u \geq 0$$

Singularity estimates [Poláčik, Q., Souplet 2007]

If $p < \frac{n+2}{n-2}$ then $\exists\, C = C(p, n, \omega) > 0$ such that any positive solution u of (1) satisfies $u(x) \leq C(1 + \mathrm{dist}(x, \partial\Omega)^{-2/(p-1)})$

Proof based on doubling arguments, scaling and

Liouville theorem [Gidas, Spruck 1981; Chen, Li 1991]

If $p < \frac{n+2}{n-2}$ then $-\Delta u = u^p$ does not possess positive solutions in $\mathbb{R}^n$.

This approach works for general elliptic and parabolic systems;
it also yields gradient and decay estimates.

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The study of standing waves of Schrödinger **systems** (nonlinear optics, Bose-Einstein condensates) leads to scaling invariant systems of the form

$$\left.\begin{aligned} -\Delta u &= \lambda u^{2q+1} + \beta u^q v^{q+1} \\ -\Delta v &= \lambda v^{2q+1} + \beta v^q u^{q+1} \end{aligned}\right\} \quad \text{in } \mathbb{R}^n. \tag{2}$$

Case $\lambda > 0$ (w.l.o.g. $\lambda = 1$):

Theorem 1 [Q., Souplet 2012]

Let $\lambda = 1$ and $n \leq 4$.
If $\beta > -1$ and $2q + 1 < \frac{n+2}{n-2}$ then (2) does not possess positive solutions.

- Conditions $\beta > -1$ and $2q + 1 < \frac{n+2}{n-2}$ are optimal.
- Results for general variational systems.
- Singularity and decay estimates as in the scalar case.
- Proof completely different from the scalar case:
  "Scalar" arguments do not apply if $\beta < 0$.

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Standing waves of nonlinear Schrödinger systems ($\lambda \leq 0$)

Case $\lambda \leq 0$, $\beta > 0$ (w.l.o.g. $\beta = 1$) and $q = 1$
(models of Bose-Einstein condensates with repulsive self-interaction and attractive interspecific interaction $\rightarrow$ phenomenon of symbiotic solitons):

$$\left.\begin{aligned} -\Delta u &= \lambda u^3 + uv^2 \\ -\Delta v &= \lambda v^3 + vu^2 \end{aligned}\right\} \quad \text{in } \mathbb{R}^n \tag{3}$$

Theorem 2 [Q., Souplet 2012]

If $0 \geq \lambda > -\frac{1}{n-2}$ *then any positive solution of* (3) *satisfies* $u = v$.

- $n = 3$ ... Liouville theorem ($\Rightarrow$ estimates as above)
- $n > 3$ ... critical or supercritical nonlinearity ($\exists$ positive solutions);
  open: Is the condition $\lambda > -\frac{1}{n-2}$ necessary?
- Results for more general systems:

$$-\Delta u = f(u,v), \quad -\Delta v = g(u,v),$$

$$(f(u,v) - g(u,v))(u - v) \leq 0 \tag{$*$}$$

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Symmetry of components — basic idea in the general case

$$-\Delta u = f(u,v), \quad -\Delta v = g(u,v),$$
$$(f(u,v) - g(u,v))(u - v) \le 0 \qquad (*)$$

If v decays at infinity, then $w := v - u$ satisfies

$$\left.\begin{aligned}\int_{w>0} |\nabla w|^2\,dx &= \int_{w>0} (-\Delta w) w\,dx \\ &= \int_{w>0} (g(u,v) - f(u,v)) w\,dx \le 0\end{aligned}\right\} \quad \Rightarrow \quad w \le 0$$

Corollary 1: *Assume* $(*)$. *If* u, v *decay at infinity, then* $u = v$.

Decay of u, v: Relatively easy if $f, g \ge 0$ (see below).
Otherwise: $z = u^a v^{1-a}$ satisfies $-\Delta z \ge cz^k$ for various $a \in (0,1)$

$$\Rightarrow \quad z \ge c|x|^{2-n}, \quad \int_{|x|<R} z^k\,dx \le CR^{n-2k/(k-1)}$$

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Results for $f, g \geq 0$

$$\left.\begin{aligned} -\Delta u &= u^r v^p \\ -\Delta v &= v^s u^q \end{aligned}\right\} \quad \text{in } \mathbb{R}^n \tag{4}$$

Theorem 3 [Q., Souplet 2012]

If $p - s = q - r \geq 0$ *and* $0 \leq r, s \leq \frac{n}{n-2}$ *then* $u = v$.

- If $r = s = 1$ and $p = q = 2$ then (4) is equivalent to (3) with $\lambda = 0$.
- Assumptions on r, s are not just technical.
- Assumption $p - s = q - r \geq 0$ guarantees

$$(u^r v^p - v^s u^q)(u - v) = u^r v^s (v^{p-s} - u^{q-r})(u - v) \leq 0 \tag{$*$}$$

Corollary 1 $\Rightarrow$ it is sufficient to prove the decay of u, v

- Theorem 3 was motivated by [Li, Ma 2008] who used moving planes for the corresponding integral system. Their method required, in particular, integrability assumptions on u, v and $p + r \leq \frac{n+2}{n-2}$.

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If $p - s = q - r \geq 0$ *and* $0 \leq r, s \leq \frac{n}{n-2}$ *then* $u = v$.

Proof of Theorem 3: It is sufficient to prove the decay of spherical averages of v (and u)

$$\bar{v}(R) := \frac{1}{|S_R|} \int_{S_R} v \, dS, \quad \text{where} \quad S_R := \{x \in \mathbb{R}^n : |x| = R\}.$$

Lemma: $-\Delta v \geq 0 \;\Rightarrow\; \bar{v}$ is nonincreasing and $v \geq L := \lim_{R\to\infty} \bar{v}(R)$.

Assume on the contrary $L > 0$. Then

$$-\Delta u = v^p u^r \geq L^p u^r \quad \ldots \quad \text{contradiction}$$

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Example 1:

Let $r = s = \frac{n}{n-2}$, $\quad p = q = r - 1$ and $\quad -\Delta w = w^{(n+2)/(n-2)}$.

Then $(u, v) = (cw, \frac{1}{c}w)$ is a solution of (4) for any $c > 0$.

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If $p - s = q - r \geq 0$ and $0 \leq r, s \leq \frac{n}{n-2}$ then $u = v$.

Example 2: Let $p = s$, $q = r$; look for solutions of the form $(v+1, v)$.
(4) is equivalent to

$$-\Delta v = v^s (1+v)^r \quad \text{in } \mathbb{R}^n. \tag{5}$$

Existence for $r \geq 0$, $s \geq \frac{n+2}{n-2}$ (Pohozaev's identity)

Lin, Ni 1988: Explicit solution for $n = 5$, $r = 1$, $s = 2$ $\left(\frac{n}{n-2} < s < \frac{n+2}{n-2}\right)$.

Bamón, Flores, del Pino 2000–2004: Existence for $n = 5$, $r = 1$ and $s \sim 2$.
Their arguments seem to guarantee existence also in other cases, for example if $n = 4$, $r = 1$ and $s = \frac{n}{n-2} + \varepsilon$ (Dávila 2012)

Counterexamples II

$$\left.\begin{array}{l} -\Delta u = u^r v^p \\ -\Delta v = v^s u^q \end{array}\right\} \quad \text{in } \mathbb{R}^n \qquad (4)$$

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Use of dynamical systems [Bamón, Flores, del Pino 2000–2004]

Existence and nonexistence of radial solutions $v = v(r)$ of

$$-\Delta v = v^p + v^q \quad \text{in } \mathbb{R}^n, \tag{6}$$

where $\frac{n}{n-2} < p < \frac{n+2}{n-2} < q$.
Emden-Fowler transformation: $x(t) = r^{2/(q-1)} v(r)$, $r = e^t$, leads to

$$x'' + \alpha x' + x_+^q + e^{\gamma t} x_+^p - \beta x = 0, \quad t \in \mathbb{R},$$

$$\text{where} \quad \alpha, \beta, \gamma > 0 \quad \text{and} \quad x(t) \to 0 \text{ as } t \to \pm\infty.$$

Introduce $y = x'$ and $z(t) = e^{\gamma t}$: (x, y, z) satisfy system (S1) of ODE's with $(0, 0, 0)$ as a steady state with two-dimensional unstable manifold.

• In order to study the behavior for $t \to \infty$, it is more convenient to use system (S2) for $\tilde{x}(t) = r^{2/(p-1)} v(r)$, $\tilde{y} = \tilde{x}'$, $\tilde{z}(t) = e^{-\tilde{\gamma} t}$.

• One looks for solutions in the unstable manifold of $(0, 0, 0)$ of (S1) which simultaneously belong to a stable manifold of a steady state of (S2).

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$$\left.\begin{aligned} x' &= y, \\ y' &= -\alpha y + \beta x - x_+^q - z x_+^p, \\ z' &= \gamma z. \end{aligned}\right\} \tag{S1}$$

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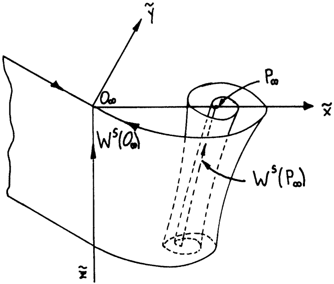

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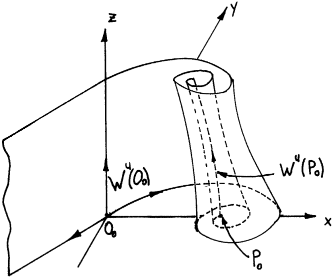
$$(r = e^t)$$

$$x(t) = r^{2/(q-1)} v(r)$$



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