Periodic Travelling Waves in Diatomic Granular Chains

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Introduction

- Granular crystal chains are chains of densely packed, elastically interacting particles.
- Recent work focuses on periodic travelling waves in granular chains; said to be more relevant to physical experiments.
- Periodic travelling waves in homogeneous granular chains (**monomers**) were approximated numerically
  - Yu. Starosvetsky and A.F. Vakakis, Urbana-Champneys
  - G. James, Grenoble
- Our work focuses on the periodic travelling waves in chains of beads of alternating masses (**dimers**).

On solitary travelling waves in homogeneous granular chains

Proofs of existence of solitary waves were developed from the variational theory based on the differential–difference equation.

- G. Friesecke and J. Wattis (1994) - general FPU (Fermi–Pasta–Ulam) lattice
- R. MacKay (1999) - adaptation of this method to granular chains
- J. English and R. Pego (2005) - proof of the double-exponential tails
- A. Stefanov and P. Kevrekidis (2012) - proof of the bell-shaped profile.

Existence problem for solitary waves in diatomic granular crystals has been studied.

Experimental setups (CalTECH)

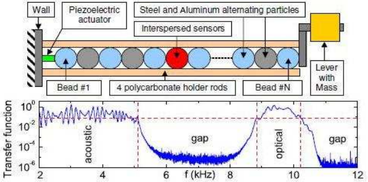


Figure : N. Boechler, G. Theocharis, S. Job, P.G. Kevrekidis, M.A. Porter, and C. Daraio, PRL 104, 244302 (2010)

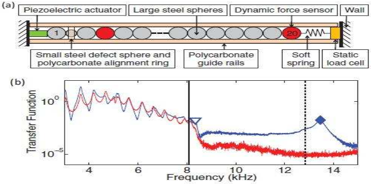
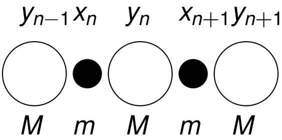


Figure : Y. Man, N. Boechler, G. Theocharis, P.G. Kevrekidis, and C. Daraio, Phys. Rev. E **85**, 037601 (2012)

The Dimer Model



Newton's equations define the FPU (Fermi-Pasta-Ulam) lattice:

$$\begin{cases} m\ddot{x}_n = V'(y_n - x_n) - V'(x_n - y_{n-1}), \\ M\ddot{y}_n = V'(x_{n+1} - y_n) - V'(y_n - x_n), \end{cases} \qquad n \in \mathbb{Z},$$

where the interaction potential for spherical beads is

$$V(x) = \frac{1}{1+\alpha}|x|^{1+\alpha}H(-x), \quad \alpha = \frac{3}{2}$$

and H is the step (Heaviside) function.

H. Hertz, J. Reine Angewandte Mathematik, 92 (1882), 156

Small mass ratio

To study small mass ratios $\varepsilon^2 = \frac{m}{M}$, we make the substitutions:

$$n \in \mathbb{Z}: \quad x_n(t) = u_{2n-1}(\tau), \quad y_n(t) = \varepsilon w_{2n}(\tau), \quad t = \sqrt{m}\tau$$

The FPU lattice is transformed into the equivalent form:

$$\left\{ \begin{array}{l} \ddot{u}_{2n-1} = V'(\varepsilon w_{2n} - u_{2n-1}) - V'(u_{2n-1} - \varepsilon w_{2n-2}), \\ \ddot{w}_{2n} = \varepsilon V'(u_{2n+1} - \varepsilon w_{2n}) - \varepsilon V'(\varepsilon w_{2n} - u_{2n-1}), \end{array} \right. \qquad n \in \mathbb{Z}.$$

The anti-continuum limit corresponds formally $\varepsilon = 0$:

$$\left\{ \begin{array}{l} \ddot{u}_{2n-1} = V'(-u_{2n-1}) - V'(u_{2n-1}) = -|u_{2n-1}|^{\alpha-1} u_{2n-1}, \\ \ddot{w}_{2n} = 0. \end{array} \right.$$

K. Yoshimura, Nonlinearity 24 (2011), 293.

Periodic travelling waves

Periodicity conditions:

$$u_{2n-1}(\tau) = u_{2n-1}(\tau + 2\pi), \quad w_{2n}(\tau) = w_{2n}(\tau + 2\pi), \quad \tau \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

Travelling wave conditions:

$$u_{2n+1}(\tau) = u_{2n-1}(\tau + 2q), \quad w_{2n+2}(\tau) = w_{2n}(\tau + 2q), \quad \tau \in \mathbb{R}, \quad n \in \mathbb{Z},$$

where $q \in [0, \pi]$ is a free parameter.

Equivalent form for periodic travelling waves:

$$u_{2n-1}(\tau) = u_*(\tau + 2qn), \quad w_{2n}(\tau) = w_*(\tau + 2qn), \quad \tau \in \mathbb{R}, \quad n \in \mathbb{Z},$$

where u_* and w_* are 2π-periodic functions.

The Monomer Model

In the limit of equal mass ratio, $\varepsilon = 1$ we apply the reduction:

$$n \in \mathbb{Z}: \quad u_{2n-1}(\tau) = U_{2n-1}(\tau), \quad w_{2n}(\tau) = U_{2n}(\tau).$$

This substitution reduces the dimer system to the monomer system:

$$\ddot{U}_n = V'(U_{n+1} - U_n) - V'(U_n - U_{n-1}), \quad n \in \mathbb{Z}.$$

Periodic travelling waves were considered recently in:
G. James, J. Nonlinear Science 22 (2012).

Remark: Travelling waves of the dimer model with $\varepsilon = 1$ do not have to obey the reductions to the monomer model.

Differential Advance-Delay Equation

Expressing the travelling waves as:

$$u_{2n-1}(\tau) = u_*(\tau + 2qn), \quad w_{2n}(\tau) = w_*(\tau + 2qn), \quad \tau \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

we obtain the differential advance-delay equations for (u_*, w_*):

$$\begin{cases} \ddot{u}_*(\tau) = V'(\varepsilon w_*(\tau) - u_*(\tau)) - V'(u_*(\tau) - \varepsilon w_*(\tau - 2q)), \\ \ddot{w}_*(\tau) = \varepsilon V'(u_*(\tau + 2q) - \varepsilon w_*(\tau)) - \varepsilon V'(\varepsilon w_*(\tau) - u_*(\tau)), \end{cases} \quad \tau \in \mathbb{R}.$$

Remark: For particular values $q = \frac{\pi}{N}$, the differential advance-delay equation is equivalently represented by the system of $2N$ second-order differential equations closed subject to the periodic boundary conditions.

Anti-continuum Limit

Let φ be a solution of the nonlinear oscillator equation,

$$\ddot{\varphi} = V'(-\varphi) - V'(\varphi) \quad \rightarrow \quad \ddot{\varphi} + |\varphi|^{\alpha-1}\varphi = 0.$$

For a unique 2π-periodic solution we set:

$$\varphi(0) = 0, \quad \dot{\varphi}(0) > 0$$

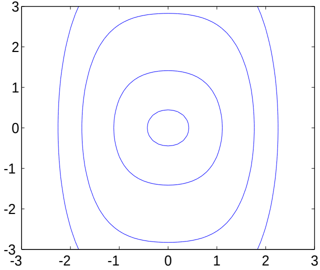


Figure : Phase portrait of the nonlinear oscillator in the $(\varphi, \dot{\varphi})$-plane.

Special Solutions

For $\varepsilon = 0$, we can construct a limiting solution to the differential advance-delay equations:

$$\varepsilon = 0: \quad u_*(\tau) = \varphi(\tau), \quad w_*(\tau) = 0, \quad \tau \in \mathbb{R},$$

Two solutions are known exactly for all $\varepsilon \geq 0$:

$$q = \frac{\pi}{2}: \quad u_*(\tau) = \varphi(\tau), \quad w_*(\tau) = 0$$

and

$$q = \pi: \quad u_*(\tau) = \frac{\varphi(\tau)}{(1+\varepsilon^2)^3}, \quad w_*(\tau) = \frac{-\varepsilon\varphi(\tau)}{(1+\varepsilon^2)^3}.$$

Goals are to consider persistence and stability of the limiting solutions in ε for any fixed $q \in [0, \pi]$.

Symmetries and Spaces

If $\{u_{2n-1}(\tau), w_{2n}(\tau)\}_{n\in\mathbb{Z}}$ is a solution, then

- $\{u_{2n-1}(\tau+c), w_{2n}(\tau+c)\}_{n\in\mathbb{Z}}$ is a solution for any $c \in \mathbb{R}$ because of the translational invariance
- $\{u_{2n-1}(\tau)+c\varepsilon, w_{2n}(\tau)+c\}_{n\in\mathbb{Z}}$ is a solution for any $c \in \mathbb{R}$ because of the symmetry w.r.t. the change of coordinates.

For persistence analysis based on the Implicit Function Theorem, we shall work in the following spaces for u and w:

$$H_u^2 = \left\{ u \in H^2_{\rm per}(0,2\pi) : \quad u(-\tau) = -u(\tau), \; \tau \in \mathbb{R} \right\},$$

and

$$H_w^2 = \left\{ w \in H^2_{\rm per}(0,2\pi) : \quad w(\tau) = -w(-\tau-2q) \right\},$$

Theorem 1

Fix $q \in [0, \pi]$. *There is a unique* C^1 *continuation of* 2π*-periodic travelling wave in* ε. *In other words, there is an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *there exist a positive constant* C *and a unique solution* $(u_*, w_*) \in H_u^2 \times H_w^2$ *of the system of differential advance-delay equations* (14) *such that*

$$\|u_* - \varphi\|_{H^2_{\mathrm{per}}} \leq C\varepsilon^2, \quad \|w_*\|_{H^2_{\mathrm{per}}} \leq C\varepsilon.$$

Theorem 1

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$$\|u_* - \varphi\|_{H^2_{\rm per}} \leq C\varepsilon^2, \quad \|w_*\|_{H^2_{\rm per}} \leq C\varepsilon.$$

Remark: By Theorem 1, the continuation of exact solutions is unique for small values of ε:

$$q = \frac{\pi}{2}: \quad u_*(\tau) = \varphi(\tau), \quad w_*(\tau) = 0$$

and

$$q = \pi: \quad u_*(\tau) = \frac{\varphi(\tau)}{(1+\varepsilon^2)^3}, \quad w_*(\tau) = \frac{-\varepsilon\varphi(\tau)}{(1+\varepsilon^2)^3}.$$

However, other solutions may coexist for large values of ε.

Formal expansion

Differential advance-delay equations:

$$\begin{cases} \ddot{u}_*(\tau) = V'(\varepsilon w_*(\tau) - u_*(\tau)) - V'(u_*(\tau) - \varepsilon w_*(\tau - 2q)), \\ \ddot{w}_*(\tau) = \varepsilon V'(u_*(\tau + 2q) - \varepsilon w_*(\tau)) - \varepsilon V'(\varepsilon w_*(\tau) - u_*(\tau)), \end{cases} \quad \tau \in \mathbb{R}.$$

If we expand solutions into the perturbation series

$$u_* = \varphi + \varepsilon^2 u_*^{(2)} + o(\varepsilon^2), \quad w_* = \varepsilon w_*^{(1)} + o(\varepsilon^2),$$

we can get nice equations for the first corrections

$$\ddot{w}_*^{(1)}(\tau) = V'(\varphi(\tau + 2q)) - V'(-\varphi(\tau))$$

and

$$\ddot{u}_*^{(2)}(\tau) + \alpha|\varphi(\tau)|^{\alpha-1} u_*^{(2)}(\tau) = V''(-\varphi(\tau)) w_*^{(1)}(\tau) + V''(\varphi(\tau)) w_*^{(1)}(\tau - 2q),$$

but will run into problem of continuation of the perturbation expansions.

Nevertheless, we can solve the linearized inhomogeneous equations

$$\left(\frac{d^2}{d\tau^2} + \alpha|\varphi|^{\alpha-1}\right) u_*^{(2)} = F_u^{(2)}, \quad \frac{d^2}{d\tau^2} w_*^{(1)} = F_w^{(1)}$$

if

$$F_u^{(2)} \in L_u^2 = \left\{u \in L^2_{\text{per}}(0, 2\pi) : \quad u(-\tau) = -u(\tau), \; \tau \in \mathbb{R}\right\},$$

and

$$F_w^{(1)} \in L_w^2 = \left\{w \in L^2_{\text{per}}(0, 2\pi) : \quad w(\tau) = -w(-\tau - 2q)\right\},$$

Under these conditions

$$F_u^{(2)} \perp \mathrm{Ker}(L_u) = \mathrm{span}(\dot{\varphi}), \quad F_w^{(1)} \perp \mathrm{Ker}(L_w) = \mathrm{span}(1).$$

Proof

To apply the Implicit Function Theorem, we rewrite the existence problem as the root-finding problem for the nonlinear operators:

$$\begin{cases} f_u(u,w,\varepsilon) := \frac{d^2u}{d\tau^2} - F_u(u,w,\varepsilon), \\ f_w(u,w,\varepsilon) := \frac{d^2w}{d\tau^2} - F_w(u,w,\varepsilon). \end{cases}$$

where

$$\begin{cases} F_u(u(\tau),w(\tau),\varepsilon) := V'(\varepsilon w(\tau) - u(\tau)) - V'(u(\tau) - \varepsilon w(\tau - 2q)), \\ F_w(u(\tau),w(\tau),\varepsilon) := \varepsilon V'(u(\tau + 2q) - \varepsilon w(\tau)) - \varepsilon V'(\varepsilon w(\tau) - u(\tau)), \end{cases}$$

- f_u and f_w are C^1 maps from $H_u^2 \times H_w^2 \times \mathbb{R}$ to $L_u^2 \times L_w^2$ since $V \in C^2$.

- At $(\varphi, 0, 0)$, $(f_u, f_w) = (0, 0)$.
- The Jacobian operator

$$\begin{bmatrix} D_u f_u & D_u f_w \\ D_w f_u & D_w f_w \end{bmatrix}_{(u,w,\varepsilon)=(\varphi,0,0)} = \begin{bmatrix} \frac{d^2}{d\tau^2} + \alpha|\varphi|^{\alpha-1} & 0 \\ 0 & \frac{d^2}{d\tau^2} \end{bmatrix}$$

  is invertible in the constrained spaces since the linear operators have zero-dimensional kernels in H_u^2 and H_w^2 respectively.

The result follows by the Implicit Function Theorem.

Linearization

To analyze stability of travelling waves, we linearize the dimer lattice equations around the travelling waves:

$$\left\{ \begin{array}{l} \ddot{u}_{2n-1} = V''(\varepsilon w_*(\tau+2qn) - u_*(\tau+2qn))(\varepsilon w_{2n} - u_{2n-1}) \\ \qquad - V''(u_*(\tau+2qn) - \varepsilon w_*(\tau+2qn-2q))(u_{2n-1} - \varepsilon w_{2n-2}), \\ \ddot{w}_{2n} = \varepsilon V''(u_*(\tau+2qn+2q) - \varepsilon w_*(\tau+2qn))(u_{2n+1} - \varepsilon w_{2n}) \\ \qquad - \varepsilon V''(\varepsilon w_*(\tau+2qn) - u_*(\tau+2qn))(\varepsilon w_{2n} - u_{2n-1}), \end{array} \right.$$

We use Floquet Theory for the chain of second-order ODEs:

$$\mathbf{u}(\tau+2\pi) = \mathcal{M}\mathbf{u}(\tau), \quad \tau \in \mathbb{R},$$

where $\mathbf{u} := [\cdots, w_{2n-2}, u_{2n-1}, w_{2n}, u_{2n+1}, \cdots]$ and $\mathcal{M}$ is the monodromy operator.

Eigenvalues of the monodromy operator, $\mathcal{M}$ are found via the substitution:

$$u_{2n-1}(\tau) = U_{2n-1}(\tau)e^{\lambda\tau}, \quad w_{2n}(\tau) = W_{2n}(\tau)e^{\lambda\tau}, \quad \tau \in \mathbb{R},$$

where (U_{2n-1}, W_{2n}) are 2π-periodic functions of τ.

Admissible λ are called the **characteristic exponents**. They define Floquet multipliers μ:

$$\mu = e^{2\pi\lambda}$$

For $\varepsilon = 0$, the only characteristic exponent is $\lambda = 0$. It splits for $\varepsilon \neq 0$ and the **goal** here is to study the splitting of the zero eigenvalue.

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Challenges: The spectrum of linearization is continuous.
V'' is only continuous.

Theorem 2

Fix $q = \frac{\pi}{N}$ for some positive integer N. Let $(u_*, w_*) \in H^2_u \times H^2_w$ be defined by Theorem 1. For a sufficiently small ε, there exists $q_0 \in (0, \pi/2)$ such that the travelling periodic waves in the linear eigenvalue problem closed at the $2N$-periodic boundary conditions are:

$$0 < q < q_0, \quad \pi - q_0 < q < \pi \quad \Rightarrow \textit{stable}$$
$$q_0 < q < \pi - q_0 \quad \Rightarrow \textit{unstable}$$

- Special solution with $q = \pi$ is stable.
- Special solution with $q = \pi/2$ is unstable.

Formal expansions

We expand the eigenvalue

$$\lambda = \varepsilon\Lambda + o(\varepsilon)$$

and the eigenvectors

$$\begin{cases} U_{2n-1} = c_{2n-1}\dot{\varphi}(\tau + 2qn) + \varepsilon U^{(1)}_{2n-1} + \varepsilon^2 U^{(2)}_{2n-1} + o(\varepsilon^2), \\ W_{2n} = a_{2n} + \varepsilon W^{(1)}_{2n} + \varepsilon^2 W^{(2)}_{2n} + o(\varepsilon^2), \end{cases}$$

where $\{c_{2n-1}, a_{2n}\}_{n\in\mathbb{Z}}$ and Λ are to be computed from the reduced eigenvalue problem:

$$\begin{cases} K\Lambda^2 c_{2n-1} = M_1(c_{2n+1} + c_{2n-3} - 2c_{2n-1}) + L_1\Lambda(a_{2n} - a_{2n-2}), \\ \Lambda^2 a_{2n} = M_2(a_{2n+2} + a_{2n-2} - 2a_{2n}) + L_2\Lambda(c_{2n+1} - c_{2n-1}), \end{cases}$$

where $K > 0$, $M_1(q)$, $M_2, L_1, L_2 < 0$ are numerical coefficients (computed from projections). Only M_1 depends on q.

Analysis of the reduced eigenvalue problem

Using a band-limited Fourier transform,

$$c_{2n-1} = \int_0^{\pi} C(\theta) e^{i\theta(2n-1)} d\theta, \quad a_{2n} = \int_0^{\pi} A(\theta) e^{i2\theta n} d\theta,$$

we transform the quadratic eigenvalue problem to the finite-dimensional form:

$$\begin{cases} K\Lambda^2 C = 2M_1(\cos(2\theta)-1)C + 2iL_1\Lambda\sin(\theta)A, \\ \Lambda^2 A = 2M_2(\cos(2\theta)-1)A + 2iL_2\Lambda\sin(\theta)C. \end{cases}$$

Eigenvalues are defined by roots of the characteristic polynomial:

$$D(\Lambda;\theta) = K\Lambda^4 + 4\Lambda^2(M_1 + KM_2 + L_1L_2)\sin^2(\theta) + 16M_1M_2\sin^4(\theta) = 0.$$

To classify the nonzero roots of $D(\Lambda;\theta)$, we define

$$\Gamma := M_1 + KM_2 + L_1L_2, \quad \Delta := 4KM_1M_2.$$

Roots of the bi-quadratic equation

The characteristic polynomial

$$D(\Lambda;\theta) = K^2\Lambda^4 + 4\Lambda^2 K\Gamma \sin^2(\theta) + 4\Delta \sin^4(\theta) = 0$$

has two pairs of roots, which are determined in the following table:

Coefficients	Roots	q Values
$\Delta < 0$	$\Lambda_1^2 < 0 < \Lambda_2^2$	$q_0 < q < \pi - q_0$
$0 < \Delta \leq \Gamma^2$, $\Gamma > 0$	$\Lambda_1^2 \leq \Lambda_2^2 < 0$	$0 < q < q_0$
$0 < \Delta \leq \Gamma^2$, $\Gamma < 0$	$\Lambda_1^2 \geq \Lambda_2^2 > 0$	
$\Delta > \Gamma^2$	$\mathrm{Re}(\Lambda_1^2) > 0,\ \mathrm{Re}(\Lambda_2^2) < 0$	

where $q_0 \approx 0.915$

Krein signature of eigenvalues

- Because of $2N$-periodic boundary conditions, the admissible values of θ are discrete and finite:

$$\theta = \frac{\pi k}{N} \equiv \theta_k(N), \quad k = 0, 1, \dots, N-1.$$

  We count $4N$ eigenvalues $\lambda = \varepsilon\Lambda + \mathrm{o}(\varepsilon)$ but some are double because $\sin(\theta) = \sin(\pi - \theta)$.

- The semi-simple eigenvalues $\lambda \in i\mathbb{R}$ have nonzero Krein signature:

$$\begin{aligned} \sigma &= i \sum_{n \in \mathbb{Z}} \left[u_{2n-1} \dot{\bar{u}}_{2n-1} - \bar{u}_{2n-1} \dot{u}_{2n-1} + w_{2n} \dot{\bar{w}}_{2n} - \bar{w}_{2n} \dot{w}_{2n} \right] \\ &= \varepsilon \sigma^{(1)} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

  Semi-simple eigenvalues $\lambda \in i\mathbb{R}$ are structurally stable w.r.t. ε.

Renormalization technique

Challenges: if V'' is only continuous, the $O(\varepsilon^2)$ computations involving computations of V''' need to be justified.

A renormalization is performed by using the derivative expansion,

$$\begin{aligned} \ddot{u}_*(\tau) &= V''(\varepsilon w_*(\tau) - u_*(\tau))(\varepsilon \dot{w}_*(\tau) - \dot{u}_*(\tau)) \\ &\quad - V''(u_*(\tau) - \varepsilon w_*(\tau - 2q))(\dot{u}_*(\tau) - \varepsilon \dot{w}_*(\tau - 2q)). \end{aligned}$$

Using now

$$U_{2n-1} = c_{2n-1}\dot{u}_*(\tau + 2qn) + \mathcal{U}_{2n-1}, \quad W_{2n} = \mathcal{W}_{2n},$$

we obtain the linear eigenvalue problem, for which $O(\varepsilon^2)$ terms of the perturbation expansions are computed without computing V'''.

Numerical Results

We close the infinite chain of beads into a chain of $2N$ (i.e. $q = \frac{\pi}{N}$) beads with periodic boundary conditions:

$$\begin{cases} \ddot{u}_{2n-1}(t) = (\varepsilon w_{2n}(t) - u_{2n-1}(t))_+^{\alpha} - (u_{2n-1}(t) - \varepsilon w_{2n-2}(t))_+^{\alpha}, \\ \ddot{w}_{2n}(t) = \varepsilon(u_{2n-1}(t) - \varepsilon w_{2n}(t))_+^{\alpha} - \varepsilon(\varepsilon w_{2n}(t) - u_{2n+1}(t))_+^{\alpha}, \end{cases}$$

where $1 \leq n \leq N$ and the periodic boundary conditions are used:

$$u_{-1} = u_{2N-1}, \quad u_{2N+1} = u_1, \quad w_0 = w_{2N}, \quad w_{2N+2} = w_2.$$

- We use the shooting method with N shooting parameters to approximate the travelling wave solutions.
- Then, we compute Floquet multipliers from the monodromy matrix of the linearized system.

$N = 1$

For $q = \pi$ ($N = 1$), the results are trivial:

$$\begin{cases} \ddot{u}_1(t) = (\varepsilon w_2(t) - u_1(t))_+^\alpha - (u_1(t) - \varepsilon w_2(t))_+^\alpha, \\ \ddot{w}_2(t) = \varepsilon(u_1(t) - \varepsilon w_2(t))_+^\alpha - \varepsilon(\varepsilon w_2(t) - u_1(t))_+^\alpha, \end{cases}$$

The exact solution is:

$$q = \pi : \quad u_*(\tau) = \frac{\varphi(\tau)}{(1+\varepsilon^2)^3}, \quad w_*(\tau) = \frac{-\varepsilon\varphi(\tau)}{(1+\varepsilon^2)^3}.$$

The branch of solutions is unique for all $\varepsilon \in [0,1]$. At $\varepsilon = 1$, it matches the periodic wave in monomers studied by G. James (2012):

$$q = \pi, \varepsilon = 1 : \quad u_*(\tau) = \frac{1}{8}\varphi(\tau), \quad w_*(\tau) = -\frac{1}{8}\varphi(\tau).$$

The branch of solution is stable for all $\varepsilon \in [0,1]$.

Existence for $N = 2$

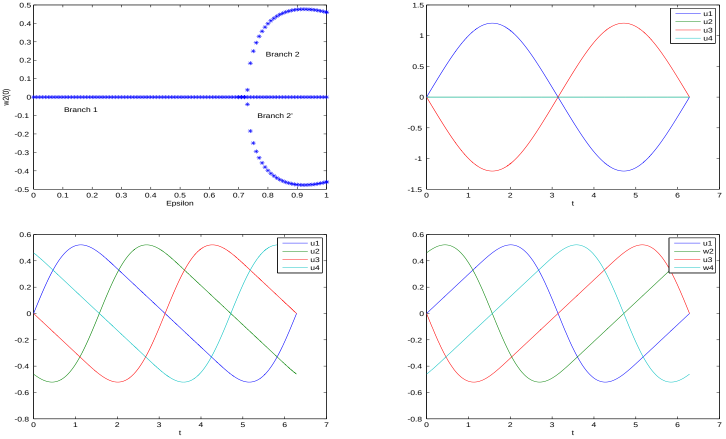

Figure : Travelling wave solutions for $q = \frac{\pi}{2}$ ($N = 2$): branch 1 (top right), branch 2 (bottom left), and branch 2' (bottom right) at $\varepsilon = 1$.

Stability for $N = 2$

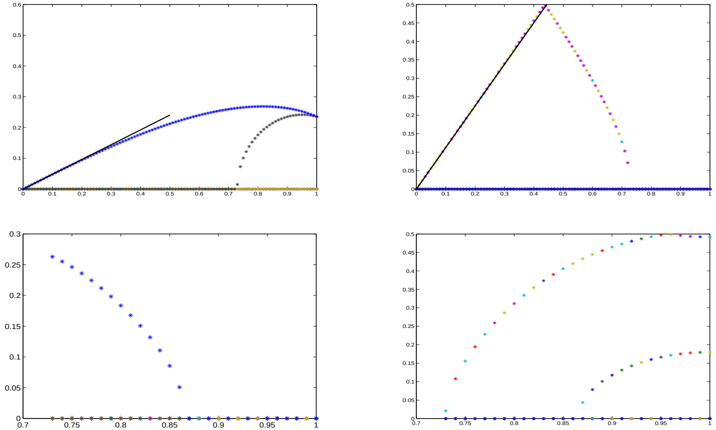

Figure : Real (left) and imaginary (right) parts of the characteristic exponents λ versus ε for $q = \frac{\pi}{2}$ for branch 1 (top) and branch 2 (bottom).

Existence for $N = 3$

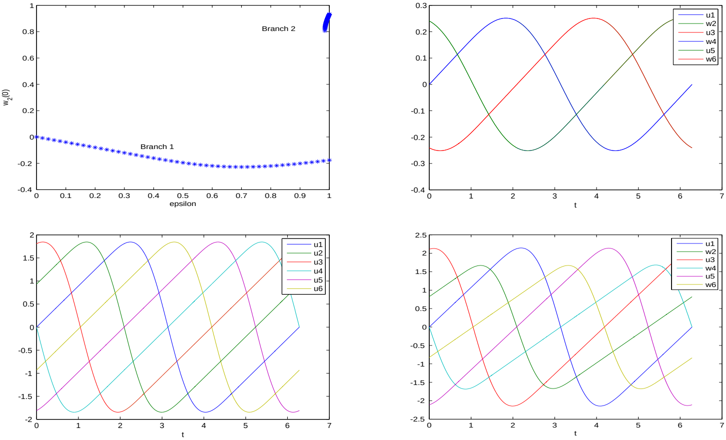

Figure : Travelling wave solutions for $q = \frac{\pi}{3}$: the solution of branch 1 is continued from $\varepsilon = 0$ to $\varepsilon = 1$ (top right) and the solution of branch 2 is continued from $\varepsilon = 1$ (bottom left) to $\varepsilon = 0.985$ (bottom right).

Stability for $N = 3$

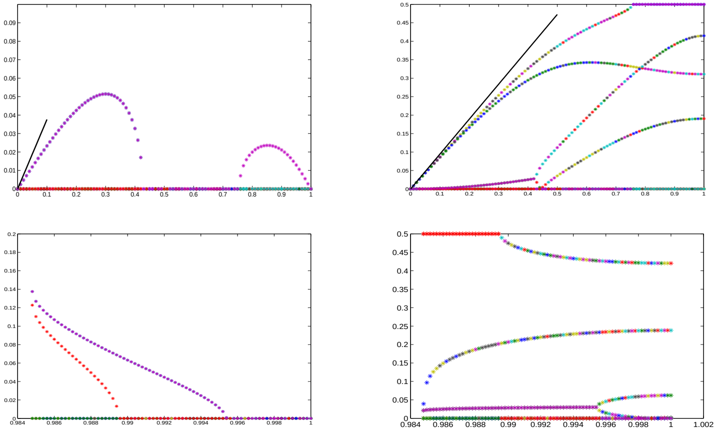

Figure : Real (left) and imaginary (right) parts of the characteristic exponents λ versus ε for $q = \frac{\pi}{3}$ for branch 1 (top) and branch 2 (bottom).

Stability for $N \geq 4$

Recall that branch 1 is stable for $0 < q < q_0 \approx 0.915$, that is, for $N \geq 4$.

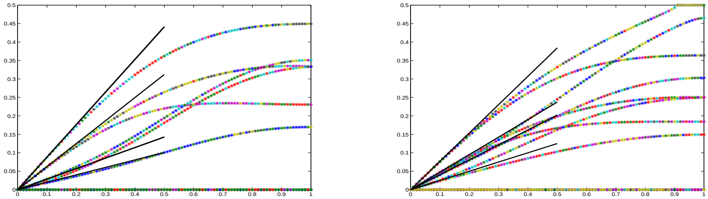

Figure : Imaginary parts of the characteristic exponents λ versus ε for $q = \frac{\pi}{4}$ (left) and $q = \frac{\pi}{5}$ (right). The real part of all the exponents is zero.

Conclusions

- We proved that the limiting periodic waves are uniquely continued from the anti-continuum limit for small mass ratio parameters.
- We proved that periodic waves with wavelengths larger than a certain critical value are spectrally stable for small mass ratios.
- We used numerical techniques to show that for larger wavelengths the stability of these periodic travelling waves with $N \geq 4$ persists all the way to the limit of equal mass ratio.
- We showed numerically that another branch of solutions bifurcates from the limit of equal mass ratio but it is unstable for $N \geq 4$.

Open Problems

- We would like to generalize Theorem 2 for continuous values of q in $[0, \pi]$.

  bifurcations of continuous spectrum?

- The nature of the bifurcations where Branch 2 terminates at $\varepsilon_* \in (0, 1)$ needs to be clarified for $N \geq 3$. We have been unsuccessful in our attempts to find another solution branch nearby for $\varepsilon \gtrapprox \varepsilon_*$.

  discontinuity-induced bifurcation?

- We would like to understand the hidden symmetry which explains why coalescent eigenvalues remain stable for branch 1 for all $\varepsilon \in [0, 1]$.

  different invariant subspaces?