First and Second Order Semi-strong Interaction in Reaction-Diffusion Systems

IMA, Minneapolis, June 2013
Jens Rademacher
Quasi-stationary sharp interfaces

Prototype: Allen-Cahn model for phase separation

\[ V_t = \varepsilon^2 V_{xx} + V(1 - V^2), \]

\[ x \in \mathbb{R}, \quad 0 < \varepsilon \ll 1. \]

Interface/front: on small scale \( y = x/\varepsilon \) as \( \varepsilon \to 0. \)
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Weak interface interaction: through exponentially small tails – motion is exponentially slow in \( \varepsilon^{-1} \). Carr & Pego, Fusca & Hale.

More general: Ei; Sandstede; Promislow; Zelik & Mielke.
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Quasi-stationary ‘semi-sharp’ interfaces

Weak coupling to linear equation (FitzHugh-Nagumo type system):

\[ \partial_t U = \partial_{xx} U - U + V \]

\[ \partial_t V = \varepsilon^2 \partial_{xx} V + V(1 - V^2) + \varepsilon U. \]
Quasi-stationary ‘semi-sharp’ interfaces

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Nonlocal coupling: \( U \)-component globally couples \( V \)-interfaces.
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Nonlocal coupling: $U$-component globally couples $V$-interfaces.

Interface: problem on both slow/large $x$-scale and fast/small $y$-scale.

Multiple steady patterns: replacing $\varepsilon U$ by $\varepsilon g(U)$ arbitrary singularities can be imbedded in existence problem [manuscript].
Quasi-stationary ‘semi-sharp’ interfaces

Weak coupling to linear equation (FitzHugh-Nagumo type system):

\[ \frac{\partial t}{\partial t} U = \partial_{xx} U - U + V \]
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Stability: Evans function in singular limit (‘NLEP’) [Doelman, Gardner, Kaper]; for this system: van Heijster’s results. (For other singular perturbation regime: SLEP method [Nishiura, Ikeda & Fuji, 80-90’s])
Semi-strong interaction

Interface motion: Now of order $\varepsilon^2$.

$\varepsilon = 0.01, T = 2000$

$\varepsilon = 0.005, T = 8000$
Semi-strong interaction

Interface motion: Now of order $\varepsilon^2$.

Semi-strong interaction laws: Leading order form

$$\frac{d}{dt} r_j = -\varepsilon^2 \langle u_{0,j}, \partial_y v_0 \rangle / \| \partial_y v_0 \|^2, \quad u_{0,j} = a_j(r_1, \ldots, r_N)$$
Semi-strong interaction

Interface motion: Now of order $\varepsilon^2$.

![Graph showing interface motion](image1)

![Graph showing interface motion](image2)

Semi-strong interaction laws: Leading order form

$$\frac{d}{dt} r_j = -\varepsilon^2 \langle u_{0,j}, \partial_y v_0 \rangle / \|\partial_y v_0\|_2^2, \quad u_{0,j} = a_j(r_1, \ldots, r_N)$$

Rigorously [Doelman, van Heijster, Kaper, Promislow]
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Rigorously [Doelman, van Heijster, Kaper, Promislow]

Strong interaction: numerics as in scalar case, monotone & coarsening
The large and small scale problem

\[ \partial_t U = \partial_{xx} U - U + V \]

\[ \partial_t V = \varepsilon^2 \partial_{xx} V + V(1 - V^2) + \varepsilon U. \]
The large and small scale problem

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**Large scale:** Assume stationary to leading order in \( \varepsilon \)

\[ 0 = \partial_{xx} U_0 - U_0 + V_0 \]

\[ 0 = V_0(1 - V_0^2). \]
The large and small scale problem

\[ \partial_t U = \partial_{xx} U - U + V \]
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\[ 0 = \partial_{xx} U_0 - U_0 + V_0 \]
\[ 0 = V_0(1 - V_0^2). \]

**Small scale:** \( y = x/\varepsilon \)

\[ \varepsilon^2 \partial_t u = \partial_{yy} u - \varepsilon^2 (u + v) \]
\[ \partial_t v = \partial_{yy} v + v(1 - v^2) + \varepsilon u. \]
The large and small scale problem

\[ \partial_t U = \partial_{xx} U - U + V \]
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**Large scale:** Assume stationary to leading order in \( \varepsilon \)

\[ 0 = \partial_{xx} U_0 - U_0 + V_0 \]
\[ 0 = V_0(1 - V_0^2). \]

**Small scale:** \( y = x/\varepsilon \), assume stationary to leading order

\[ 0 = \partial_{yy} u_0 \]
\[ 0 = \partial_{yy} v_0 + v_0(1 - v_0^2). \]
A 3-component FHN-type system

\[ \tau \partial_t U = \partial_{xx} U - U + V \]

\[ \theta \partial_t W = \partial_{xx} W - W + V \]

\[ \partial_t V = \varepsilon^2 \partial_{xx} V + V(1 - V^2) + \varepsilon(\gamma + \alpha U + \beta W). \]

Front patterns studied in semi-strong regime by van Heijster (with Doelman, Kaper, Promislow; also in 2D with Sandstede).

Already single front behaves different from Allen-Cahn: ‘butterfly catastrophe’ and Hopf bifurcation.

[Chirilius-Bruckner, Doelman, van Heijster, R.; manuscript]
Is this typical for localised solutions?

Consider \((u, v) \in \mathbb{R}^{N+M}\) and systems of the form

\[
\begin{align*}
\partial_t u &= D_u \partial_{xx} u + F(u, v; \varepsilon) \\
\partial_t v &= \varepsilon^2 D_v \partial_{xx} v + G(u, v; \varepsilon)
\end{align*}
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\end{align*}

**Fronts:** localisation to jump in $v$ as $\varepsilon \to 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{figure}
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\end{align*}
\]

**Fronts:** localisation to jump in \(v\) as \(\varepsilon \to 0\).

**Pulse:** localisation to Dirac mass in \(v\) as \(\varepsilon \to 0\).
‘Semi-sharp’ pulses / spikes

A major motivation for semi-strong regime: Pulse motion and pulse-splitting in Gray-Scott model. Numerics and asymptotic matching by Reynolds, Pearson & Ponce-Dawson in early 90’s. Continued by Osipov, Doelman, Kaper, Ward, Wei, ...

Weak interaction: ‘edge splitting’

Semi-strong interaction: ‘$2^n$-splitting’
Example: simplified Schnakenberg model

\[ \partial_t U = \partial_{xx} U + \alpha - V \]
\[ \partial_t V = \varepsilon^2 \partial_{xx} V - V + UV^2. \]

Leading order existence, stability, interaction?
Two regimes within semi-strong regime

\[ \partial_t u = \partial_{xx} u + \alpha - v \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]

For Dirac-mass on \( x \)-scale set: \( u = \hat{u}, v = \varepsilon^{-1} \hat{v} \rightarrow \)

\[ \partial_t \hat{u} = \partial_{xx} \hat{u} + \hat{\alpha} - \varepsilon^{-1} \hat{v} \]
\[ \partial_t \hat{v} = \varepsilon^2 \partial_{xx} \hat{v} - \hat{v} + \varepsilon^{-1} \hat{u} \hat{v}^2. \]

We will see that here motion is order \( \varepsilon \).
Two regimes within semi-strong regime

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\end{align*}
\]

We will see that here motion is order \(\varepsilon\).

Embedded motion of order \(\varepsilon^2\) analogous to front for \(\alpha = \sqrt{\varepsilon \hat{\alpha}}: \)

\(u = \sqrt{\varepsilon \hat{u}}, \ v = \sqrt{\varepsilon^{-1} \hat{v}} \rightarrow\)

\[
\begin{align*}
\partial_t \hat{u} &= \partial_{xx} \hat{u} + \hat{\alpha} - \varepsilon^{-1} \hat{v} \\
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\end{align*}
\]
Generally: Two regimes within semi-strong regime

\[ \partial_t u = \partial_{xx} u + \alpha - u - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]
Generally: Two regimes within semi-strong regime

\[ \partial_t u = \partial_{xx} u + \alpha - u - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]

Case \( \alpha = \hat{\alpha} = O(1) \): \( u = \hat{u}, \ v = \hat{v}/\varepsilon \rightarrow \text{‘1st order standard form’} \)

\[ \partial_t \hat{u} = \partial_{xx} \hat{u} + \hat{\alpha} - \varepsilon^{-1} (\hat{u} + \varepsilon^{-1} \hat{u} \hat{v}^2) \]
\[ \partial_t \hat{v} = \varepsilon^2 \partial_{xx} \hat{v} - \hat{v} + \varepsilon^{-1} \hat{u} \hat{v}^2. \]
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$$\partial_t \hat{v} = \varepsilon^2 \partial_{xx} \hat{v} - \hat{v} + \varepsilon^{-1} \hat{u} \hat{v}^2.$$ 

Case $\alpha = \sqrt{\varepsilon \hat{\alpha}}$: $u = \sqrt{\varepsilon} \check{u}, v = \check{v}/\sqrt{\varepsilon} \rightarrow$ ‘2nd order standard form’

$$\partial_t \check{u} = \partial_{xx} \check{u} + \check{\alpha} - \varepsilon^{-1} (\check{u} + \check{u} \check{v}^2)$$

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Can distinguish interaction type in general systems via ‘standard forms’

[R. SIADS ’13]
1st order semi-strong interaction

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha - uv^2 \\
\partial_t v &= \varepsilon^2 \partial_{xx} v - v + uv^2.
\end{align*}
\]

\[\varepsilon = 0.01, \quad T = 200\]

\[\alpha = 2.95\]
Asymptotics for 1st order interaction

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]

Expand \( u = u_0 + \varepsilon u_1 + O(\varepsilon^2) \), \( v = v_0 + O(\varepsilon) \)
Asymptotics for 1st order interaction

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
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Expand \( u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2) \), \( v = v_0 + \mathcal{O}(\varepsilon) \)

Large scale: \( V_0 = 0, \quad 0 = \partial_{xx} \hat{U}_0 + \hat{\alpha} \to \text{parabola} \)
Asymptotics for 1st order interaction

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Large scale: \( V_0 = 0 \), \( 0 = \partial_{xx} \hat{U}_0 + \hat{\alpha} \rightarrow \text{parabola} \)

Small scale: \( \hat{u}_0 = 0 \), \( \partial_{yy} \hat{u}_1 = \hat{u}_1 \hat{v}_0^2 \)

(‘core problem’) \( \partial_{yy} \hat{v}_0 = \hat{v}_0 - \hat{u}_1 \hat{v}_0^2 \).
Asymptotics for 1st order interaction

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
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(‘core problem’) \( \partial_{yy} \hat{v}_0 = \hat{v}_0 - \hat{u}_1 \hat{v}_0^2 \)

Matching:

\[ \hat{U}_0(x_j) = 0 (!) \]
\[ \partial_y \hat{u}_1(\pm\infty) = \partial_x \hat{U}_0(x_j \pm 0) \]
\[ \hat{v}_0(\pm\infty) = 0. \]
Asymptotics for 1st order interaction

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Expand \( u = u_0 + \varepsilon u_1 + O(\varepsilon^2) \), \( v = v_0 + O(\varepsilon) \)

Large scale: \( V_0 = 0 \), \( 0 = \partial_{xx} \hat{U}_0 + \hat{\alpha} \rightarrow \) parabola

Small scale: \( \hat{u}_0 = 0 \), \( \partial_{yy} \hat{u}_1 = \hat{u}_1 \hat{v}^2_0 \)

(‘core problem’) \( \partial_{yy} \hat{v}_0 = \hat{v}_0 - \hat{u}_1 \hat{v}^2_0 \).

Matching:

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\begin{align*}
\hat{U}_0(x_j) &= 0 (\!\!\!) \\
\partial_y \hat{u}_1(\pm \infty) &= \partial_x \hat{U}_0(x_j \pm 0) \\
\hat{v}_0(\pm \infty) &= 0.
\end{align*}
\]

\( \Rightarrow \) one parameter missing \( \rightarrow \) allow for \( dx_j/dt = \varepsilon c + O(\varepsilon^2) \).
Asymptotics for 1st order interaction

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]

Expand \( u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2), \ v = v_0 + \mathcal{O}(\varepsilon) \)

Large scale: \( V_0 = 0, \ 0 = \partial_{xx} \hat{U}_0 + \hat{\alpha} \rightarrow \) parabola

Small scale: \( \hat{u}_0 = 0, \ \partial_{yy} \hat{u}_1 = \hat{u}_1 \hat{v}_0^2 \)

('core problem') \( \partial_{yy} \hat{v}_0 = \hat{v}_0 - \hat{u}_1 \hat{v}_0^2 + c \partial_y \hat{v}_0. \)

Matching:

\[ \hat{U}_0(x_j) = 0 \ (\!) \]
\[ \partial_y \hat{u}_1(\pm \infty) = \partial_x \hat{U}_0(x_j \pm 0) \]
\[ \hat{v}(\pm \infty) = 0. \]

\[ \Rightarrow \] one parameter missing \( \rightarrow \) allow for \( \frac{dx_j}{dt} = \varepsilon c + \mathcal{O}(\varepsilon^2). \)
Large and small scale problems

\[ \hat{U}_0(x_j) = 0 \]
\[ \partial_y \hat{u}_1(\pm\infty) = \partial_x \hat{U}_0(x_j \pm 0) \]
Large and small scale problems

\[ \hat{U}_0(x_j) = 0 \]
\[ \partial_y \hat{u}_1(\pm \infty) = \partial_x \hat{U}_0(x_j \pm 0) \]

Existence problem local:

nearest neighbor coupling

\( \Rightarrow \) use single small scale problem,
parameters \( c, p_{\pm} = \partial_x \hat{U}_0(x_j \pm 0) \).

\[ \partial_{yy} \hat{u}_1 = \hat{u}_1 \hat{v}_0^2 \]
\[ \partial_{yy} \hat{v}_0 = c \partial_y \hat{v}_0 + \hat{v}_0 - \hat{u}_1 \hat{v}_0^2. \]

Motion law not projection
onto translation mode!
Small-scale pulse manifold

Numerically compute by continuation in $p_s = p_+ - p_-$, $p_a = p_+ + p_-$:

$\begin{align*}
\text{norm} & \quad \text{unimodal} \\
\text{bimodal} & \quad c > 0 \\
\text{unimodal} & \quad c < 0 \\
\end{align*}$

$\begin{align*}
\text{folds of bimodal} & \quad c < 0 \\
\text{folds of unimodal} & \quad c > 0 \\
\end{align*}$

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\text{norm} & \quad \text{unimodal} \\
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\text{unimodal} & \quad c < 0 \\
\end{align*}$
Small-scale pulse manifold

Numerically compute by continuation in $p_s = p_+ - p_-$, $p_a = p_+ + p_-$:

Reduced 1-pulse motion towards symmetric configuration for $p_s < p_s^*$.
Pulse-splitting near $p_a = 0$ for $p_s > p_s^*$. 
Small-scale pulse manifold

Numerically compute by continuation in $p_s = p_+ - p_-$, $p_a = p_+ + p_-:

Reduced 1-pulse motion towards symmetric configuration for $p_s < p_s^*$. Pulse-splitting near $p_a = 0$ for $p_s > p_s^*$. Monotonicity of $c$ in $p_a \Rightarrow$ Abstract theorem applies: e.g. largest pulse distance is Lyapunov functional (until splitting). [R. SIADS ‘13]
Existence & stability map for pulse patterns

Solve boundary value problem formulation for eigenfunctions again by numerical continuation:

\[ \text{Hopf stable region} \]

\[ \text{fold} \]

\[ c < 0 \]

\[ c = 0 \]
Crossing the boundary: Pulse-replication

Numerics by J. Ehrt (WIAS/HU)
1st and 2nd order semi-strong interaction

with M. Wolfrum & J. Ehrt (WIAS/HU)

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha - uv^2 \\
\partial_t v &= \varepsilon^2 \partial_{xx} v - v + uv^2.
\end{align*}
\]

1st order semi-strong:
velocity \( c = \mathcal{O}(\varepsilon) \), \( \alpha = 0.9 \).

2nd order semi-strong:
velocity \( c = \mathcal{O}(\varepsilon^2) \), \( \alpha = 1.3\sqrt{\varepsilon} \).

Small ‘production’: slow motion and coarsening,
Large ‘production’: fast motion and splitting
Stability boundary in 2nd order case

PDE numerics when crossing boundary: annihilation (‘overcrowding’)

Numerics delicate: delayed Hopf-bifurcation...
Crossing unstable region

\[ \alpha = 1.2 \]
Crossing unstable region

Only for relatively large $\varepsilon$: bifurcation (appears to be) subcritical.
A class of examples

$$\partial_t u = \partial_{xx} u + \alpha - \mu u + \gamma v - uv^2$$

$$\partial_t v = \varepsilon^2 \partial_{xx} v + \beta - v + uv^2,$$
A class of examples

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha - \mu u + \gamma v - uv^2 \]
\[ \frac{\partial v}{\partial t} = \varepsilon^2 \frac{\partial^2 v}{\partial x^2} + \beta - v + uv^2, \]

Schnakenberg model: \[ \mu = \gamma = 0, \]
Gray-Scott model: \[ \alpha = \mu, \gamma = \beta = 0, \]
Brusselator model: \[ \alpha = \mu = 0. \]
A class of examples

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha - \mu u + \gamma v - uv^2 \\
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\]

Schnakenberg model: \( \mu = \gamma = 0 \),

Gray-Scott model: \( \alpha = \mu, \gamma = \beta = 0 \),

Brusselator model: \( \alpha = \mu = 0 \).

Scalings:

1st order semi-strong: \( v = \varepsilon^{-1} \hat{v} \)

2nd order semi-strong: \( \alpha = \varepsilon^{1/2} \hat{\alpha}, u = \varepsilon^{1/2} \hat{u}, v = \varepsilon^{-1/2} \hat{v} \)
Second order case ($\alpha = \varepsilon \tilde{\alpha}$):
Existence and stability from Doelman-Kaper ‘normal form’ approach.
For fronts in FHN-type system: van Heijster.
Interaction laws rigorously for FHN-type and Gierer-Meinhardt model variants [Doelman, Kaper, Promislow; van Heijster; Bellsky].

First order case ($\alpha = \mathcal{O}(1)$):
Numerically: ‘core problem’ existence up to critical value (fold) – proof?
Rich solution set [Doelman, Kaper, Peletier ’06].
Interaction laws: asymptotics for Schnakenberg model [Ward et al].
Proofs?
Model independent view

Semi-strong interaction can occur for $0 < \varepsilon \ll 1$ in systems of the form

$$\partial_t u = D_u \partial_{xx} u + F(u, v; \varepsilon)$$

$$\partial_t v = \varepsilon^2 D_v \partial_{xx} v + G(u, v; \varepsilon)v + \varepsilon E(u, v; \varepsilon)$$

**Pulse:** localisation to Dirac mass in $v$ as $\varepsilon \to 0$.

**Fronts:** localisation to jump in $v$ as $\varepsilon \to 0$. 

[Graphs showing pulse and front solutions]
Model independent view

Semi-strong interaction can occur for $0 < \varepsilon \ll 1$ in systems of the form

$$\begin{align*}
\partial_t u & = D_u \partial_{xx} u + F(u, v; \varepsilon) \\
\partial_t v & = \varepsilon^2 D_v \partial_{xx} v + G(u, v; \varepsilon)v + \varepsilon E(u, v; \varepsilon)
\end{align*}$$

**Pulse:** localisation to Dirac mass in $v$ as $\varepsilon \to 0$.

**Fronts:** localisation to jump in $v$ as $\varepsilon \to 0$.

Expand and apply natural constraints to obtain boundedness as $\varepsilon \to 0$:

$$\begin{align*}
\partial_t u & = D_u \partial_{xx} u + H(u, v; \varepsilon) + \varepsilon^{-1}(F^s(u, v) + \varepsilon^{-1}F^f(u, v)u)v, \\
\partial_t v & = \varepsilon^2 D_v \partial_{xx} v + \varepsilon E(u, v; \varepsilon) + G^s(u, v)v + \varepsilon^{-1}G^f(u, v)uv.
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Second order semi-strong interaction: $F^f \equiv G^f \equiv 0.$
Summary

- Semi-strong interaction comes in different types.
- Unified framework for fronts, pulses and 1st, 2nd order interaction.
- Can read off the laws of motion (formally).
- In 1st order case: conditions for gradient-like pulse interaction.

[R. SIADS ’13]
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[S. SIADS ’13]

Sidenote: In semi-strong regime also rich single interface bifurcations & pencil and paper analysis possible also for nonlocalized solutions...
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Thank you!