Existence and uniqueness results for the MFG system

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Based on Lasry-Lions (2006) and Lions’ lectures at Collège de France

IMA Special Short Course
“Mean Field Games”,
November 12-13, 2012
The Mean Field Game system

The general MFG system takes the form (with unknown \((u, m) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^2) :

\[
(MFG) \begin{cases}
(i) & -\partial_t u - \sigma^2 \Delta u + H(x, Du, m) = 0 \\
(ii) & \partial_t m - \sigma^2 \Delta m - \text{div}(m \, D_p H(x, Du, m)) = 0 \\
(iii) & m(0) = m_0, \ u(x, T) = G(x, m(T))
\end{cases}
\]

where

- \(\sigma \in \mathbb{R}\),
- \(H = H(x, p, m)\) is a convex Hamiltonian (in \(p\)) depending on the density \(m\),
- \(G = G(x, m(T))\) is a function depending on the position \(x\) and the density \(m(T)\) at time \(T\).
- \(m_0\) is a probability density on \(\mathbb{R}^d\).
System introduced in

- Lasry-Lions ’06, as a finite dimensional reduction of a “master equation”,
- Huang-Caines-Malhame, ’06

to model large population differential games.
Outline

1. First order MFG equations with nonlocal coupling
   - Semi-concavity properties of HJ equations
   - The Kolmogorov equation
   - Existence/uniqueness of a solution for the MFG system

2. First order MFG systems with local coupling
   - The quasilinear elliptic equation
   - MFG as optimality conditions
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First order MFG equations with nonlocal coupling

1. Semi-concavity properties of HJ equations
2. The Kolmogorov equation
3. Existence/uniqueness of a solution for the MFG system

2. First order MFG systems with local coupling
1. The quasilinear elliptic equation
2. MFG as optimality conditions
Discuss existence and uniqueness for the first order, non local system

\[
(MFG) \begin{cases}
(i) & -\partial_t u + \frac{1}{2} |Du|^2 = F(x, m(t, \cdot)) \\
(ii) & \partial_t m - \text{div} (mDu) = 0 \\
(iii) & m(0) = m_0, \ u(x, T) = u_f(x)
\end{cases}
\]

where

- the first equation is backward in time
- the second one is forward in time
- the map $F$ is nonlocal and regularizing.
Approach by fixed point

We fix a family \( (m(t))_{t \in [0,T]} \).
Let \( u \) be the solution of the (backward) HJ equation

\[
(HJ) \quad \begin{cases}
(i) & -\partial_t u + \frac{1}{2} |Du|^2 = F(x, m(t, \cdot)) \\
(ii) & u(x, T) = u_f(x)
\end{cases}
\text{ in } \mathbb{R}^d \times (0, T)
\]

and \( \tilde{m} \) be the solution of the (forward) Kolmogorov equation

\[
(K) \quad \begin{cases}
(i) & \partial_t \tilde{m} - \text{div} (\tilde{m} Du) = 0 \\
(ii) & \tilde{m}(0) = m_0
\end{cases}
\text{ in } \mathbb{R}^d \times (0, T)
\]

Problem

Prove that the map \( m \rightarrow \tilde{m} \) has a fixed point.
Difficulties

- (HJ) has a Lipschitz continuous solution, but not $C^1$ solutions in general (shocks).
- (K) is ill-posed for vector fields $Du$ which are only $L^\infty$ (multiple solutions).

A system arising in geometric optics: The (forward-forward) system

\[
\begin{align*}
(i) & \quad \partial_t u + \frac{1}{2} |Du|^2 = F(x, m(t, \cdot)) \\
& \quad \text{in } \mathbb{R}^d \times (0, T) \\
(ii) & \quad \partial_t m - \text{div}(m Du) = 0 \\
& \quad \text{in } \mathbb{R}^d \times (0, T) \\
(iii) & \quad m(0) = m_0, \quad u(0, x) = u_0(x)
\end{align*}
\]

was studied by Gosse and James (2002), Ben Moussa and Kossioris (2003) and Strömberg (2007).

$\rightarrow$ Well-posed of measured-valued solutions.
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A system arising in geometric optics: The (forward-forward) system

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\end{align*}
\]

was studied by Gosse and James (2002), Ben Moussa and Kossioris (2003)) and Strömberg (2007).

$\longrightarrow$ Well-posed of measured-valued solutions.
Existence and stability properties of the (MFG) system rely on:

- Semi-concavity estimate for HJ equations
- Well-posedness of the Kolmogorov equation for gradient vector fields of semi-concave functions

Further simplifying assumption:

\[ m_0 \text{ is bounded, with bounded support.} \]
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Semi-concave functions

Definition

A map \( w : \mathbb{R}^d \to \mathbb{R} \) is semi-concave if there is some \( C > 0 \) such that

\[
w(\lambda x + (1 - \lambda)y) \geq \lambda w(x) + (1 - \lambda)w(y) - C\lambda(1 - \lambda)|x - y|^2
\]

for any \( x, y \in \mathbb{R}^d \), \( \lambda \in [0, 1] \).
**Proposition**

$w : \mathbb{R}^d \rightarrow \mathbb{R}$ is semi-concave iff one of the following conditions is satisfied:

1. The map $x \mapsto w(x) - \frac{C}{2} |x|^2$ is concave in $\mathbb{R}^d$,

2. $D^2 w \leq C I_d$ in the sense of distributions,

3. $\langle p - q, x - y \rangle \leq C|x - y|^2$ for any $x, y \in \mathbb{R}^d$, $t \in [0, T]$, $p \in D_x^+ w(x)$ and $q \in D_y^+ w(y)$, where $D_x^+ w$ denotes the super-differential of $w$ with respect to the $x$ variable, namely

$$D_x^+ w(x) = \left\{ p \in \mathbb{R}^d ; \limsup_{y \to x} \frac{w(y) - w(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}$$

In particular, a semi-concave function is locally Lipschitz continuous in $\mathbb{R}^d$. 
Lemma [Stability]

Let \((w_n)\) be a sequence of uniformly semi-concave maps on \(\mathbb{R}^d\) which point-wisely converge to a map \(w : \mathbb{R}^d \to \mathbb{R}\).

- Then the convergence is locally uniform and \(w\) is semi-concave.
- Moreover, \(Dw_n(x)\) converge to \(Dw(x)\) for a.e. \(x \in \mathbb{R}^d\).
First order MFG equations with nonlocal coupling

Semi-concavity properties of HJ equations

Semi-concavity and HJ equations

Semi-concavity is the natural framework for HJ equations of the form

\[
\begin{align*}
-\partial_t u + \frac{1}{2} |Du|^2 &= f(t, x) & \text{in } [0, T] \times \mathbb{R}^d \\
u(T, \cdot) &= g(x)
\end{align*}
\]

for data such that \( \sup_{t \in [0, T]} \| f(\cdot, t) \|_{C^2} \leq \tilde{C}, \quad \| g \|_{C^2} \leq \tilde{C} \).

**Proposition**

The unique viscosity solution \( u \) of \((HJ)\) is semi-concave in space with modulus \( C = C(\tilde{C}) \) and given by the representation formula:

\[
u(x, t) = \inf_{\alpha \in L^2([t, T], \mathbb{R}^d)} \int_t^T \frac{1}{2} |\alpha(s)|^2 + f(s, x(s))ds + g(x(T)),
\]

where \( x(s) = x + \int_t^s \alpha(\tau)d\tau \).
Lemma (Optimal synthesis)

Let \((x, t) \in \mathbb{R}^d \times [0, T)\) and \(x(\cdot)\) be an a.c. solution of

\[
\begin{aligned}
    (\text{grad} - \text{eq}) \quad & \quad \begin{cases} 
        x'(s) = -D_x u(s, x(s)) & \text{a.e. in } [t, T] \\
        x(t) = x
    \end{cases}
\end{aligned}
\]

Then the control \(\alpha := x'\) is optimal for the problem

\[
\inf_{\alpha \in L^2([t, T], \mathbb{R}^d)} \int_t^T \frac{1}{2} |\alpha(s)|^2 + f(s, x(s)) ds + g(x(T))
\]

If \(u(\cdot, t)\) is differentiable at \(x\), then equation \((\text{grad} - \text{eq})\) has a unique solution, corresponding to the optimal trajectory.

In particular, for a.e. \(x \in \mathbb{R}^d\), equation \((\text{grad} - \text{eq})\) has a unique solution.
We denote by $\Phi = \Phi(x, t, s)$ the flow of $(\operatorname{grad} - \operatorname{eq})$:

$$(\operatorname{grad} - \operatorname{eq}) \quad \left\{ \begin{array}{l}
\frac{\partial}{\partial s} \Phi(x, t, s) = -D_xu(s, \Phi(x, t, s)) \quad \text{a.e. in } [t, T] \\
\Phi(x, t, t) = x
\end{array} \right.$$ 

**Remark**: $\Phi(x, t, s)$ is defined for a.e. $x \in \mathbb{R}^d$.

**Lemma**

*The flow $\Phi$ has the semi-group property*

$$\Phi(x, t, s') = \Phi(\Phi(x, t), s, s') \quad \forall t \leq s \leq s' \leq T$$
Lemma (contraction property of the flow)

There is a constant $C = C(\bar{C})$ such that

$$|x - y| \leq C|\Phi(x, t, s) - \Phi(y, t, s)| \quad \forall 0 \leq t < s \leq T, \forall x, y \in \mathbb{R}^d.$$ 

In particular the map $x \to \Phi(x, s, t)$ has a Lipschitz continuous inverse on the set $\Phi(\mathbb{R}^d, t, s)$. 
**Proof:** Let \( x(\tau) = \Phi(x, t, s - \tau) \) and \( y(\tau) = \Phi(y, t, s - \tau) \). Then \( x(\cdot) \) and \( y(\cdot) \) solve

\[
x'(\tau) = D_x u(x(\tau), s - \tau), \quad y'(\tau) = D_x u(y(\tau), s - \tau)
\]

on \([0, s - t]\) with initial condition \( x(0) = \Phi(x, t, s) \) and \( y(0) = \Phi(y, t, s) \). Hence

\[
\frac{d}{d\tau} \left( \frac{1}{2} |x - y(\tau)|^2 \right) = \langle (x' - y')(\tau), (x - y)(\tau) \rangle \leq C |(x - y)(\tau)|^2
\]

since \( u(\tau, \cdot) \) is \( C \)-semiconcave. Therefore

\[
|x(0) - y(0)| \geq e^{-C(s-\tau)} |(x - y)(\tau)| \quad \forall \tau \in [0, s - t]
\]

which proves the claim. \( \square \)
Proof: Let \( x(\tau) = \Phi(x, t, s - \tau) \) and \( y(\tau) = \Phi(y, t, s - \tau) \). Then \( x(\cdot) \) and \( y(\cdot) \) solve

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\[
    \frac{d}{d\tau} \left( \frac{1}{2} |(x - y)(\tau)|^2 \right) = \langle (x' - y')(\tau), (x - y)(\tau) \rangle \leq C |(x - y)(\tau)|^2
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Fix $u = u(t, x)$ the solution of (HJ). Recall that $D^2 u \leq C$ and $Du \in L^\infty$ where $C = C(\bar{C})$.

We study the Kolmogorov equation

$$(K) \quad \begin{cases} \partial_t \mu - \text{div} (\mu Du) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ \mu(x, 0) = m_0(x) & \text{in } \mathbb{R}^d \end{cases}$$

**Goal** : show that $(K)$ has a unique solution given by the flow $\Phi$. 
Definition

Let $\mu \in L^1_{loc}([0, T] \times \mathbb{R}^d)$. Then $\mu$ is a weak solution to

\[
\begin{cases}
    \partial_t \mu - \text{div}(\mu Du) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\
    \mu(x, 0) = m_0(x) & \text{in } \mathbb{R}^d
\end{cases}
\]

if, for any $\varphi \in C_\infty^\infty([0, T] \times \mathbb{R}^d)$,

\[
\int_0^T \int_{\mathbb{R}^d} \left( -\partial_t \varphi + \langle Du, D\varphi \rangle \right) \mu \, dxdt = \int_{\mathbb{R}^d} \varphi(0, x)m_0(x)dx
\]
Pull-back of a measure: Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ and $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ be a Borel map. The measure $\nu := \Psi_\# \mu$ is defined by one of the equivalent statements:

- $\int_{\mathbb{R}^d} f(x) \, d\nu(x) = \int_{\mathbb{R}^d} f(\Psi(x)) \, d\mu(x)$ for any $f \in C^0_b(\mathbb{R}^d)$.
- $\nu(E) = \mu(\Psi^{-1}(E))$ for any Borel set $E \subset \mathbb{R}^d$.

The Monge-Kantorovitch distance: Let $\mathcal{P}_1(\mathbb{R}^d)$ be the set of Borel probability measures $m$ on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |x| \, dm(x) < +\infty$. The Monge-Kantorovitch distance between measures $m, m' \in \mathcal{P}_1(\mathbb{R}^d)$ is given by

$$d_1(m, m') = \sup \left\{ \int_Q h \, d(m - m') \right\}.$$ 

where the supremum is taken over the maps $h : \mathbb{R}^d \to \mathbb{R}$ which are $1-$Lipschitz continuous.
Pull-back of a measure: Let \( \mu \) be a Borel probability measure on \( \mathbb{R}^d \) and \( \Psi : \mathbb{R}^d \to \mathbb{R}^d \) be a Borel map. The measure \( \nu := \Psi_#\mu \) is defined by on of the equivalent statements:

- \( \int_{\mathbb{R}^d} f(x) d\nu(x) = \int_{\mathbb{R}^d} f(\Psi(x)) d\mu(x) \) for any \( f \in C_0^b(\mathbb{R}^d) \).
- \( \nu(E) = \mu(\Psi^{-1}(E)) \) for any Borel set \( E \subset \mathbb{R}^d \).

The Monge-Kantorovitch distance: Let \( \mathcal{P}_1(\mathbb{R}^d) \) be the set of Borel probability measures \( m \) on \( \mathbb{R}^d \) such that \( \int_{\mathbb{R}^d} |x| dm(x) < +\infty \).

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\[
d_1(m, m') = \sup \left\{ \int_Q h \, d(m - m') \right\}.
\]

where the supremum is taken over the maps \( h : \mathbb{R}^d \to \mathbb{R} \) which are \( 1 \)–Lipschitz continuous.
Pull-back of a measure: Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ and $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be a Borel map. The measure $\nu := \psi_#\mu$ is defined by one of the equivalent statements:

1. $\int_{\mathbb{R}^d} f(x) d\nu(x) = \int_{\mathbb{R}^d} f(\psi(x)) d\mu(x)$ for any $f \in C_0^b(\mathbb{R}^d)$.

2. $\nu(E) = \mu(\psi^{-1}(E))$ for any Borel set $E \subset \mathbb{R}^d$.

The Monge-Kantorovitch distance: Let $P_1(\mathbb{R}^d)$ be the set of Borel probability measures $m$ on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |x| dm(x) < +\infty$. The Monge-Kantorovitch distance between measures $m, m' \in P_1(\mathbb{R}^d)$ is given by

$$d_1(m, m') = \sup \left\{ \int_Q h \ d(m - m') \right\}.$$

where the supremum is taken over the maps $h : \mathbb{R}^d \to \mathbb{R}$ which are $1$–Lipschitz continuous.
Lemma

Any weak solution $\mu$ of

\[
(K) \quad \begin{cases}
\partial_t \mu - \text{div} (\mu Du) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\
\mu(x, 0) = m_0(x) & \text{in } \mathbb{R}^d
\end{cases}
\]

such that $\mu(t) \in \mathcal{P}_1(\mathbb{R}^d)$ for any $t$, satisfies

\[
d_1(\mu(t_1), \mu(t_2)) \leq \|Du\|_{\infty} |t_2 - t_1|
\]
**Proof:** Fix $0 < t_1 < t_2 \leq T$ and let $h \in C^\infty_c(\mathbb{R}^d)$ be $1-$Lipschitz continuous. Let $\varepsilon > 0$ and

$$
\varphi_{\varepsilon}(t, x) = \begin{cases} 
(t - t_1)h(x)/\varepsilon & \text{if } t \in [t_1, t_1 + \varepsilon] \\
h(x) & \text{if } t \in [t_1 + \varepsilon, t_2 - \varepsilon] \\
(t_2 - t)h(x)/\varepsilon & \text{if } t \in [t_2 - \varepsilon, t_2] \\
0 & \text{otherwise}
\end{cases}
$$

As

$$
\int_0^T \int_{\mathbb{R}^d} (-\partial_t \varphi_{\varepsilon} + \langle Du, D\varphi_{\varepsilon} \rangle) \mu = \int_{\mathbb{R}^d} \varphi_{\varepsilon}(0, x) m_0(x) dx
$$

we have

$$
\int_{t_1}^{t_1 + \varepsilon} \int_{\mathbb{R}^d} h\mu \frac{1}{\varepsilon} + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \langle Du, Dh \rangle \mu + \int_{t_2 - \varepsilon}^{t_2} \int_{\mathbb{R}^d} -h\mu \frac{1}{\varepsilon} \approx 0
$$
Letting $\varepsilon \to 0$ gives for a.e. $0 < t_1 < t_2 < T$:

$$
\int_{\mathbb{R}^d} h(\mu(t_1) - \mu(t_2)) + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \langle Du, Dh \rangle \mu = 0.
$$

So

$$
\int_{\mathbb{R}^d} h(\mu(t_1) - \mu(t_2)) \leq \|Du\|_\infty \|Dh\|_\infty |t_2 - t_1|.
$$

Take the sup over $h$:

$$
d_1(\mu(t_1), \mu(t_2)) \leq \|Du\|_\infty |t_2 - t_1|.
$$
Letting $\varepsilon \to 0$ gives for a.e. $0 < t_1 < t_2 < T$:

$$\int_{\mathbb{R}^d} h(\mu(t_1) - \mu(t_2)) + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \langle Du, Dh \rangle \mu = 0 .$$

So

$$\int_{\mathbb{R}^d} h(\mu(t_1) - \mu(t_2)) \leq \| Du \|_\infty \| Dh \|_\infty |t_2 - t_1| .$$

Take the sup over $h$:

$$d_1(\mu(t_1), \mu(t_2)) \leq \| Du \|_\infty |t_2 - t_1| .$$
Recall that $\Phi = \Phi(t, x, s)$ is the flow of $(\text{grad} - \text{eq})$:

$$
(\text{grad} - \text{eq}) \quad \begin{cases} 
\frac{\partial}{\partial s} \Phi(t, x, s) = -D_x u(s, \Phi(t, x, s)) & \text{a.e. in } [t, T] \\
\Phi(t, x, t) = x 
\end{cases}
$$

Assume that $m_0$ has compact support contained in $B(0, \bar{C})$.

**Lemma**

There is $C = C(\bar{C})$ such that, for any $s \in [0, T]$, the measure $\mu(s) := \Phi(\cdot, 0, s)\#m_0$ satisfies

- has a support contained in the ball $B(0, C)$,
- is absolutely continuous,
- and its density $\mu(s, \cdot)$ satisfies $\|\mu(s, \cdot)\|_{L^\infty} \leq C$. 
Proof

Since \( \|D_x u\|_{\infty} \leq C \) and \( Spt(m_0) \subset B(0, \bar{C}) \), the support of \( (\mu(s)) \) is contained in \( B(0, R) \) where \( R = \bar{C} + TC \).

- Fix \( s \in [0, T] \). From the contraction Lemma, the map \( x \rightarrow \Phi(x, 0, s) \) has a \( C \)-Lipschitz continuous inverse \( \Psi(s, \cdot) \) on the set \( \Phi(\mathbb{R}^d, 0, s) \).

- If \( E \) is a Borel subset of \( \mathbb{R}^d \),

\[
\mu(s, E) = m_0(\Phi^{-1}(\cdot, 0, s)(E)) = m_0(\Psi(s, E)) \\
\leq \|m_0\|_{\infty} \mathcal{L}^d(\Psi(s, E)) \leq \|m_0\|_{\infty} C \mathcal{L}^d(E).
\]

Therefore \( \mu(s) \) is a.c. with a density \( \mu(t, \cdot) \) satisfying

\[
\|\mu(t, \cdot)\|_{\infty} \leq \|m_0\|_{\infty} C \quad \forall t \in [0, T].
\]
Proof  

Since $\|D_x u\|_{\infty} \leq C$ and $Spt(m_0) \subset B(0, \bar{C})$, the support of $(\mu(s))$ is contained in $B(0, R)$ where $R = \bar{C} + TC$.

Fix $s \in [0, T]$. From the contraction Lemma, the map $x \rightarrow \Phi(x, 0, s)$ has a $C$–Lipschitz continuous inverse $\psi(s, \cdot)$ on the set $\Phi(\mathbb{R}^d, 0, s)$.

If $E$ is a Borel subset of $\mathbb{R}^d$,

$$
\mu(s, E) = m_0(\Phi^{-1}(\cdot, 0, s)(E)) = m_0(\psi(s, E)) \\
\leq \|m_0\|_{\infty} \mathcal{L}^d(\psi(s, E)) \leq \|m_0\|_{\infty} C \mathcal{L}^d(E).
$$

Therefore $\mu(s)$ is a.c. with a density $\mu(t, \cdot)$ satisfying

$$
\|\mu(t, \cdot)\|_{\infty} \leq \|m_0\|_{\infty} C \quad \forall t \in [0, T].
$$
Proof

- Since $\|D_x u\|_\infty \leq C$ and $\text{Spt}(m_0) \subset B(0, \bar{C})$, the support of $(\mu(s))$ is contained in $B(0, R)$ where $R = \bar{C} + TC$.
- Fix $s \in [0, T]$. From the contraction Lemma, the map $x \mapsto \Phi(x, 0, s)$ has a $C-$Lipschitz continuous inverse $\Psi(s, \cdot)$ on the set $\Phi(\mathbb{R}^d, 0, s)$.
- If $E$ is a Borel subset of $\mathbb{R}^d$,

$$
\mu(s, E) = m_0(\Phi^{-1}(\cdot, 0, s)(E)) = m_0(\Psi(s, E)) \leq \|m_0\|_\infty \mathcal{L}^d(\Psi(s, E)) \leq \|m_0\|_\infty C \mathcal{L}^d(E).
$$

Therefore $\mu(s)$ is a.c. with a density $\mu(t, \cdot)$ satisfying

$$
\|\mu(t, \cdot)\|_\infty \leq \|m_0\|_\infty C \quad \forall t \in [0, T].
$$
Proof

Since $\|D_x u\|_\infty \leq C$ and $\text{Spt}(m_0) \subset B(0, \bar{C})$, the support of $(\mu(s))$ is contained in $B(0, R)$ where $R = \bar{C} + TC$.

Fix $s \in [0, T]$. From the contraction Lemma, the map $x \mapsto \Phi(x, 0, s)$ has a $C$–Lipschitz continuous inverse $\psi(s, \cdot)$ on the set $\Phi(\mathbb{R}^d, 0, s)$.

If $E$ is a Borel subset of $\mathbb{R}^d$,

$$
\mu(s, E) = m_0(\Phi^{-1}(\cdot, 0, s)(E)) = m_0(\psi(s, E)) \\
\leq \|m_0\|_\infty \mathcal{L}^d(\psi(s, E)) \leq \|m_0\|_\infty C \mathcal{L}^d(E). 
$$

Therefore $\mu(s)$ is a.c. with a density $\mu(t, \cdot)$ satisfying

$$
\|\mu(t, \cdot)\|_\infty \leq \|m_0\|_\infty C \quad \forall t \in [0, T]. 
$$
The map \( s \mapsto \mu(s) := \Phi(\cdot, 0, s)\#m_0 \) is a weak solution of

\[
(K) \quad \begin{cases} 
\partial_t \mu - \text{div} (\mu Du) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\
\mu(0, x) = m_0(x) & \text{in } \mathbb{R}^d 
\end{cases}
\]
Proof: Let $\varphi \in C^\infty_c(\mathbb{R}^N \times [0, T))$. Then

$$
\frac{d}{ds} \int_{\mathbb{R}^d} \varphi(s, x) \mu(s, x)
= \frac{d}{ds} \int_{\mathbb{R}^d} \varphi(s, \Phi(x, 0, s)) m_0(x) dx
= \int_{\mathbb{R}^d} \left( \partial_s \varphi(s, \Phi(x, 0, s)) + \langle D_x \varphi(s, \Phi(x, 0, s)), \partial_s \Phi(x, 0, s) \rangle \right) m_0(x) dx
= \int_{\mathbb{R}^d} \left( \partial_s \varphi(s, \Phi(x, 0, s)) - \langle D_x \varphi(s, \Phi(x, 0, s)), D_x u(s, \Phi(x, 0, s)) \rangle \right) m_0(x) dx
= \int_{\mathbb{R}^d} \left( \partial_s \varphi(s, y) - \langle D_x \varphi(s, y), D_x u(s, y) \rangle \right) \mu(s, y)
$$

We conclude by integrating between 0 and $T$. \qed
Theorem

Given a solution $u$ to $(HJ)$, the map $s \rightarrow \mu(s) := \Phi(\cdot, 0, s)^\# m_0$ is the unique weak solution of $(K)$.

Theorem

Given a solution $u$ to (HJ), the map $s \rightarrow \mu(s) := \Phi(\cdot,0,s)^\#m_0$ is the unique weak solution of (K).

Outline

1. First order MFG equations with nonlocal coupling
   - Semi-concavity properties of HJ equations
   - The Kolmogorov equation
   - Existence/uniqueness of a solution for the MFG system

2. First order MFG systems with local coupling
   - The quasilinear elliptic equation
   - MFG as optimality conditions
We go back to the MFG system:

\[
\begin{align*}
(i) \quad & -\partial_t u + \frac{1}{2} |Du|^2 = F(x, m(t, \cdot)) \\
(ii) \quad & \partial_t m - \text{div} (mDu) = 0 \\
(iii) \quad & m(0) = m_0, \ u(0, x) = G(x, m(T))
\end{align*}
\]
Standing assumptions:

1. $F$ and $G$ are continuous over $\mathbb{R}^d \times \mathcal{P}_1$.

2. There is a constant $C$ such that, for any $m \in \mathcal{P}_1$,

$$
\|F(\cdot, m)\|_{C^2} \leq C, \quad \|G(\cdot, m)\|_{C^2} \leq C \quad \forall m \in \mathcal{P}_1,
$$

where $C^2$ is the space of function with continuous second order derivatives endowed with the norm

$$
\|f\|_{C^2} = \sup_{x \in \mathbb{R}^d} \left[ |f(x)| + |Df(x)| + |D^2f(x)| \right].
$$

3. Finally we suppose that $m_0$ is absolutely continuous, with a density still denoted $m_0$ which is bounded and has a compact support.
Definition

A pair \((u, m)\) is a solution to the (MFG) system

\[
\begin{aligned}
(i) & \quad \frac{1}{2} \left| \frac{\partial u}{\partial t} + \frac{1}{2} |Du|^2 \right| = F(x, m(t, \cdot)) \\
(ii) & \quad \frac{\partial t}{m} - \text{div}(mDu) = 0 \\
(iii) & \quad m(0) = m_0, \quad u(0, x) = G(x, m(T))
\end{aligned}
\]

if \(u \in W_{loc}^{1,\infty}([0, T] \times \mathbb{R}^d)\) and \(m \in L_{loc}^1([0, T] \times \mathbb{R}^d)\),

- \(u\) is a (backward) viscosity solution to \((MFG - (i))\) with terminal condition \(u(0, x) = G(x, m(T))\).
- \(m\) is a weak solution to \((MFG - (ii))\) with initial condition \(m(0) = m_0\).
A stability property

Let \((m_n)\) be a sequence of \(C([0, T], \mathcal{P}_1)\) which uniformly converges to \(m \in C([0, T], \mathcal{P}_1(\mathbb{R}^d))\).

Let \(u_n\) be the solution to

\[
\begin{cases}
-\partial_t u_n + \frac{1}{2} |Du_n|^2 = F(x, m_n(t)) & \text{in } \mathbb{R}^d \times (0, T) \\
u_n(x, T) = g(x, m_n(T)) & \text{in } \mathbb{R}^d
\end{cases}
\]

and \(u\) be the solution to

\[
\begin{cases}
-\partial_t u + \frac{1}{2} |Du|^2 = F(x, m(t)) & \text{in } \mathbb{R}^d \times (0, T) \\
u(x, T) = g(x, m(T)) & \text{in } \mathbb{R}^d
\end{cases}
\]

Let \(\Phi_n\) (resp. \(\Phi\)) the flow associated to \(u_n\) (resp. to \(u\)) as above and let us set \(\mu_n(s) = \Phi_n(\cdot, 0, s)\#m_0\) and \(\mu(s) = \Phi(\cdot, 0, s)\#m_0\).
Lemma (Stability)

The solution \((u_n)\) locally uniformly converges \(u\) in \(\mathbb{R}^d \times [0, T]\) while \((\mu_n)\) converges to \(\mu\) in \(C([0, T], \mathcal{P}_1(\mathbb{R}^d))\).
Proof:

- As the $F(x, m_n(t))$ and the $g(x, m_n(T))$ locally uniformly converge to $F(x, m(t))$ and $g(x, m(T))$, local uniform convergence of $(u_n)$ to $u$ is a consequence of the standard stability of viscosity solutions.

- Since the $u_n$ are uniformly semi-concave, $Du_n$ converges almost everywhere in $\mathbb{R}^d \times (0, T)$ to $Du$.

- The $(\mu_n(t))$ have a uniformly bounded support (say in some compact $K \subset \mathbb{R}^d$) and are uniformly Lipschitz continuous:

$$d_1(\mu_n(t_1), \mu_n(t_2)) \leq \|Du_n\|_\infty |t_2 - t_1| \leq C|t_2 - t_1| .$$

and uniformly bounded in $L^\infty$.

- By Ascoli and Banach-Alaoglu, a subsequence (still denoted $(\mu_n)$) of the $(\mu_n)$ converges in $C([0, T], \mathcal{P}_1(K))$ and in $L^\infty$–weak-* to some $m$ which has a support in $K \times [0, T]$, belongs to $L^\infty(\mathbb{R}^d \times [0, T])$ and to $C([0, T], \mathcal{P}_1(K))$.

- One can pass to the limit in the Kolmogorov equation to get that $\mu$ is the unique solution associated to $u : \mu(s) = \Phi(\cdot, 0, s)^\#m_0$. 

□
Proof:

- As the $F(x, m_n(t))$ and the $g(x, m_n(T))$ locally uniformly converge to $F(x, m(t))$ and $g(x, m(T))$, local uniform convergence of $(u_n)$ to $u$ is a consequence of the standard stability of viscosity solutions.

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- One can pass to the limit in the Kolmogorov equation to get that $\mu$ is the unique solution associated to $u : \mu(s) = \Phi(\cdot, 0, s)\#m_0$.  \qed
First order MFG equations with nonlocal coupling

Existence/uniqueness of a solution for the MFG system

Theorem (Lasry and Lions, 2006)

There is at least one solution to the MFG system

\[
\begin{align*}
(i) & \quad -\partial_t u + \frac{1}{2} |Du|^2 = F(x, m(t, \cdot)) \quad \text{in } \mathbb{R}^d \times (0, T) \\
(ii) & \quad \partial_t m - \text{div} (mDu) = 0 \quad \text{in } \mathbb{R}^d \times (0, T) \\
(iii) & \quad m(0) = m_0, \ u(T, x) = G(x, m(T))
\end{align*}
\]
Proof: Let $C$ be the convex subset of maps $m \in C([0, T], \mathcal{P}_1)$ such that $m(0) = m_0$. To any $m \in C$ one associates the unique solution $u$ to

$$
\begin{cases}
-\partial_t u + \frac{1}{2} |Du|^2 = F(x, m(t)) & \text{in } \mathbb{R}^d \times (0, T) \\
u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^d
\end{cases}
$$

and to this solution one associates the unique solution to the continuity equation

$$
\begin{cases}
\partial_t \tilde{m} - \text{div} (\tilde{m} Du) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\
\tilde{m}(0) = m_0
\end{cases}
$$

Then $\tilde{m} \in C$ and, from the stability Lemma, the mapping $m \rightarrow \tilde{m}$ is continuous. It is also compact because there is a constant $C = C(\overline{C})$ such that, for any $s \in [0, T]$, $\tilde{m}(s)$ has a support in $B(0, C)$ and satisfies

$$
d_1(\tilde{m}(t_1), \tilde{m}(t_2)) \leq C |t_1 - t_2| \quad \forall t_1, t_2 \in [0, T].
$$

One completes the proof by Schauder fix point Theorem. □
Comments

- Semi-concavity properties of value function in optimal control are borrowed from the monograph by Cannarsa and Sinestrari 2004.

- For the analysis of transport equations with discontinuous vector fields, see, in particular,
  - Di Perna-Lions, 1989, for vector fields in Sobolev spaces,
Uniqueness

Assume furthermore that $F$ and $G$ satisfy the monotonicity condition:

$$\int_{\mathbb{R}^d} \left( F(x, m_1) - F(x, m_2) \right) d(m_1 - m_2)(x) \geq 0 \quad \forall m_1, m_2 \in \mathcal{P}_1 ,$$

and

$$\int_{\mathbb{R}^d} \left( G(x, m_1) - G(x, m_2) \right) d(m_1 - m_2)(x) \geq 0 \quad \forall m_1, m_2 \in \mathcal{P}_1 .$$

Theorem

Under the above conditions, the (MFG) system has a unique solution.
**Proof:** Let \((u_1, m_1)\) and \((u_2, m_2)\) two solutions of \((MFG)\). Set \(\tilde{u} = u_1 - u_2\) and \(\tilde{m} = m_1 - m_2\). Then \(\tilde{u}\) solves

\[
(1) \quad -\partial_t \tilde{u} + \frac{1}{2} (|Du_1|^2 - |Du_2|^2) - (F(x, m_1) - F(x, m_2)) = 0
\]

while \(\tilde{m}\) solves

\[
(2) \quad \partial_t \tilde{m} - \text{div} (m_1 Du_1 - m_2 Du_2) = 0.
\]

We compute

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \tilde{u}(t) \tilde{m}(t) = \int_{\mathbb{R}^d} (\partial_t \tilde{u}(t)) \tilde{m}(t) + \tilde{u}(t) (\partial_t m(t))
\]

\[
= \int_{\mathbb{R}^d} \left( \frac{1}{2} (|Du_1|^2 - |Du_2|^2) - (F(x, m_1) - F(x, m_2)) \right) \tilde{m}
\]

\[
- \int_{\mathbb{R}^d} \langle D\tilde{u}, (m_1 Du_1 - m_2 Du_2) \rangle
\]

Note that

\[
\frac{\tilde{m}}{2} (|Du_1|^2 - |Du_2|^2) - \langle D\tilde{u}, m_1 Du_1 - m_2 Du_2 \rangle = - \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2
\]
**Proof:** Let \((u_1, m_1)\) and \((u_2, m_2)\) two solutions of \((MFG)\). Set \(\tilde{u} = u_1 - u_2\) and \(\tilde{m} = m_1 - m_2\). Then \(\tilde{u}\) solves

\[
(1)\quad - \partial_t \tilde{u} + \frac{1}{2} (|Du_1|^2 - |Du_2|^2) - (F(x, m_1) - F(x, m_2)) = 0
\]

while \(\tilde{m}\) solves

\[
(2)\quad \partial_t \tilde{m} - \text{div} (m_1 Du_1 - m_2 Du_2) = 0.
\]

We compute

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \tilde{u}(t) \tilde{m}(t) = \int_{\mathbb{R}^d} (\partial_t \tilde{u}(t)) \tilde{m}(t) + \tilde{u}(t) (\partial_t m(t))
\]

\[
= \int_{\mathbb{R}^d} \left( \frac{1}{2} (|Du_1|^2 - |Du_2|^2) - (F(x, m_1) - F(x, m_2)) \right) \tilde{m}
\]

\[
- \int_{\mathbb{R}^d} \langle D\tilde{u}, (m_1 Du_1 - m_2 Du_2) \rangle
\]

Note that

\[
\frac{\tilde{m}}{2} (|Du_1|^2 - |Du_2|^2) - \langle D\tilde{u}, m_1 Du_1 - m_2 Du_2 \rangle = - \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2
\]
As

\[
\int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2)) \tilde{m} \geq 0 ,
\]

we get

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \tilde{u}(t) \tilde{m}(t) \leq - \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2
\]

Integrate over \([0, T] :\)

\[
\int_{\mathbb{R}^d} \tilde{m}(T)\tilde{u}(T) - \int_{\mathbb{R}^d} \tilde{m}(0)\tilde{u}(0) \leq - \int_0^T \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2
\]

As

\[
\int_{\mathbb{R}^d} \tilde{m}(T)\tilde{u}(T) = \int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2))(m_1 - m_2) \geq 0
\]

while \(\tilde{m}(0) = 0\), we get

\[
0 \leq - \int_0^T \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2
\]
As
\[ \int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2)) \tilde{m} \geq 0, \]
we get
\[ \frac{d}{dt} \int_{\mathbb{R}^d} \tilde{u}(t) \tilde{m}(t) \leq - \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |D\tilde{u}_1 - D\tilde{u}_2|^2 \]
Integrate over \([0, T]\):
\[ \int_{\mathbb{R}^d} \tilde{m}(T) \tilde{u}(T) - \int_{\mathbb{R}^d} \tilde{m}(0) \tilde{u}(0) \leq - \int_0^T \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |D\tilde{u}_1 - D\tilde{u}_2|^2 \]
As
\[ \int_{\mathbb{R}^d} \tilde{m}(T) \tilde{u}(T) = \int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2))(m_1 - m_2) \geq 0 \]
while \(\tilde{m}(0) = 0\), we get
\[ 0 \leq - \int_0^T \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |D\tilde{u}_1 - D\tilde{u}_2|^2 \]
As

\[ \int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2)) \tilde{m} \geq 0, \]

we get

\[ \frac{d}{dt} \int_{\mathbb{R}^d} \tilde{u}(t) \tilde{m}(t) \leq - \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2 \]

Integrate over \([0, T]\) :

\[ \int_{\mathbb{R}^d} \tilde{m}(T) \tilde{u}(T) - \int_{\mathbb{R}^d} \tilde{m}(0) \tilde{u}(0) \leq - \int_0^T \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2 \]

As

\[ \int_{\mathbb{R}^d} \tilde{m}(T) \tilde{u}(T) = \int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2))(m_1 - m_2) \geq 0 \]

while \( \tilde{m}(0) = 0 \), we get

\[ 0 \leq - \int_0^T \int_{\mathbb{R}^d} \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2 \]
So

\[ Du_1 = Du_2 \text{ in } \{m_1 > 0\} \cup \{m_2 > 0\} \]

Hence \( m_1 \) is a solution of

\[
\begin{cases}
\partial_t \mu - \text{div} (\mu Du_2) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\
\mu(x, 0) = m_0(x) & \text{in } \mathbb{R}^d
\end{cases}
\]

By uniqueness, \( m_1 = m_2 \).

Then \( u_1 = u_2 \) because \( u_1 \) and \( u_2 \) solve the same (HJ) equation.
Conclusion for first order, nonlocal (MFG) systems

- The existence/uniqueness theory can be easily generalized to general Hamiltonians,
- Stability of solutions can be proved along the same lines,
- Other existence proof: by vanishing viscosity,
Outline

1. First order MFG equations with nonlocal coupling
   - Semi-concavity properties of HJ equations
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2. First order MFG systems with local coupling
   - The quasilinear elliptic equation
   - MFG as optimality conditions
The Mean Field Game system

We now study the MFG system

\[
\begin{align*}
& (MFG) \\
& (i) -\partial_t u + H(x, Du) = f(x, m(x, t)) \\
& (ii) \partial_t m - \text{div}(mD_pH(x, Du)) = 0 \\
& (iii) m(0) = m_0, \quad u(x, T) = u_T(x)
\end{align*}
\]

where \( f: \mathbb{R}^d \times [0, +\infty) \to [0, +\infty) \) is a local coupling term.

Problem: Existence/uniqueness of a solutions - smoothness properties.
Similar systems

- **Adjoint methods for Hamilton-Jacobi PDE**: analysis of the vanishing viscosity limit:

\[
\begin{align*}
\partial_t u^\varepsilon + H(Du^\varepsilon) &= \varepsilon \Delta u^\varepsilon, \\
-\partial_t m^\varepsilon - \text{div}\left(DH(Du^\varepsilon)m^\varepsilon\right) &= \varepsilon \Delta m^\varepsilon, \\
u^\varepsilon(0, x) &= u_0(x), \quad m^\varepsilon(1, \cdot) = m_1.
\end{align*}
\]

→ better understand the convergence of the vanishing viscosity method when $H$ is nonconvex.
(Evans (2010))
A congestion model: An optimal transport problem related to congestion yields to the following system of PDEs: for $\alpha \in (0, 1)$,

$$
\begin{align*}
\partial_t u + \frac{\alpha}{4} m^{\alpha-1} |Du|^2 &= 0, \\
\partial_t m + \text{div}(\frac{1}{2} m^{\alpha} Du) &= 0, \\
m(0,.) &= m_0, \quad m(1,.) = m_1.
\end{align*}
$$

(Benamou-Brenier formulation of Wasserstein distance: $\alpha = 1$ - Dolbeault-Nazaret-Savaré, 2009)

→ analysis of the system: Existence and uniqueness of a weak solution

(C.-Carlier-Nazaret 2012.)
First order MFG systems with local coupling

Back to the MFG system

\[
\begin{align*}
(MFG) \quad \left\{ \begin{array}{ll}
(i) & \quad -\partial_t u + H(x, Du) = f(x, m(x, t)) \\
(ii) & \quad \partial_t m - \text{div}(mD_p H(x, Du)) = 0 \\
(iii) & \quad m(0) = m_0, \quad u(x, T) = u_T(x)
\end{array} \right.
\end{align*}
\]

under the assumptions that

- \( f : \mathbb{R}^d \times [0, +\infty) \to [0, +\infty) \) is a local coupling term, strictly increasing in \( m \) with \( f(x, 0) = 0 \),
- \( H \) is uniformly convex in \( p \),
- all the data are \( \mathbb{Z}^d \)-periodic in space : denoted by \( \# \).
For nonlocal coupling, existence/regularity of the MFG system come from

- semi-concavity estimates for HJ equations with $C^2$ RHS,
- preservation of the absolute continuity of $m_0$ by Kolmogorov equation for gradient flows of semi-concave functions.

For local equations,

- the RHS is a priori only measurable (or integrable),
- the HJ with discontinuous RHS is poorly understood,
- the Kolmogorov equation with $Du \in L^\infty$ is ill-posed as well.

→ requires a completely different approach.
Difficulties for the MFG with local coupling

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  - semi-concavity estimates for HJ equations with $C^2$ RHS,
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→ requires a completely different approach.
Ideas for the MFG system

Look at the MFG system

- as a quasilinear elliptic equation for $u$ in time-space.

- or/and as necessary condition for optimal control problems. This is the case
  - For an optimal control problem for an Hamilton-Jacobi equation
  - For an optimal control problem for a Kolmogorov equation
  - The second one being the dual of the first one.
Outline

1. First order MFG equations with nonlocal coupling
   - Semi-concavity properties of HJ equations
   - The Kolmogorov equation
   - Existence/uniqueness of a solution for the MFG system

2. First order MFG systems with local coupling
   - The quasilinear elliptic equation
   - MFG as optimality conditions
For simplicity of presentation, we assume that $H = H(p)$, $f = f(m)$. The MFG system becomes

\[
\begin{align*}
(MFG) \quad \left\{ 
\begin{array}{l}
(i) & - \partial_t u + H(Du) = f(m(x, t)) \\
(ii) & \partial_t m - \text{div}(mH'(Du)) = 0 \\
(iii) & m(0) = m_0, \; u(x, T) = u_T(x)
\end{array}
\right.
\end{align*}
\]

So

\[m(x, t) = f^{-1} (- \partial_t u + H(Du))\]
Therefore

\[ 0 = \partial_t m - \text{div}(mH'(Du)) \]

\[ = \partial_t (f^{-1}(-\partial_t u + H(Du))) - \text{div} \left( (f^{-1}(-\partial_t u + H(Du))H'(Du) \right) \]

\[ = (f^{-1})'(-\partial_t u + H(Du)) - f^{-1}(-\partial_t u + H(Du)) \]

\[ - f^{-1}(-\partial_t u + H(Du)) \text{Tr} (H''(Du)D^2 u) \]

\[ = - \text{Tr} (A(D_t, x)D^2_{t,x} u) \]

where

\[ A(D_t, x) = (f^{-1})'(-\partial_t u + H(Du)) \begin{pmatrix} 1 & H'(Du) \\ H'(Du)^T & H'(Du) \otimes H'(Du) \end{pmatrix} \]

\[ + f^{-1}(-\partial_t u + H(Du)) \begin{pmatrix} 0 & 0 \\ 0 & H''(Du) \end{pmatrix} \geq 0 \]

(because \((f^{-1})' > 0\) and \(f^{-1}(-\partial_t u + H(Du)) = m \geq 0\))
The MFG systems reduces to the **quasilinear elliptic equation**

\[
\begin{cases}
- \text{Tr} \left( A(D_{t,x}u) D_{t,x}^2 u \right) = 0 \text{ in } (0,1) \times \mathbb{T}^d \\
- \partial_t u + H(Du) = f(m_0) \text{ at } t = 0 \\
u(T, \cdot) = u_T \text{ at } t = T
\end{cases}
\]
A priori estimates

1. The map $\Phi(t, x) := -\partial_t u + H(Du)$ is bounded above.

   Indeed:
   - $\Phi$ satisfies an equation of the form
     
     $$-\text{Tr}(A(D_t, x)\Phi) - B.\Delta \phi \leq 0 \text{ in } (0, 1) \times \mathbb{T}^d$$

     So $\max \Phi$ reached at the boundary,
   - $\Phi(0, \cdot) = f(m_0)$ is bounded,
   - $\Phi(T, x)$ is bounded (barrier argument).

2. $|Du|$ is bounded (Bernstein method)
First order MFG systems with local coupling

The quasilinear elliptic equation

Theorem (Lasry-Lions, 2009)

1. Existence/uniqueness of solution for the MFG system with \( u \in W^{1,\infty} \) and \( m \in L^\infty \).

2. If \( f(m) \sim \log(m) \) at 0, then the solution is smooth.
Outline

1. First order MFG equations with nonlocal coupling
   - Semi-concavity properties of HJ equations
   - The Kolmogorov equation
   - Existence/uniqueness of a solution for the MFG system

2. First order MFG systems with local coupling
   - The quasilinear elliptic equation
   - MFG as optimality conditions
Some assumptions

We study the MFG system

\[
\begin{align*}
(i) & \quad -\partial_t u + H(x, Du) = f(x, m(x, t)) \\
(ii) & \quad \partial_t m - \text{div}(mD_p H(x, Du)) = 0 \\
(iii) & \quad m(0) = m_0, \ u(x, T) = u_T(x)
\end{align*}
\]

under the following conditions:

- \( f : \mathbb{R}^d \times [0, +\infty) \rightarrow \mathbb{R} \) is smooth and increasing w.r. to \( m \) with \( f(x, 0) = 0 \), with

\[-\bar{C} + \frac{1}{C} |m|^{q-1} \leq f(x, m) \leq \bar{C}(1 + |m|^{q-1}) \quad \text{(where } q > 1 \text{)}.
\]

- There is \( r > d(q - 1) \) such that

\[\frac{1}{C} |\xi|^r - \bar{C} \leq H(x, \xi) \leq \bar{C}(|\xi|^r + 1) \quad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.\]

- + technical conditions of \( D_x H \)...
The optimal control of HJ equation

\[
\inf \left\{ \int_0^T \int_Q F^*(x, \alpha(t, x)) \, dx \, dt - \int_Q u(0, x) \, dm_0(x) \right\}
\]

where \( u \) is the solution to the HJ equation

\[
\begin{cases}
-\partial_t u + H(x, Du) = \alpha & \text{in } (0, T) \times \mathbb{R}^d \\
u(T, \cdot) = u_T & \text{in } \mathbb{R}^d
\end{cases}
\]

and \( F^*(x, a) = \sup_{m \in \mathbb{R}} (am - F(x, m)) \) where

\[
F(x, m) = \begin{cases}
  \int_0^m f(x, m') \, dm' & \text{if } m \geq 0 \\
  +\infty & \text{otherwise}
\end{cases}
\]

Note that \( F^*(x, a) = 0 \) for \( a \leq 0 \).
The optimal control of Kolmogorov equation

\[
\inf \left\{ \int_0^T \int_Q mH^*(x, -v) + F(x, m) \, dx \, dt + \int_Q u_T(x)m(T, x) \, dx \right\}
\]

where the infimum is taken over the pairs \((m, v)\) such that

\[
\partial_t m + \text{div}(mv) = 0, \quad m(0) = m_0
\]

in the sense of distributions.
Aim:

- Provide a framework in which both problems are well-posed and in duality,

- Write the MFG system as optimality conditions.

Requires estimates on HJ

\[
\begin{aligned}
(HJ) & \quad \left\{ \begin{array}{l}
- \partial_t u + H(x, Du) = \alpha \\
\quad \text{in } (0, T) \times \mathbb{R}^d \\
\quad u(T, x) = u_T(x) \quad \text{in } \mathbb{R}^d
\end{array} \right.
\end{aligned}
\]

where \( \alpha \) is in some \( L^p \).
Assume that $p > 1 + d/r$ and $r > 1$ such that

$$
\frac{1}{\tilde{C}}|\xi|^r - \tilde{C} \leq H(x, \xi) \leq \tilde{C}(|\xi|^r + 1) \quad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.
$$

and $u$ be a subsolution to

$$
-\partial_t u + H(x, Du) \leq \alpha \quad \text{in } (0, T) \times \mathbb{R}^d.
$$

**Lemma (Upper bounds)**

There is a universal constant $C$ such that

$$
u(t_1, x) \leq u(t_2, x) + C(t_2 - t_1)^{\frac{p-1-d/r}{p+d/r}} \|(\alpha)_+\|_p
$$

for any $0 \leq t_1 < t_2 \leq T$ (where $p - 1 - d/r > 0$ by assumption).
Theorem (Regularity, C.-Silvestre, 2012)

Let $u$ be a bounded viscosity solution of

$$\begin{cases}
- \partial_t u + H(x, Du) = \alpha & \text{in } (0, T) \times \mathbb{R}^d \\
u(T, x) = u_T(x) & \text{in } \mathbb{R}^d
\end{cases}$$

where $\alpha \geq 0$, $\alpha \in L^p$ with $p > 1 + \frac{d}{r}$.

Then, for any $\delta > 0$, $u$ is Hölder continuous in $[0, T - \delta] \times \mathbb{R}^d$:

$$|u(t, x) - u(s, y)| \leq C |(t, x) - (s, y)|^\gamma$$

where

$$\gamma = \gamma(\|u\|_\infty, \|\alpha\|_p, d, r), \quad C = C(\|u\|_\infty, \|\alpha\|_p, d, r, \delta).$$
Related results

- Capuzzo Dolcetta-Leoni-Porretta (2010), Barles (2010) : stationary equations, bounded RHS,
- C. (2009), Cannarsa-C. (2010), C. Rainer (2011) : evolution equations, bounded RHS,
- Dall’Aglio-Porretta (preprint) : stationary setting, unbounded RHS,
- C.-Silvestre (2012) : evolution equations, unbounded RHS.
Summary

Let \((u, \alpha)\) solve

\[
\begin{aligned}
\begin{cases}
-\partial_t u + H(x, Du) &= \alpha & \text{in } (0, T) \times \mathbb{R}^d \\
u(T, \cdot) &= u_T & \text{in } \mathbb{R}^d
\end{cases}
\end{aligned}
\]

with \(\alpha = \alpha(t, x) \geq 0\). Then

- **(Upper bound)**
  \[
  u(t_1, \cdot) \leq u(t_2, \cdot) + C(t_2 - t_1)^\frac{p-1-d/r}{d+d/r} \|\alpha\|_p \quad \text{a.e.}
  \]
  for any \(0 \leq t_1 < t_2 \leq T\).

- **(Lower bound)** \(u(t, x) \geq u_T(x) - C(T - t)\).

- **(Regularity)** \(u\) is locally Hölder continuous in \([0, T) \times \mathbb{R}^d\) with a modulus depending only on \(\|\alpha\|_p\) and \(\|u_T\|_\infty\).
Back to the optimal control of HJB

We study the optimal control of the HJ equation:

\[
(HJ - Pb) \inf \left\{ \int_0^T \int_Q F^*(x, \alpha(t, x)) \, dx \, dt - \int_Q u(0, x) \, dm_0(x) \right\}
\]

where \( u \) is the solution to the HJ equation

\[
\begin{align*}
-\partial_t u + H(x, Du) &= \alpha \\
u(T, \cdot) &= u_T
\end{align*}
\]

in \((0, T) \times \mathbb{R}^d\)

Proposition

If \((u_n, \alpha_n)\) is a minimizing sequence, then

- the \((\alpha_n)\) are bounded in \(L^p\).
- the \((u_n)\) are uniformly continuous in \([0, T] \times \mathbb{R}^d\).
- (Integral bounds) \(Du_n\) is bounded in \(L^r\) and \((\partial_t u_n)\) is bounded in \(L^1\).
The relaxed problem

Let $\mathcal{K}$ be the set $(u, \alpha) \in BV_\#((0, T) \times \mathbb{R}^d) \times L^p_\#((0, T) \times \mathbb{R}^d)$ such that

- $Du \in L^r_\#((0, T) \times \mathbb{R}^d)$
- $\alpha \geq 0$ a.e.
- $u(T, x) = u_T(x)$ and

$$-\partial_t u + H(x, Du) \leq \alpha \quad \text{in } (0, T) \times \mathbb{R}^d.$$ 

holds in the sense of distribution.

Note that $\mathcal{K}$ is a convex set.

The relaxed problem is

$$(\text{HJ} - \text{Rpb}) \inf_{(u, \alpha) \in \mathcal{K}} \left\{ \int_0^T \int_{\Omega} F^*(x, \alpha(t, x)) \, dx \, dt - \int_{\Omega} u(0, x) \, dm_0(x) \right\}$$
Theorem

- The relaxed problem has (HJ-Rpb) at least one minimum \((u, \alpha)\), where \(u\) is continuous and satisfies in the viscosity sense

\[-\partial_t u + H(x, Du) \geq 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^d\]

- The value of the relaxed problem (HJ-Rpb) is equal to the value of the optimal control of the HJ equation (HJ-Pb).
The optimal control problem of HJ equation (HJ-Pb) can be rewritten as

\[
\inf \left\{ \int_0^T \int_Q F^*(x, -\partial_t u + H(x, Du)) \, dx dt - \int_Q u(0, x) dm_0(x) \right\}
\]

with constraint \( u(T, \cdot) = u_T \).

This is a convex problem.

\( \rightarrow \) Suggests a duality approach.
The Fenchel-Rockafellar duality Theorem

Let

- \( E \) and \( F \) be two normed spaces,
- \( \Lambda \in \mathcal{L}_c(E,F) \) and
- \( \mathcal{F} : E \to \mathbb{R} \cup \{+\infty\} \) and \( \mathcal{G} : F \to \mathbb{R} \cup \{+\infty\} \) be two lsc proper convex maps.

**Theorem**

Under the **qualification condition**: there is \( x_0 \) such that \( \mathcal{F}(x_0) < +\infty \) and \( \mathcal{G} \) continuous at \( \Lambda(x_0) \), we have

\[
\inf_{x \in E} \{ \mathcal{F}(x) + \mathcal{G}(\Lambda(x)) \} = \max_{y^* \in F'} \{-\mathcal{F}^*(\Lambda^*(y^*)) - \mathcal{G}^*(-y^*)\}
\]

as soon as the LHS is finite.
Remark: If $\bar{x}$ and $\bar{y}^*$ are respectively optimal in

$$\inf_{x \in E} \{ \mathcal{F}(x) + \mathcal{G}(\Lambda(x)) \} \text{ and } \sup_{y^* \in F'} \{-\mathcal{F}^*(\Lambda^*(y^*)) - \mathcal{G}^*(-y^*)\}$$

then

$$\mathcal{F}(\bar{x}) + \mathcal{F}^*(\Lambda^*(\bar{y}^*)) = \langle \Lambda^*(\bar{y}^*), \bar{x} \rangle$$

and

$$\mathcal{G}(\Lambda(\bar{x})) + \mathcal{G}^*(-\bar{y}^*) = \langle -\bar{y}^*, \Lambda(\bar{x}) \rangle$$

i.e.,

$$\Lambda^*(\bar{y}^*) \in \partial \mathcal{F}(\bar{x})$$

and

$$\Lambda(\bar{x}) \in \partial \mathcal{G}^*(-\bar{y}^*)$$
The optimal control problem of HJ equation (\textbf{HJ-Pb})

\[
\inf \left\{ \int_0^T \int_Q F^*(x, -\partial_t u + H(x, Du)) \, dx \, dt - \int_Q u(0, x) \, dm_0(x) \right\}
\]

(with constraint \( u(T, \cdot) = u_T \)) can be rewritten as

\[
\inf_{u \in E} \left\{ \mathcal{F}(u) + \mathcal{G}(\Lambda(u)) \right\}
\]

where

- \( E = C^1_\sharp([0, T] \times \mathbb{R}^d) \),
- \( F = C^0_\sharp([0, T] \times \mathbb{R}^d, \mathbb{R}) \times C^0_\sharp([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \),
- \( \mathcal{F}(u) = -\int_Q m_0(x) u(0, x) \, dx \) if \( u(T, \cdot) = u_T \) \((+\infty\) otherwise). \)
- \( \mathcal{G}(a, b) = \int_0^T \int_Q F^*(x, -a(t, x) + H(x, b(t, x))) \, dx \, dt \).
- \( \Lambda(u) = (\partial_t u, Du) \).
Then $\mathcal{F}$ is convex and lower semi-continuous on $E$ while $\mathcal{G}$ is convex and continuous on $F$. Moreover $\Lambda$ is bounded and linear.

The qualification condition is satisfied by $u(t, x) = u_T(x)$.

By Fenchel-Rockafellar duality theorem we have

$$\inf_{u \in E} \{ \mathcal{F}(u) + \mathcal{G}(\Lambda(u)) \} = \max_{(m, w) \in F'} \{ -\mathcal{F}^*(\Lambda^*(m, w)) - \mathcal{G}^*(-(m, w)) \}$$

where $F'$ is the set of Radon measures $(m, w) \in M_\#([0, T] \times \mathbb{R}^d, \mathbb{R} \times \mathbb{R}^d)$ and $\mathcal{F}^*$ and $\mathcal{G}^*$ are the convex conjugates of $\mathcal{F}$ and $\mathcal{G}$. 
Recall that

$$F(u) = - \int_Q m_0(x)u(0, x)dx \quad \text{if } u(T, \cdot) = u_T \quad (+\infty \text{ otherwise})$$

and

$$G(a, b) = \int_0^T \int_Q F^*(x, -a(t, x) + H(x, b(t, x))) \, dxdt.$$ 

**Lemma**

$$F^*(\Lambda^*(m, w)) = \begin{cases} 
\int_Q u_T(x)dm(T, x) & \text{if } \partial_t m + \text{div}(w) = 0, \ m(0) = m_0 \\
+\infty & \text{otherwise}
\end{cases}$$

and

$$G^*(m, w) = \begin{cases} 
\int_0^T \int_Q -F(x, m) - mH^*(x, -\frac{w}{m}) \, dt\, dx & \text{if } (m, w) \in L_1^\# \\
+\infty & \text{otherwise}
\end{cases}$$
Consequence of the Lemma:

\[
\max_{(m,w) \in F'} \left\{ -\mathcal{F}^* (\Lambda^* (m, w)) - \mathcal{G}(m, w) \right\}
\]

\[
= \max \left\{ \int_0^T \int_Q -F(x, m) - mH^*(x, -\frac{w}{m}) \, dt \, dx - \int_Q u_T(x)m(T, x) \, dx \right\}
\]

where the maximum is taken over the \(L^1_\#\) maps \((m, w)\) such that \(m \geq 0\) a.e. and

\[
\partial_t m + \text{div}(w) = 0, \quad m(0) = m_0
\]
Idea of proof:

$$F^*(\Lambda^*(m, w)) = \sup_{u(T)=u_T} \langle \Lambda^*(m, w), u \rangle - F(u)$$

$$= \sup_{u(T)=u_T} \langle (m, w), \Lambda(u) \rangle + \int_Q m_0(x)u(0, x)dx$$

$$= \sup_{u(T)=u_T} \int_0^T \int_Q (m \partial_t u + \langle w, Du \rangle) + \int_Q m_0(x)u(0, x)dx$$

$$= \sup_{u(T)=u_T} \int_0^T \int_Q -u(-\partial_t m + \text{div}(w))$$

$$+ \int_Q m(T)u_T + \int_Q (m_0 - m(0))u(0)dx$$

$$= \begin{cases} \int_Q u_T(x)dm(T, x) & \text{if } \partial_t m + \text{div}(w) = 0, \ m(0) = m_0 \\ +\infty & \text{otherwise} \end{cases}$$
The dual of the optimal control of HJ eqs

**Theorem**

The dual of the optimal control of HJ (HJ-Pb) equation is given by

\[
(K - Pb) \inf \left\{ \int_0^T \int_Q mH^*(x, -\frac{w}{m}) + F(x, m) \, dx\,dt + \int_Q u_T(x)m(T, x)\,dx \right\}
\]

where the infimum is taken over the pairs \((m, w) \in L^1_\#((0, T) \times \mathbb{R}^d) \times L^1_\#((0, T) \times \mathbb{R}^d, \mathbb{R}^d)\) such that

\[
\partial_t m + \text{div}(w) = 0, \quad m(0) = m_0
\]

in the sense of distributions. Moreover this dual problem has a unique minimum.
(K-Pb) as an optimal control problem for Kolmogorov equation:

Set $v = w/m$. Then (K-Pb) becomes

$$\inf \left\{ \int_0^T \int_Q mH^*(x, -v) + F(x, m) \, dxdt + \int_Q u_T(x) m(T, x) dx \right\}$$

where the infimum is taken over the pairs $(m, v)$ such that

$$\partial_t m + \text{div}(mv) = 0, \quad m(0) = m_0$$

in the sense of distributions.
We now define **weak solutions** of the \((MFG)\) system

\[
\begin{align*}
(i) & \quad -\partial_t u + H(x, Du) = f(x, m(x, t)) \\
(ii) & \quad \partial_t m - \text{div}(m D_p H(x, Du)) = 0 \\
(iii) & \quad m(0) = m_0, \quad u(x, T) = u_T(x)
\end{align*}
\]

and explain the relation with the two optimal control problems

- for the HJ equation (problem (\textbf{HJ-Pb}))
- for the Kolmogorov equation (problem (\textbf{K-Pb})))
Definition

A pair \((m, u) \in L^1((0, T) \times \mathbb{R}^d) \times BV((0, T) \times \mathbb{R}^d)\) (with \(u\) continuous in \([0, T] \times \mathbb{R}^d\), \(Du \in L^r((0, T) \times \mathbb{R}^d, m)\), \(mD_pH(x, Du) \in L^1_\#\)) is a weak solution of (MFG) if

(i) \(\partial_t u + H(x, Du) \leq f(x, m)\) holds in the sense of distribution, with \(u(T, x) = u_T(x)\) in the sense of trace,

(iii) \(\partial_t m - \text{div}(mD_pH(x, Du)) = 0\) holds in the sense of distribution in \((0, T) \times \mathbb{R}^d\) and \(m(0) = m_0\),

(iv) Equality \(\int_0^T \int_Q m(\partial_t u_{ac} - \langle Du, D_pH(x, Du) \rangle) = \int_Q m(T)u_T - m_0u(0)\) holds.

(where \(\partial_t u_{ac}\) is the a.c. part of the measure \(\partial_t u\)).
Remarks

- If (i) holds with an equality and if $u$ is in $W^{1,1}$, then (ii) implies (iii).

- Conditions (i) and (iii) imply that

$$-\partial_t u^{ac}(t, x) + H(x, Du(t, x)) = f(x, m(t, x))$$

holds a.e. in $\{m > 0\}$. 
Existence/uniqueness of weak solutions

Theorem (Cannarsa, C., in preparation)

There is a unique weak solution \((m, u)\) of \((MFG)\) such that \(u\) is locally Hölder continuous in \([0, T) \times \mathbb{R}^d\) and which satisfies in the viscosity sense

\[-\partial_t u + H(x, Du) \geq 0 \quad \text{in } (0, T) \times \mathbb{R}^d.
\]

Idea of proof:

- Let \((m, w)\) is a minimizer of \((K-Pb)\) and \((u, \alpha)\) is a minimizer of \((HJ-Rpb)\) such that \(u\) is continuous. Then one can show that \((m, u)\) is a solution of mean field game system \((MFG)\) and \(w = -mD_pH(\cdot, Du)\) while \(\alpha = f(\cdot, m)\) a.e..

- Conversely, any solution of \((MFG)\) such that \(u\) is continuous is such that the pair \((m, -mD_pH(\cdot, Du))\) is the minimizer of \((K-Pb)\) while \((u, f(\cdot, m))\) is a minimizer of \((HJ-Rpb)\).
Existence/uniqueness of weak solutions

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- Conversely, any solution of (MFG) such that \(u\) is continuous is such that the pair \((m, -mD_pH(\cdot, Du))\) is the minimizer of (K-Pb) while \((u, f(\cdot, m))\) is a minimizer of (HJ-Rpb).
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- Let \((m, w)\) is a minimizer of \((K-Pb)\) and \((u, \alpha)\) is a minimizer of \((HJ-Rpb)\) such that \(u\) is continuous. Then one can show that \((m, u)\) is a solution of mean field game system \((MFG)\) and \(w = -mD_p H(\cdot, Du)\) while \(\alpha = f(\cdot, m)\) a.e..

- Conversely, any solution of \((MFG)\) such that \(u\) is continuous is such that the pair \((m, -mD_p H(\cdot, Du))\) is the minimizer of \((K-Pb)\) while \((u, f(\cdot, m))\) is a minimizer of \((HJ-Rpb)\).
Link with the quasilinear equation

**Proposition**

If \((u, m)\) is a weak solution of the MFG system, then \(u\) is a viscosity solution of

\[
\begin{align*}
G(x, \partial_t \phi, D\phi, \partial_{tt} \phi, D\partial_t \phi, D^2 \phi) &= 0 \text{ in } (0, T) \times \mathbb{R}^d \\
\phi(T, \cdot) &= \phi_T \text{ in } \mathbb{R}^d \\
-\partial_t \phi + H(x, D\phi) &= f(m_0) \text{ in } \mathbb{R}^d
\end{align*}
\]

where

\[
G(x, p_t, p_x, a, b, C) = -\text{Tr} \left( A(x, p_t, p_x) \begin{pmatrix} a & b^T \\ b & C \end{pmatrix} \right) - F^*_{\alpha, \alpha} \langle H_p, H_x \rangle - \langle F^*_{x, \alpha}, H_p \rangle - F^*_{\alpha} \text{Tr}(H_{x,p})
\]

with

\[
A(x, p_t, p_x) = F^*_{\alpha, \alpha} \begin{pmatrix} 1 & -H_p^T \\ -H_p & H_p \otimes H_p \end{pmatrix} + F^*_{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & H_{pp} \end{pmatrix}
\]
Conclusion

Summary

- Existence/uniqueness is well understood for first order MFG systems with nonlocal coupling,
- For local coupling, existence of smooth solutions in some cases, otherwise weak solutions.

Open problems

- Understand the regularity of solutions for 1st order, local MFG systems in full generality.
- Existence/uniqueness for the MFG system of congestion type $(\alpha \in (0, 2))$

\[
\begin{align*}
(i) \quad & -\partial_t u + \frac{|Du|^2}{2m^\alpha} = f(x, m(x, t)) \\
(ii) \quad & \partial_t m - \text{div}(m^{1-\alpha} Du)) = 0 \\
(iii) \quad & m(0) = m_0, \ u(x, T) = u_T(x)
\end{align*}
\]

- Application to $N$–player games, periodic solutions...