Parallel Sections and Related Problems in Convex Geometry

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May 30, 2012
Definitions and notation

**Definition**
A body $K$ is a compact subset of $\mathbb{R}^n$ with non-empty interior.

**Definition**
A body $K$ is centrally symmetric if $K = -K$.

**Definition**
A body $K$ is convex if given any two points $P, Q \in K$, the segment $PQ$ is contained in $K$.
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$$\xi \rightarrow A_{K,\xi}(t)$$

is called the parallel section function of $K$. 
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General problem

Reconstruct \( K \) from information about the sections of \( K \).
Minkowski Uniqueness:
Every centrally-symmetric convex (or star) body $K$ is determined by
$\{A_{K,\xi(0)}\}_{\xi \in S^{n-1}}$. 
Reconstruction from Sections

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Busemann-Petty problem: Given two centrally-symmetric convex 
figures $K, L$, such that for all $\xi \in S^{n-1}$, 
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Parallel section problem: If for a fixed $t$ and all $\xi \in S^{n-1}$,
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$$\|x\|_K = \sup\{\lambda \geq 0 : x \in \lambda K\}.$$
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Theorem (Minkowski Uniqueness Theorem)

Every centrally-symmetric star body $K$ is determined by the volumes of their central hyperplane sections.

Proof. A Fourier Analytical proof by Koldobsky is based on the fact that
\[ A_K,\xi(0) = \text{vol}^{n-1}(K \cap \xi^\perp) = \frac{1}{\pi^{(n-1)}\|\cdot\|^{n+1}K \wedge (\xi)} \].

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Maximal Section Problem

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In dimension 2, both problems have long been known to have a negative answer: there exist bodies of constant width that are not discs. The answer is also no in higher dimensions: for problem 1, Nazarov, Ryabogin, Zvavitch (2012). For problem 2, Gardner, Ryabogin, Yaskin, Zvavitch (2011).
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The Busseman-Petty Problem (1956)

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- Larman and Rogers (1975): No for $n \geq 12$. 

Ball (1987): No for $n \geq 10$.

Giannopoulos, Bourgain, Papadimitrakis, Gardner, Zhang (1990s): Counterexamples in several dimensions $n \geq 5$.

Gardner, Koldobsky, Schlumprecht (1999), analytic proof in all dimensions: Yes if $n = 2, 3, 4$; No for $n \geq 5$. 

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Parallel Sections

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![Diagram of centrally-symmetric convex bodies](image-url)
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- We will present a partial result in dimension 4.
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Nazarov, Ryabogin, Zvavitch (2012) prove the following formula for sections of bodies of revolution:

$$vol_{n-1}(K \cap H(L)) = \kappa_{n-2} \sqrt{1 + s^2} \int_{-x}^{y} (f^2(x_1) - L^2(t, s, x_1))^{(n-2)/2} \, dx_1.$$
Let $t > 0$ and let $K$ be a 4-dimensional body of revolution containing the ball of radius $t$ centered at the origin. We assume that all 3-dimensional $t$-sections of $K$ have constant volume.
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By dilating $K$, we may assume $(t, \sqrt{1 - t^2}) \in \partial K$ (and then the point $(-t, \sqrt{1 - t^2})$ must also be on $\partial K$).
In dimension 4, the formula for sections becomes

$$\text{vol}_3(K \cap H(L)) = \pi \sqrt{1 + s^2} \int_{-x}^{y} (f^2(x_1) - L^2(s, t, x_1)) \, dx_1$$
In dimension 4, the formula for sections becomes

$$vol_3(K \cap H(L)) = \pi \sqrt{1 + s^2} \int_{-x}^{y} (f^2(x_1) - L^2(s, t, x_1)) \, dx_1 = \frac{4}{3} \pi (1 - t^2)^{3/2}.$$
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Differentiating with respect to $s$, we obtain a cubic equation in terms of $x$ and $y$. 
In dimension 4, the formula for sections becomes

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\]

Differentiating with respect to \( s \), we obtain a cubic equation in terms of \( x \) and \( y \), that does not depend on \( f \).
\[
(y^3 + x^3) \pm \frac{3t(1 + 2s^2)}{2s\sqrt{1 + s^2}} (y^2 - x^2) + 3t^2(y + x) = 2 \left( \frac{1 - t^2}{1 + s^2} \right)^{3/2}.
\]
Given a point \((y, f(y)) \in \partial K\), we can solve the equation and find \((-x, -f(-x)) \in \partial K\). Then we can iterate, starting now at \((-x, -f(-x))\).
Iteration

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If \((y, f(y))\) is on the unit sphere, so is \((-x, -f(-x))\).
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- If \((y, f(y))\) is on the unit sphere, so is \((-x, -f(-x))\).
- By assumption, \(\partial K\) contains four points that are on the unit sphere.

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- Given a point \((y, f(y)) \in \partial K\), we can solve the equation and find \((-x, -f(-x)) \in \partial K\). Then we can iterate, starting now at \((-x, -f(-x))\).
- If \((y, f(y))\) is on the unit sphere, so is \((-x, -f(-x))\).
- By assumption, \(\partial K\) contains four points that are on the unit sphere. If \(\arccos(t)\) is an irrational multiple of \(\pi\), the iteration will give us a dense set of points, both on \(\partial K\) and on the unit sphere, and hence \(K = S^3\).
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Thank you!