Milnor Conjectures and Quadratic Forms

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IMA, University of Minnesota

"Milnor had had a deep and affectionate interest in quadratic form theory. . . . His paper Algebraic K-theory and quadratic forms continues to guide much of the current research in the algebraic theory of quadratic forms."

— H. Bass (1993)

Witt rings and filtration

F field, $\mathrm{char}(F) \neq 2$

(V, q) quadratic form over F

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$b_q : V \times V \to k$ associated bilinear form

$$b_q(v, w) = \frac{1}{2}\big(q(v + w) - q(v) - q(w)\big)$$

$T(q) = \big(b_q(e_i, e_j)\big)$ Gram matrix of q, $\{e_1, \cdots, e_n\}$ a basis of V

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(V, q) is *nondegenerate* if $T(q)$ is invertible.

Witt rings and filtration

$h = (F^2, xy)$ hyperbolic plane

Theorem. (Witt decomposition) $(V, q) = (V_0, q_0) \perp h^r$
q_0 anisotropic, uniquely determined by q
$r =$ Witt index of q.

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Witt equivalence. $q \sim q' \iff q \perp h^m \simeq q' \perp h^n \iff q_0 \simeq q'_0$

$W(F) =$ Witt equivalence classes of quadratic forms under $\perp$ and $\otimes$.

$I(F)$ ideal of $W(F)$ of even-dimensional forms

$I^n(F) = I(F)^n$

Witt rings and filtration

n-fold *Pfister form*:

$$\langle\langle a_1, a_2, \ldots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$$

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$I^n(F)$ is additively generated by n-fold Pfister forms

Galois cohomology

$\Gamma_F = \mathrm{Gal}(F_s/F)$

$$\begin{aligned} H^n(F, \mathbb{Z}/2\mathbb{Z}) &= H^n(\Gamma_F, \mathbb{Z}/2\mathbb{Z}) \\ &= \varinjlim_{L/F \text{ finite Galois}} H^n(\mathrm{Gal}(L/F), \mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

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- $H^0(F, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$
- $H^1(F, \mathbb{Z}/2\mathbb{Z}) = F^\times/F^{\times 2}$ (Kummer isomorphism)
  $(a) \in H^1(F, \mathbb{Z}/2\mathbb{Z})$ denotes the square class of $a \in F^\times$

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  $(a) \in H^1(F, \mathbb{Z}/2\mathbb{Z})$ denotes the square class of $a \in F^\times$
- $H^2(F, \mathbb{Z}/2\mathbb{Z}) = {}_2\mathrm{Br}(F)$
  The cup product $(a).(b)$ represents the quaternion algebra with generators i, j and relations
  $i^2 = a$, $j^2 = b$, $ij = -ji$.

Classical invariants for quadratic forms

The *dimension* mod 2

$$e_0 \colon W(F) \to \mathbb{Z}/2\mathbb{Z} = H^0(F, \mathbb{Z}/2\mathbb{Z})$$

$e_0(V, q) = \dim V \pmod 2$

$\ker(e_0) = I(F)$

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The *discriminant*

$$e_1\colon\ I(F) \to F^\times/F^{\times 2} = H^1(F, \mathbb{Z}/2\mathbb{Z})$$

$e_1(V, q) = \big((-1)^n \det T(q)\big)$, $2n = \dim V$

$\ker(e_1) = I^2(F)$

Classical invariants for quadratic forms

The *Clifford invariant*

$$e_2 \colon I^2(F) \to {}_2\mathrm{Br}(F) = H^2(F, \mathbb{Z}/2\mathbb{Z})$$

$e_2(V, q) = [C(q)] \in {}_2\mathrm{Br}(F)$

$C(q) =$ Clifford algebra of $(V, q) = T(V)/\langle v \otimes v - q(v) \rangle$.

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$C(q) =$ Clifford algebra of $(V, q) = T(V)/\langle v \otimes v - q(v) \rangle$.

Example: $e_2(\langle 1, -a \rangle \otimes \langle 1, -b \rangle) = (a) \cdot (b) \in H^2(F, \mathbb{Z}/2\mathbb{Z})$

Classical invariants for quadratic forms

Classical questions.

- Is e_2 onto?
  Is ${}_2\mathrm{Br}(F)$ generated by quaternion algebras?

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- Is e_2 onto?
  Is ${}_2\mathrm{Br}(F)$ generated by quaternion algebras?

- Is $\ker(e_2) = I^3(F)$?

Milnor Conjecture (MC)

Theorem. (Arason 1975) The assignment

$$e_n(\langle\langle a_1, \ldots, a_n \rangle\rangle) = (a_1) \cdot (a_2) \cdot \cdots \cdot (a_n) \in H^n(F, \mathbb{Z}/2\mathbb{Z})$$

is well-defined on the isomorphism classes of n-fold Pfister forms.

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is well-defined on the isomorphism classes of n-fold Pfister forms.

Milnor conjecture (1970). The map e_n extends to a homomorphism

$$e_n \colon I^n(F) \to H^n(F, \mathbb{Z}/2\mathbb{Z})$$

which is onto with kernel $I^{n+1}(F)$.

Milnor Conjecture (MC)

Equivalently, there is an isomorphism

$$(e_n)\colon \bigoplus_{n\geq 0} I^n(F)/I^{n+1}(F) \longrightarrow \bigoplus_{n\geq 0} H^n(F, \mathbb{Z}/2\mathbb{Z})$$

of the graded Witt ring and the graded cohomology ring.

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- MC is a theorem due to Voevodsky (2003) and Orlov–Vishik–Voevodsky (2007).
- MC as stated above is a consequence of two conjectures of Milnor relating Milnor ring K_*F with mod 2 Galois cohomology ring and the graded Witt ring.

Milnor K-theory

$K_0(F) = \mathbb{Z}$

$K_1(F) = F^\times$

$K_*(F) = T(F^\times)/\langle a \otimes (1-a),\ a \in F^\times,\ a \neq 1\rangle$

$\{a_1, \ldots, a_n\}$ denotes the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n(F)$

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Milnor's definition of K_2 is inspired by Steinberg, Moore, and Matsumoto. His definition of K_n for $n > 2$ is "purely ad hoc"!

Norm residue map and MC1

p prime, $p \neq \operatorname{char} F$

Definition. (Bass–Tate) The Kummer isomorphism $F^\times/F^{\times p} \xrightarrow{\sim} H^1(F, \mu_p)$ extends to a homomorphism called the *norm residue map*

$$h_{n,F} \colon K_n(F)/p\,K_n(F) \longrightarrow H^n(F, \mu_p^{\otimes n})$$

sending $\{a_1, a_2, \ldots, a_n\}$ to $(a_1) \cdot (a_2) \cdot \cdots \cdot (a_n)$.

Norm residue map and MC1

Milnor Conjecture (MC1). For $p = 2$, $h_{n,F}$ is an isomorphism for all n.

Theorem. (Merkurjev 1981) For $p = 2$, $h_{2,F}$ is an isomorphism.

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Theorem. (Merkurjev 1981) For $p = 2$, $h_{2,F}$ is an isomorphism.

Theorem. (Voevodsky 2003) MC1 holds.

Norm residue map and Bloch–Kato conjecture

Bloch–Kato conjecture. For general p, $h_{n,F}$ is an isomorphism.

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Theorem. (Merkurjev–Suslin, Rost 1990) For general p, $h_{3,F}$ is an isomorphism.

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Theorem. (Voevodsky, Rost 2009) BKC holds.

Milnor ring and graded Witt ring—MC2

$k_n(F) = K_n(F)/2K_2(F)$

$$\begin{aligned} s_{1,F}\colon k_1(F) &\to I(F)/I^2(F) \\ (a) &\mapsto \langle 1, -a\rangle \end{aligned}$$

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Theorem (Orlov, Vishik, Voevodsky 2007) MC2 holds.

The Milnor Conjectures

$$
\begin{array}{ccc}
 & K_n(F)/2K_n(F) & \\
 \swarrow^{s_{n,F}}_{\sim} & & \searrow^{h_{n,F}}_{\sim} \\
I^n(F)/I^{n+1}(F) & \xrightarrow[\sim]{e_{n,F}} & H^n(F, \mathbb{Z}/2\mathbb{Z})
\end{array}
$$

Classification of quadratic forms by cohomological invariants

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Together with MC, one has:

Theorem. Given anisotropic quadratic forms q_1, q_2, if $e_n(q_1 - q_2) = 0$ for all n, then $q_1 \simeq q_2$.

Symbol length in Galois cohomology

An element of the form $(a_1) \cdot (a_2) \cdot \cdots \cdot (a_n) \in H^n(F, \mathbb{Z}/2\mathbb{Z})$ is called a *symbol*.

MC $\Longrightarrow H^n(F, \mathbb{Z}/2\mathbb{Z})$ is generated by symbols.

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The *symbol length* of $\xi \in H^n(F, \mathbb{Z}/2\mathbb{Z})$ is the least r such that ξ is a sum of r symbols.

The *n-symbol length* of F is the maximum symbol length of elements in $H^n(F, \mathbb{Z}/2\mathbb{Z})$.

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Boundedness of symbol lengths is an arithmetic property of F.

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Question. Is there a bound for the symbol length of finitely generated fields?

This question has interesting consequences for the u-invariant of a field.

u-invariant and symbol length

$u(F) = \max\{\dim(q) \ : \ q \text{ anisotropic quadratic form over } F\}$

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Examples.

- $u(\mathbb{F}_p) = 2$
- $u(\mathbb{Q}_p) = 4$
- $u(\mathbb{Q}(\sqrt{-1}) = 4$
- $u(\mathbb{R}) = \infty$

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Easy consequence of Voevodsky's theorem:

Theorem. Suppose $H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \geq n_0$ and the i-symbol length of F is bounded for $i \leq n_0$. Then $u(F) < \infty$.

u-invariant and symbol length

Idea of Proof. Supposing that $H^3(F, \mathbb{Z}/2\mathbb{Z}) = 0$ and the 2-symbol length of $F \leq r$.

Let q be an anisotropic quadratic form of dimension $2n$, $\mathrm{disc}(q) = 1$, and

$$e_2(q) = \sum_{1 \leq i \leq r} (a_i) \cdot (b_i).$$

then MC implies

$$q - \sum_{1 \leq i \leq r} \langle 1, -a_i \rangle \otimes \langle 1, -b_i \rangle \in I^3(F) = 0.$$

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Thus

$$q \sim \sum_{1 \leq i \leq r} \langle 1, -a_i \rangle \otimes \langle 1, -b_i \rangle;$$

Thus $\dim(q) \leq 4r$ and $u(F) \leq 2 + 4r$.

u-invariant and symbol length

Recall. $u(\mathbb{Q}_p) = 4$

Finiteness of $u(\mathbb{Q}_p(t))$ was open until late 90s.

Open question list of Lam's book: $u(\mathbb{Q}_p(t)) = 8$?

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The finiteness of $u(\mathbb{Q}_p(t))$ came as a consequence of symbol length bounds.

u-invariant and symbol length

Theorem. (Saltman 1997) For $p \neq 2$, the 2-symbol length of $\mathbb{Q}_p(t)$ is 2.

Theorem. (Parimala–Suresh 1998) For $p \neq 2$, the 3-symbol length of $\mathbb{Q}_p(t)$ is one.

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Theorem. (Parimala–Suresh 2007) For $p \neq 2$, $u(\mathbb{Q}_p(t)) = 8$.

Theorem. (Heath-Brown, Leep 2010) For all p, $u(\mathbb{Q}_p(t)) = 8$.
(Methods very different from Galois cohomology)

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Converse Question. Suppose $u(F) < \infty$. Is the n-symbol length of F bounded for all n?

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Observation. If $u(F) < \infty$ then $H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \gg 0$.

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Converse Question. Suppose $u(F) < \infty$. Is the n-symbol length of F bounded for all n?

Observation. If $u(F) < \infty$ then $H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \gg 0$.

Proof. $u(F) < \infty \Longrightarrow$ n-fold Pfister forms are zero in $W(F)$ for all $n \gg 0$

$\Longrightarrow I^n(F) = 0$ for $n \gg 0$

$\Longrightarrow H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \gg 0$ (MC).

u-invariant and symbol length

Special case. $u(F) < \infty \implies$ 3-symbol length is bounded.

u-invariant and symbol length

Special case. $u(F) < \infty \Longrightarrow$ 3-symbol length is bounded.

Proof. Let $2n_0 > u(F)$, $q_0 = \langle x_1, x_2, \dots, x_{2n_0-1}, \epsilon x_1 x_2 \cdots x_{2n_0-1} \rangle$,
$\epsilon = \pm 1$, *generic form* of dimension $2n_0$ and discriminant 1.

$X =$ Brauer-Severi variety of $C(q_0)$ over $F(x_1, \dots, x_{2n_0-1})$

$\widetilde{F} = F(x_1, \dots, x_{2n_0-1})(X)$

$q_0 \in I^3(\widetilde{F})$: suppose $e_3(q_0)$ is a sum of r symbols in
$H^3(\widetilde{F}, \mathbb{Z}/2\mathbb{Z})$.

$\xi \in H^3(F, \mathbb{Z}/2\mathbb{Z})$, $\xi = e_3(q_1)$, $q_1 \in I^3(F)$:

$\dim(q_1) = 2n_0$, $\operatorname{disc} q_1 = 1$, $e_2(q_1) = 0$.

$\exists$ specialisation $X \to F$, $q_0 \mapsto q_1$, $e_3(q_0) \mapsto e_3(q_1) = \xi$
$\Longrightarrow \xi$ is a sum of at most r symbols.

u-invariant and symbol length

There are no analogous varieties $X/F(x_1, \ldots, x_{2n_0-1})$ to push the generic form q_0 into $I^d(F(x_1, \ldots, x_{2n_0-1})(X))$ for $d \geq 4$ with good specialization properties.

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Theorem. (Saltman 2011) There exist finitely many function fields $F_{i,d}/F(x_1, \ldots, x_{2n_0-1})$ such that

- $q_0 \in I^d(F_{i,d})$
- Given $q \in I^d(F)$, $\exists$ specialisation $F_{j,d} \to F$ for some j with $q_0 \mapsto q$

Corollary. $u(F) < \infty \implies i$-symbol length is bounded for all i.

Pythagoras number

Definition. $p(F)$ is the least n such that every sum of squares in F is a sum of n squares.

Example. (Euler 1754) $p(\mathbb{Q}) = 4$

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Theorem. (Hilbert 1888, Cassels–Ellison–Pfister 1971)

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General bounds: $n + 2 \leq p(\mathbb{R}(x_1, \ldots, x_n)) \leq 2^n$, for $n \geq 3$.

Pythagoras number

Interesting questions concern $p(\mathbb{Q}(t_1, \ldots, t_n))$.

Conjecture. (Pfister) Let F be a function field of transcendence degree d over a number field k. Then

- $p(F) \leq 5$ if $d = 1$
- $p(F) \leq 2^{d+1}$ if $d \geq 2$

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Theorem. (Landau 1906, Pourchet 1971, Hsia-Johnson 1974)
$p(\mathbb{Q}(t)) = 5$

Theorem. (J.-L. Colliot-Thélène 1986, F. Pop (unpublished))
For $d = 1$, $p(F) \leq 6$

Pythagoras number

Theorem. (Arason 2000) MC $\Longrightarrow p(F) \leq 2^{d+2}$.

Pythagoras number

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Proof. $H^n(F(\sqrt{-1}), \mathbb{Z}/2\mathbb{Z}) = 0$ for $n > d + 2$
(since $\mathrm{cd}(F(\sqrt{-1})) = d + 2$)

$\Longrightarrow I^n(F(\sqrt{-1})) = 0$ for $n > d + 2$ (MC)

$\Longrightarrow$ n-fold Pfister forms are universal over $F(\sqrt{-1})$, for
$n = d + 2$

$\Longrightarrow p(F) \leq 2^{d+2}$ (Pfister)

Pythagoras number

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Conjecture. (Kato 1986) Let k be a number field, Ω_k its set of places, and X a geometrically integral variety over k of dimension d. The restriction map

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Theorem. (Colliot-Thélène and Jannsen 1991)
MC + Kato's conjecture $\Longrightarrow$ Pfister's conjecture for $d \geq 2$.

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Concerning Kato's conjecture:

- $\dim X = 0$: the Hasse–Brauer–Noether theorem.
- $\dim X = 1$: Kato 1986.
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Remaining open case of Pfister's conjecture. $p(F) = 5$ where F has trancendence degree one over a number field.

Sketch of CT-J proof of Pfister's conjecture.

Let $F = k(X)$ and $F_v = k_v(X)$ for $v \in \Omega_k$.

Let $\phi_{d+1} = \langle\langle 1, \ldots, 1 \rangle\rangle$ be the $(d+1)$-fold Pfister form.

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Claim. $\phi_{d+1} \otimes \langle 1, -f \rangle = 0$ in $W(F_v)$ for every $v \in \Omega_k$.

This fact is easy to check over finite completions. Over a real completion, it's a consequence of Pfister's effective Hilbert 17th problem.

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Thus $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f \rangle) = 0$ in $H^{d+2}(F_v, \mathbb{Z}/2\mathbb{Z})$ for all $v \in \Omega_k$

$\Longrightarrow e_{d+2}(\phi_{d+1} \otimes \langle 1, -f \rangle) = 0$ in $H^{d+2}(F, \mathbb{Z}/2\mathbb{Z})$ (Kato conj.)

$\Longrightarrow \phi_{d+1} \otimes \langle 1, -f \rangle \in I^{d+3}(F)$ (MC)

$\Longrightarrow \phi_{d+1} \otimes \langle 1, -f \rangle = 0$ (by Arason–Pfister Hauptsatz)

$\Longrightarrow f$ is a sum of 2^{d+1} squares in F.

Norm residue isomorphism in degree 2 and Serre's Conjecture II

The norm residue isomorphism

$$h_{2,F}\colon K_2(F)/pK_2(F) \xrightarrow{\simeq} H^2(F, \mu_p^{\otimes 2})$$

due to Merkurjev–Suslin has deep consequences in the direction of a conjecture of Serre.

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Definition. F has *cohomological dimension* $\mathrm{cd}(F) \leq n$ if $H^m(F, M) = 0$ for all finite discrete Galois modules M, and all $m \geq n+1$.

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- $\mathrm{cd}(\mathbb{F}_q) = 1$
- $\mathrm{cd}(\mathbb{C}(t)) = 1$
- $\mathrm{cd}(\mathbb{Q}_p) = 2$
- $\mathrm{cd}(\mathbb{Q}(\sqrt{-1})) = 2$
- $\mathrm{cd}(\mathbb{C}(t_1, t_2)) = 2$

Norm residue isomorphism in degree 2 and Serre's Conjecture II

Conjecture II. (Serre) Let F be a perfect field of $\mathrm{cd}(F) \leq 2$ and G a semisimple simply-connected linear algebraic group defined over F. Then every principal homogeneous space X under G over F is trivial, i.e. $H^1(F, G) = \{0\}$.

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Merkurjev–Suslin theorem $\Longrightarrow$ Serre's Conjecture II for groups of inner type A_n.

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Let G be a semisimple simply-connected absolutely simple group of inner type A_n over F.

Then $G \simeq \mathrm{SL}_1(A)$ where A is a central simple algebra over F. The exact sequence

$$1 \to \mathrm{SL}_1(A) \to \mathrm{GL}_1(A) \to \mathbb{G}_m \to 1$$

yields a bijection $F^\times / \mathrm{Nrd}(A^\times) \simeq H^1(F, \mathrm{SL}_1(A))$.

Thus, $H^1(F, \mathrm{SL}_1(A)) = 1 \Longleftrightarrow \mathrm{Nrd}(A^\times) = F^\times$.

Norm residue isomorphism in degree 2 and Serre's Conjecture II

Theorem. (Merkurjev–Suslin) Let F be a perfect field. The following are equivalent:

- $\mathrm{cd}(F) \leq 2$
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The above settles Serre's Conjecture II for groups of inner type A_n and also provides a converse to Conjecture II.

The proof relies on the norm residue isomorphism

$$h_{2,F}\colon K_2(F)/pK_2(F) \xrightarrow{\simeq} H^2(F, \mu_p^{\otimes 2})$$

together with the injectivity of $h_{3,F}$ on symbols.

Norm residue isomorphism in degree 2 and Serre's Conjecture II

The above theorem of Merkurjev–Suslin was fundamental to the solution of Conjecture II for all classical groups.

Theorem. (Eva Bayer, Parimala 1994) Conjecture II holds for all classical groups and groups of type G_2 or F_4.

Norm residue isomorphism in degree 2 and Serre's Conjecture II

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Classical groups:

- special linear groups
- unitary groups
- spinor groups
- symplectic groups

The conjecture is open in general for other exceptional groups.

Function fields of surfaces and E_8

$F = \mathbb{C}(X)$ where X is an integral algebraic complex surface

Theorem. (P. Gille 2001) Serre's conjecture II holds for trialitarian D_4, E_6, and E_7 over F.

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