Milnor Conjectures and Quadratic Forms

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IMA, University of Minnesota
“Milnor had had a deep and affectionate interest in quadratic form theory. . . . His paper Algebraic $K$-theory and quadratic forms continues to guide much of the current research in the algebraic theory of quadratic forms.”

— H. Bass (1993)
Witt rings and filtration

$F$ field, $\text{char}(F) \neq 2$

$(V, q)$ quadratic form over $F$
Witt rings and filtration

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$(V, q)$ quadratic form over $F$

$b_q : V \times V \rightarrow k$ associated bilinear form

\[
b_q(v, w) = \frac{1}{2}(q(v + w) - q(v) - q(w))
\]

$T(q) = \begin{pmatrix} b_q(e_i, e_j) \end{pmatrix}$ Gram matrix of $q$, $\{e_1, \cdots, e_n\}$ a basis of $V$
Witt rings and filtration

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$T(q) = (b_q(e_i, e_j))$ Gram matrix of $q$, $\{e_1, \cdots, e_n\}$ a basis of $V$

$(V, q)$ is nondegenerate if $T(q)$ is invertible.
Witt rings and filtration

\[ h = (F^2, xy) \text{ hyperbolic plane} \]

**Theorem.** (Witt decomposition) \((V, q) = (V_0, q_0) \perp h^r\)

\(q_0\) anisotropic, uniquely determined by \(q\)

\(r = \text{Witt index of } q.\)
h = (F^2, xy) hyperbolic plane

**Theorem.** (Witt decomposition) (V, q) = (V_0, q_0) \perp h^r

q_0 anisotropic, uniquely determined by q

r = Witt index of q.

**Witt equivalence.** q \sim q' \iff q \perp h^m \sim q' \perp h^n \iff q_0 \sim q'_0

W(F) = Witt equivalence classes of quadratic forms under \perp and \otimes.

I(F) ideal of W(F) of even-dimensional forms

I^n(F) = I(F)^n
$n$-fold *Pfister form*:

$$\langle \langle a_1, a_2, \ldots, a_n \rangle \rangle = \langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$$
Witt rings and filtration

$n$-fold *Pfister form*:

\[
\langle\langle a_1, a_2, \ldots, a_n\rangle\rangle = \langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle
\]

\(I^n(F)\) is additively generated by \(n\)-fold Pfister forms
Galois cohomology

\[ \Gamma_F = \text{Gal}(F_s/F) \]

\[ H^n(F, \mathbb{Z}/2\mathbb{Z}) = H^n(\Gamma_F, \mathbb{Z}/2\mathbb{Z}) \]

\[ = \lim_{\text{L/F finite Galois}} \quad H^n(\text{Gal}(L/F), \mathbb{Z}/2\mathbb{Z}) \]
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- \[ H^0(F, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \]
- \[ H^1(F, \mathbb{Z}/2\mathbb{Z}) = F^\times / F^\times 2 \] (Kummer isomorphism) \( (a) \in H^1(F, \mathbb{Z}/2\mathbb{Z}) \) denotes the square class of \( a \in F^\times \)
Galois cohomology

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- \( H^1(F, \mathbb{Z}/2\mathbb{Z}) = F^\times / F^\times 2 \) (Kummer isomorphism)
  \( (a) \in H^1(F, \mathbb{Z}/2\mathbb{Z}) \) denotes the square class of \( a \in F^\times \)
- \( H^2(F, \mathbb{Z}/2\mathbb{Z}) = 2\text{Br}(F) \)
  The cup product \((a). (b)\) represents the quaternion algebra with generators \( i, j \) and relations \( i^2 = a, j^2 = b, ij = -ji \).
Classical invariants for quadratic forms

The dimension mod 2

\[ e_0: W(F) \rightarrow \mathbb{Z}/2\mathbb{Z} = H^0(F, \mathbb{Z}/2\mathbb{Z}) \]

\[ e_0(V, q) = \dim V \text{ (mod 2)} \]

\[ \ker(e_0) = I(F) \]
Classical invariants for quadratic forms

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The \textit{discriminant}

\[ e_1: I(F) \to F^\times/F^\times 2 = H^1(F, \mathbb{Z}/2\mathbb{Z}) \]

\[ e_1(V, q) = ((-1)^n \det T(q)), \ 2n = \dim V \]

\[ \ker(e_1) = I^2(F) \]
Classical invariants for quadratic forms

The *Clifford invariant*

\[ e_2 : l^2(F) \to 2\text{Br}(F) = H^2(F, \mathbb{Z}/2\mathbb{Z}) \]

\[ e_2(V, q) = [C(q)] \in 2\text{Br}(F) \]

\[ C(q) = \text{Clifford algebra of } (V, q) = T(V)/\langle v \otimes v - q(v) \rangle. \]
Classical invariants for quadratic forms

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\[ C(q) = \text{Clifford algebra of } (V, q) = T(V)/\langle v \otimes v - q(v) \rangle. \]

**Example:** \[ e_2(\langle 1, -a \rangle \otimes \langle 1, -b \rangle) = (a) \cdot (b) \in H^2(F, \mathbb{Z}/2\mathbb{Z}) \]
Classical invariants for quadratic forms

Classical questions.

• Is $e_2$ onto?
  Is $\text{Br}_2(F)$ generated by quaternion algebras?
Classical invariants for quadratic forms

Classical questions.

• Is $e_2$ onto?
  Is $\mathcal{2Br}(F)$ generated by quaternion algebras?

• Is $\ker(e_2) = I^3(F)$?
Theorem. (Arason 1975) The assignment

\[ e_n(\langle a_1, \ldots, a_n \rangle) = (a_1) \cdot (a_2) \cdot \cdots \cdot (a_n) \in H^n(F, \mathbb{Z}/2\mathbb{Z}) \]

is well-defined on the isomorphism classes of \( n \)-fold Pfister forms.
**Milnor Conjecture (MC)**

**Theorem.** (Arason 1975) The assignment

\[ e_n(\langle a_1, \ldots, a_n \rangle) = (a_1) \cdot (a_2) \cdot \cdots \cdot (a_n) \in H^n(F, \mathbb{Z}/2\mathbb{Z}) \]

is well-defined on the isomorphism classes of \( n \)-fold Pfister forms.

**Milnor conjecture (1970).** The map \( e_n \) extends to a homomorphism

\[ e_n: \mathcal{I}^n(F) \to H^n(F, \mathbb{Z}/2\mathbb{Z}) \]

which is onto with kernel \( \mathcal{I}^{n+1}(F) \).
Milnor Conjecture (MC)

Equivalently, there is an isomorphism

\[(e_n): \bigoplus_{n \geq 0} I^n(F)/I^{n+1}(F) \longrightarrow \bigoplus_{n \geq 0} H^n(F, \mathbb{Z}/2\mathbb{Z})\]

of the graded Witt ring and the graded cohomology ring.
Milnor Conjecture (MC)

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- MC as stated above is a consequence of two conjectures of Milnor relating Milnor ring $K_* F$ with mod 2 Galois cohomology ring and the graded Witt ring.
Milnor $K$-theory

\[ K_0(F) = \mathbb{Z} \]
\[ K_1(F) = F^\times \]
\[ K_*(F) = T(F^\times)/\langle a \otimes (1 - a), \ a \in F^\times, \ a \neq 1 \rangle \]
\{a_1, \ldots, a_n\} denotes the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n(F)$
Milnor $K$-theory

\[ K_0(F) = \mathbb{Z} \]
\[ K_1(F) = F^\times \]
\[ K_\ast(F) = T(F^\times)/\langle a \otimes (1 - a), \; a \in F^\times, \; a \neq 1 \rangle \]
\{a_1, \ldots, a_n\} denotes the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n(F)$

Milnor’s definition of $K_2$ is inspired by Steinberg, Moore, and Matsumoto. His definition of $K_n$ for $n > 2$ is "purely ad hoc"!
Norm residue map and MC1

$p$ prime, $p \neq \text{char } F$

**Definition.** (Bass–Tate) The Kummer isomorphism

$$F^\times / F^\times p \xrightarrow{\sim} H^1(F, \mu_p)$$

extends to a homomorphism called the *norm residue map*

$$h_{n,F} : K_n(F)/p K_n(F) \longrightarrow H^n(F, \mu_p^\otimes n)$$

sending \( \{a_1, a_2, \ldots, a_n\} \) to \((a_1) \cdot (a_2) \cdots \cdot (a_n)\).
Norm residue map and MC1

Milnor Conjecture (MC1). For $p = 2$, $h_{n,F}$ is an isomorphism for all $n$.

Theorem. (Merkurjev 1981) For $p = 2$, $h_{2,F}$ is an isomorphism.
Norm residue map and MC1

**Milnor Conjecture (MC1).** For $p = 2$, $h_{n,F}$ is an isomorphism for all $n$.

**Theorem.** (Merkurjev 1981) For $p = 2$, $h_{2,F}$ is an isomorphism.

**Theorem.** (Voevodsky 2003) MC1 holds.
Norm residue map and Bloch–Kato conjecture

**Bloch–Kato conjecture.** For general $p$, $h_{n,F}$ is an isomorphism.
**Norm residue map and Bloch–Kato conjecture**

**Bloch–Kato conjecture.** For general $p$, $h_{n,F}$ is an isomorphism.

**Theorem.** (Merkurjev–Suslin 1982) For general $p$, $h_{2,F}$ is an isomorphism.

**Theorem.** (Merkurjev–Suslin, Rost 1990) For general $p$, $h_{3,F}$ is an isomorphism.
Norm residue map and Bloch–Kato conjecture

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**Theorem.** (Merkurjev–Suslin, Rost 1990) For general $p$, $h_{3,F}$ is an isomorphism.

**Theorem.** (Voevodsky, Rost 2009) BKC holds.
Milnor ring and graded Witt ring—MC2

\[ k_n(F) = K_n(F)/2K_2(F) \]

\[ s_{1,F} : k_1(F) \to I(F)/I^2(F) \]

\[ (a) \mapsto \langle 1, -a \rangle \]
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\[ s_{1,F} \text{ induces a homomorphism} \]
\[ s_{n,F} : k_n(F) \to I^n(F)/I^{n+1}(F) \]
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**Milnor Conjecture (MC2).** \( s_{n,F} \) is an isomorphism for all \( n \).
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**Milnor Conjecture (MC2).** \( s_n, F \) is an isomorphism for all \( n \).

**Theorem** (Orlov, Vishik, Voevodsky 2007) MC2 holds.
The Milnor Conjectures

\[ \frac{K_n(F)}{2K_n(F)} \]

\[ S_{n,F} \sim \]

\[ l^n(F)/l^{n+1}(F) \]

\[ e_{n,F} \sim \]

\[ H^n(F, \mathbb{Z}/2\mathbb{Z}) \]

\[ h_{n,F} \sim \]
Classification of quadratic forms by cohomological invariants

Arason–Pfister Hauptsatz. \( \bigcap_{n \geq 0} l^n(F) = 0 \)
Classification of quadratic forms by cohomological invariants

**Arason–Pfister Hauptsatz.** \( \bigcap_{n \geq 0} l^n(F) = 0 \)

Together with MC, one has:

**Theorem.** Given anisotropic quadratic forms \( q_1, q_2 \), if \( e_n(q_1 - q_2) = 0 \) for all \( n \), then \( q_1 \simeq q_2 \).
An element of the form \((a_1) \cdot (a_2) \cdot \cdots \cdot (a_n) \in H^n(F, \mathbb{Z}/2\mathbb{Z})\) is called a *symbol*.

\(\text{MC} \implies H^n(F, \mathbb{Z}/2\mathbb{Z})\) is generated by symbols.
Symbol length in Galois cohomology

An element of the form \((a_1) \cdot (a_2) \cdots (a_n) \in H^n(F, \mathbb{Z}/2\mathbb{Z})\) is called a *symbol*.

MC \(\implies H^n(F, \mathbb{Z}/2\mathbb{Z})\) is generated by symbols.

The *symbol length* of \(\xi \in H^n(F, \mathbb{Z}/2\mathbb{Z})\) is the least \(r\) such that \(\xi\) is a sum of \(r\) symbols.

The *n-symbol length* of \(F\) is the maximum symbol length of elements in \(H^n(F, \mathbb{Z}/2\mathbb{Z})\).
Symbol length in Galois cohomology

Boundedness of symbol lengths is an arithmetic property of $F$.

**Example.** The 2-symbol length of a local or global field is 1.
Symbol length in Galois cohomology

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**Question.** Is there a bound for the symbol length of finitely generated fields?
Symbol length in Galois cohomology

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**Example.** The 2-symbol length of a local or global field is 1.

**Question.** Is there a bound for the symbol length of finitely generated fields?

This question has interesting consequences for the $u$-invariant of a field.
$u$-invariant and symbol length

$$u(F) = \max\{\dim(q) : q \text{ anisotropic quadratic form over } F\}$$

Examples.

- $u(F_p) = 2$
- $u(Q_p) = 4$
- $u(Q(\sqrt{-1})) = 4$
- $u(R) = \infty$

Easy consequence of Voevodsky's theorem:

Theorem. Suppose $H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \geq n_0$ and the $i$-symbol length of $F$ is bounded for $i \leq n_0$. Then $u(F) < \infty$. 
$u$-invariant and symbol length

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Easy consequence of Voevodsky's theorem:
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**Examples.**
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Easy consequence of Voevodsky’s theorem:

**Theorem.** Suppose \( H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0 \) for \( n \geq n_0 \) and the \( i \)-symbol length of \( F \) is bounded for \( i \leq n_0 \). Then \( u(F) < \infty \).
**u-invariant and symbol length**

**Idea of Proof.** Supposing that $H^3(F, \mathbb{Z}/2\mathbb{Z}) = 0$ and the 2-symbol length of $F \leq r$.

Let $q$ be an anisotropic quadratic form of dimension $2n$, $\text{disc}(q) = 1$, and

$$e_2(q) = \sum_{1 \leq i \leq r} (a_i) \cdot (b_i).$$

then MC implies

$$q - \sum_{1 \leq i \leq r} \langle 1, -a_i \rangle \otimes \langle 1, -b_i \rangle \in \mathfrak{i}^3(F) = 0.$$
**u-invariant and symbol length**

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Let $q$ be an anisotropic quadratic form of dimension $2n$, $\text{disc}(q) = 1$, and

$$ e_2(q) = \sum_{1 \leq i \leq r} (a_i) \cdot (b_i). $$

then MC implies

$$ q - \sum_{1 \leq i \leq r} \langle 1, -a_i \rangle \otimes \langle 1, -b_i \rangle \in l^3(F) = 0. $$

Thus

$$ q \sim \sum_{1 \leq i \leq r} \langle 1, -a_i \rangle \otimes \langle 1, -b_i \rangle; $$

Thus $\text{dim}(q) \leq 4r$ and $u(F) \leq 2 + 4r$. 
Recall. \( u(\mathbb{Q}_p) = 4 \)

Finiteness of \( u(\mathbb{Q}_p(t)) \) was open until late 90s.

Open question list of Lam’s book: \( u(\mathbb{Q}_p(t)) = 8? \)
Recall. $u(\mathbb{Q}_p) = 4$

Finiteness of $u(\mathbb{Q}_p(t))$ was open until late 90s.

Open question list of Lam’s book: $u(\mathbb{Q}_p(t)) = 8$?

The finiteness of $u(\mathbb{Q}_p(t))$ came as a consequence of symbol length bounds.
Theorem. (Saltman 1997) For $p \neq 2$, the 2-symbol length of $\mathbb{Q}_p(t)$ is 2.

Theorem. (Parimala–Suresh 1998) For $p \neq 2$, the 3-symbol length of $\mathbb{Q}_p(t)$ is one.
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Theorem. (Parimala–Suresh 2007) For \( p \neq 2 \), \( u(\mathbb{Q}_p(t)) = 8 \).
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Theorem. (Parimala–Suresh 2007) For $p \neq 2$, $u(\mathbb{Q}_p(t)) = 8$.

Theorem. (Heath-Brown, Leep 2010) For all $p$, $u(\mathbb{Q}_p(t)) = 8$. (Methods very different from Galois cohomology)
Converse Question. Suppose $u(F) < \infty$. Is the $n$-symbol length of $F$ bounded for all $n$?
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Observation. If $u(F) < \infty$ then $H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \gg 0$. 
Converse Question. Suppose $u(F) < \infty$. Is the $n$-symbol length of $F$ bounded for all $n$?

Observation. If $u(F) < \infty$ then $H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \gg 0$.

Proof. $u(F) < \infty \implies n$-fold Pfister forms are zero in $W(F)$ for all $n \gg 0$

$\implies I^n(F) = 0$ for $n \gg 0$

$\implies H^n(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \gg 0$ (MC).
$u$-invariant and symbol length

Special case. $u(F) < \infty \implies$ 3-symbol length is bounded.
$u$-invariant and symbol length

**Special case.** $u(F) < \infty \Rightarrow$ 3-symbol length is bounded.

**Proof.** Let $2n_0 > u(F)$, $q_0 = \langle x_1, x_2, \ldots, x_{2n_0-1}, \epsilon x_1 x_2 \cdots x_{2n_0-1} \rangle$, $\epsilon = \pm 1$, *generic form* of dimension $2n_0$ and discriminant 1.

$X =$ Brauer-Severi variety of $C(q_0)$ over $F(x_1, \ldots, x_{2n_0-1})$

$\tilde{F} = F(x_1, \ldots, x_{2n_0-1})(X)$

$q_0 \in I^3(\tilde{F})$: suppose $e_3(q_0)$ is a sum of $r$ symbols in $H^3(\tilde{F}, \mathbb{Z}/2\mathbb{Z})$.

$\xi \in H^3(F, \mathbb{Z}/2\mathbb{Z})$, $\xi = e_3(q_1)$, $q_1 \in I^3(F)$:

$\dim(q_1) = 2n_0$, $\text{disc } q_1 = 1$, $e_2(q_1) = 0$.

$\exists$ specialisation $X \rightarrow F$, $q_0 \mapsto q_1$, $e_3(q_0) \mapsto e_3(q_1) = \xi$

$\Rightarrow \xi$ is a sum of at most $r$ symbols.
$u$-invariant and symbol length

There are no analogous varieties $X/F(x_1, \ldots, x_{2n_0-1})$ to push the generic form $q_0$ into $I^d(F(x_1, \ldots, x_{2n_0-1})(X))$ for $d \geq 4$ with good specialization properties.
There are no analogous varieties $X/F(x_1, \ldots, x_{2n_0-1})$ to push the generic form $q_0$ into $I^d(F(x_1, \ldots, x_{2n_0-1})(X))$ for $d \geq 4$ with good specialization properties.

**Theorem.** (Saltman 2011) There exist finitely many function fields $F_{i,d}/F(x_1, \ldots, x_{2n_0-1})$ such that

- $q_0 \in I^d(F_{i,d})$
- Given $q \in I^d(F)$, $\exists$ specialisation $F_{j,d} \rightarrow F$ for some $j$ with $q_0 \mapsto q$

**Corollary.** $u(F) < \infty \implies$ $i$-symbol length is bounded for all $i$. 
Pythagoras number

**Definition.** $p(F)$ is the least $n$ such that every sum of squares in $F$ is a sum of $n$ squares.

**Example.** (Euler 1754) $p(\mathbb{Q}) = 4$
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**Theorem.** (Pfister’s effective Hilbert’s 17th problem 1967)

$$p(\mathbb{R}(x_1, \ldots, x_n)) \leq 2^n$$
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**Theorem.** (Hilbert 1888, Cassels–Ellison–Pfister 1971)

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General bounds: $n + 2 \leq p(\mathbb{R}(x_1, \ldots, x_n)) \leq 2^n$, for $n \geq 3$. 

Interesting questions concern $p(\mathbb{Q}(t_1, \ldots, t_n))$.

**Conjecture.** (Pfister) Let $F$ be a function field of transcendence degree $d$ over a number field $k$. Then

- $p(F) \leq 5$ if $d = 1$
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**Theorem.** (Landau 1906, Pouchet 1971, Hsia-Johnson 1974)

$p(\mathbb{Q}(t)) = 5$

**Theorem.** (J.-L. Colliot-Thélène 1986, F. Pop (unpublished))

For $d = 1$, $p(F) \leq 6$
Pythagoras number

Theorem. (Arason 2000) MC $\iff p(F) \leq 2^{d+2}$. 
**Theorem.** (Arason 2000) $\text{MC} \implies p(F) \leq 2^{d+2}$.

**Proof.** $H^n(F(\sqrt{-1}), \mathbb{Z}/2\mathbb{Z}) = 0$ for $n > d + 2$
(since $\text{cd}(F(\sqrt{-1})) = d + 2$)

$\implies I^n(F(\sqrt{-1})) = 0$ for $n > d + 2$ (MC)

$\implies n$-fold Pfister forms are universal over $F(\sqrt{-1})$, for $n = d + 2$

$\implies p(F) \leq 2^{d+2}$ (Pfister)
There are connections between Pfister’s conjecture and a conjecture of Kato.
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**Conjecture.** (Kato 1986) Let $k$ be a number field, $\Omega_k$ its set of places, and $X$ a geometrically integral variety over $k$ of dimension $d$. The restriction map

$$H^{d+2}(k(X), \mathbb{Z}/2\mathbb{Z}) \to \prod_{v \in \Omega_k} H^{d+2}(k_v(X), \mathbb{Z}/2\mathbb{Z})$$

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**Theorem.** (Colliot-Thélène and Jannsen 1991)

$MC + \text{Kato’s conjecture} \implies \text{Pfister’s conjecture for } d \geq 2.$
Concerning Kato’s conjecture:

- \( \dim X = 0 \): the Hasse–Brauer–Noether theorem.
- \( \dim X = 1 \): Kato 1986.
- \( \dim X \geq 2 \): Jannsen 1989, 2009.
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Remaining open case of Pfister’s conjecture. \( p(F) = 5 \) where \( F \) has transcendence degree one over a number field.
Kato’s conjecture

Sketch of CT-J proof of Pfister’s conjecture.

Let $F = k(X)$ and $F_v = k_v(X)$ for $v \in \Omega_k$.

Let $\phi_{d+1} = \langle 1, \ldots, 1 \rangle$ be the $(d + 1)$-fold Pfister form.

Let $f$ be a sum of squares in $F$. 
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**Claim.** $\phi_{d+1} \otimes \langle 1, -f \rangle = 0$ in $W(F_v)$ for every $v \in \Omega_k$.

This fact is easy to check over finite completions. Over a real completion, it’s a consequence of Pfister’s effective Hilbert 17th problem.
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Thus $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f \rangle) = 0$ in $H^{d+2}(F_v, \mathbb{Z}/2\mathbb{Z})$ for all $v \in \Omega_k$

$\implies e_{d+2}(\phi_{d+1} \otimes \langle 1, -f \rangle) = 0$ in $H^{d+2}(F, \mathbb{Z}/2\mathbb{Z})$ (Kato conj.)

$\implies \phi_{d+1} \otimes \langle 1, -f \rangle \in I^{d+3}(F)$ (MC)

$\implies \phi_{d+1} \otimes \langle 1, -f \rangle = 0$ (by Arason–Pfister Hauptsatz)

$\implies f$ is a sum of $2^{d+1}$ squares in $F$.  


Norm residue isomorphism in degree 2 and Serre’s Conjecture II

The norm residue isomorphism

$$h_{2,F} : K_2(F)/pK_2(F) \sim H^2(F, \mu_p \otimes^2)$$

due to Merkurjev–Suslin has deep consequences in the direction of a conjecture of Serre.
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**Definition.** $F$ has **cohomological dimension** $\text{cd}(F) \leq n$ if $H^m(F, M) = 0$ for all finite discrete Galois modules $M$, and all $m \geq n + 1$. 
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- $\text{cd}(\mathbb{F}_q) = 1$
- $\text{cd}(\mathbb{C}(t)) = 1$
- $\text{cd}(\mathbb{Q}_p) = 2$
- $\text{cd}(\mathbb{Q}(\sqrt{-1})) = 2$
- $\text{cd}(\mathbb{C}(t_1, t_2)) = 2$
Norm residue isomorphism in degree 2 and Serre’s Conjecture II

**Conjecture II.** (Serre) Let $F$ be a perfect field of $\text{cd}(F) \leq 2$ and $G$ a semisimple simply-connected linear algebraic group defined over $F$. Then every principal homogeneous space $X$ under $G$ over $F$ is trivial, i.e. $H^1(F, G) = \{0\}$. 
Norm residue isomorphism in degree 2 and Serre’s Conjecture II

Merkurjev–Suslin theorem $\iff$ Serre’s Conjecture II for groups of inner type $A_n$. 
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Let $G$ be a semisimple simply-connected absolutely simple group of inner type $A_n$ over $F$.

Then $G \simeq \text{SL}_1(A)$ where $A$ is a central simple algebra over $F$.

The exact sequence

$$1 \rightarrow \text{SL}_1(A) \rightarrow \text{GL}_1(A) \rightarrow \mathbb{G}_m \rightarrow 1$$

yields a bijection $F^\times / \text{Nrd}(A^\times) \simeq H^1(F, \text{SL}_1(A))$.

Thus, $H^1(F, \text{SL}_1(A)) = 1 \iff \text{Nrd}(A^\times) = F^\times$. 
Norm residue isomorphism in degree 2 and Serre’s Conjecture II

**Theorem.** (Merkurjev–Suslin) Let $F$ be a perfect field. The following are equivalent:

- $\text{cd}(F) \leq 2$
- $\text{Nrd}: A^\times \to L^\times$ is onto for any finite extensions $L/F$ and central simple algebra $A$ over $L$. 

The above settles Serre’s Conjecture II for groups of inner type $A_n$ and also provides a converse to Conjecture II. The proof relies on the norm residue isomorphism $h_2, F: K_2(F)/pK_2(F) \xrightarrow{\sim} H_2(F, \mu \otimes p^2)$ together with the injectivity of $h_3, F$. 

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The above theorem of Merkurjev–Suslin was fundamental to the solution of Conjecture II for all classical groups.

**Theorem.** (Eva Bayer, Parimala 1994) Conjecture II holds for all classical groups and groups of type $G_2$ or $F_4$. 
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Classical groups:

- special linear groups
- unitary groups
- spinor groups
- symplectic groups

The conjecture is open in general for other exceptional groups.
Function fields of surfaces and $E_8$

$F = \mathbb{C}(X)$ where $X$ is an integral algebraic complex surface

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