New Domain Decomposition Algorithms From Old

Olof Widlund
http://www.cs.nyu.edu/cs/faculty/widlund/
Joint projects with Clark Dohrmann.

Need for Parallel Computing

Finite element modeling of elastic problems, as in fluid mechanics, and electro-magnetics, often requires very many degrees of freedom if there is a lot of geometric detail and (or) great variations in the material parameters. Domain decomposition research addresses these issues.

Recent experiments reported in Rheinbach and Klawonn, ZAMM 90(1), 2010, illustrates what can be done with an elasticity problems in two dimensions with large coefficient contrasts, on a large BlueGene system. A 3-level FETI–DP algorithm was implemented in PETSC. The problem had about $2.7 * 10^9$ degrees of freedom and 65536 processors and 36 seconds were used. A problem with about $3 * 10^6$ degrees of freedom and 64 processors was also run. It was about 18% faster. Elastic problems in three dimensions would have had to be smaller.

Many people work on systems with hundreds of processors.

Domain Decomposition Methods

We wish to solve the large systems of algebraic equations, arising in finite element approximations, fast and reliably. Divide the domain into often many non-overlapping subdomains Ω_i.

Two main families: the *iterative substructuring algorithms*, using solvers on the Ω_i, each with tens of thousands of degrees of freedom, and the *overlapping Schwarz methods*, using solvers on an overlapping set of subdomains Ω_i', e.g., obtained by adding layers of elements to the Ω_i.

The preconditioners – approximate inverses – of the finite element problem are built from these solvers and all preconditioners considered also include a coarse, global solver with a few degrees of freedom for each subdomain. Accelerate by a conjugate gradient method.

We will discuss almost incompressible elasticity; our results also hold for the compressible case and for problems where some subdomain problems are discretized using standard finite element methods and others with mixed finite elements, with discontinuous pressure approximations. We will also touch upon recent work on H(curl), H(div), and Reissner-Mindlin plates.

We have developed overlapping Schwarz methods and theoretical results for related saddle point problems (with large parameters). Experimental work and two papers by Klawonn and Pavarino over a decade ago; their algorithms have been used relatively extensively. Our coarse spaces different and more generous; our experiments discussed in DD17 paper, SINUM vol. 47(4), and Internat. J. Numer. Meth. Engng. vol. 82 and Dohrmann's talk yesterday. Algorithms implemented in SALINAS by Dohrmann.

Desirable properties of domain decomposition solvers

- Should handle arbitrary jumps in material properties between subdomains.
- Use of approximate local solvers should affect iteration count marginally.
- Should work even if stiffness matrix is assembled.
- Straightforward to implement in parallel and scalable.
- Should be well supported by theory.
- Should be robust with respect to parameters.
- Well defined for and insensitive to irregularity of subdomains.
- Should handle jumps inside subdomains.

Assumptions in Previous Work

In previous theory for iterative substructuring methods, we typically assume that, *the partition into subdomains Ω_i is such that each subdomain is the union of shape-regular coarse tetrahedral elements of a global conforming mesh $\mathcal{T}_{\mathcal{H}}$ and the number of such tetrahedra forming an individual subdomain is uniformly bounded;*

A bit more general than assumption on geometric multigrid.

Also, *the variation of the material properties is in a relatively narrow range for each subdomain.*

In the standard theory for two-level overlapping Schwarz methods, with conventional coarse spaces, defined on a coarse triangulation, *the coefficients should not vary a lot in the entire region.*

We will deal with jumps across the interface only by using more exotic coarse spaces. We can then draw on knowledge on what works for iterative substructuring methods. The Lamé parameters can then change a lot.

Why are the geometric assumptions on the subdomains unsatisfactory? Subregions change with the mesh size and might not be Lipschitz when resulting from a mesh partitioner. Can we develop bounds that only depend on a few geometric parameters?

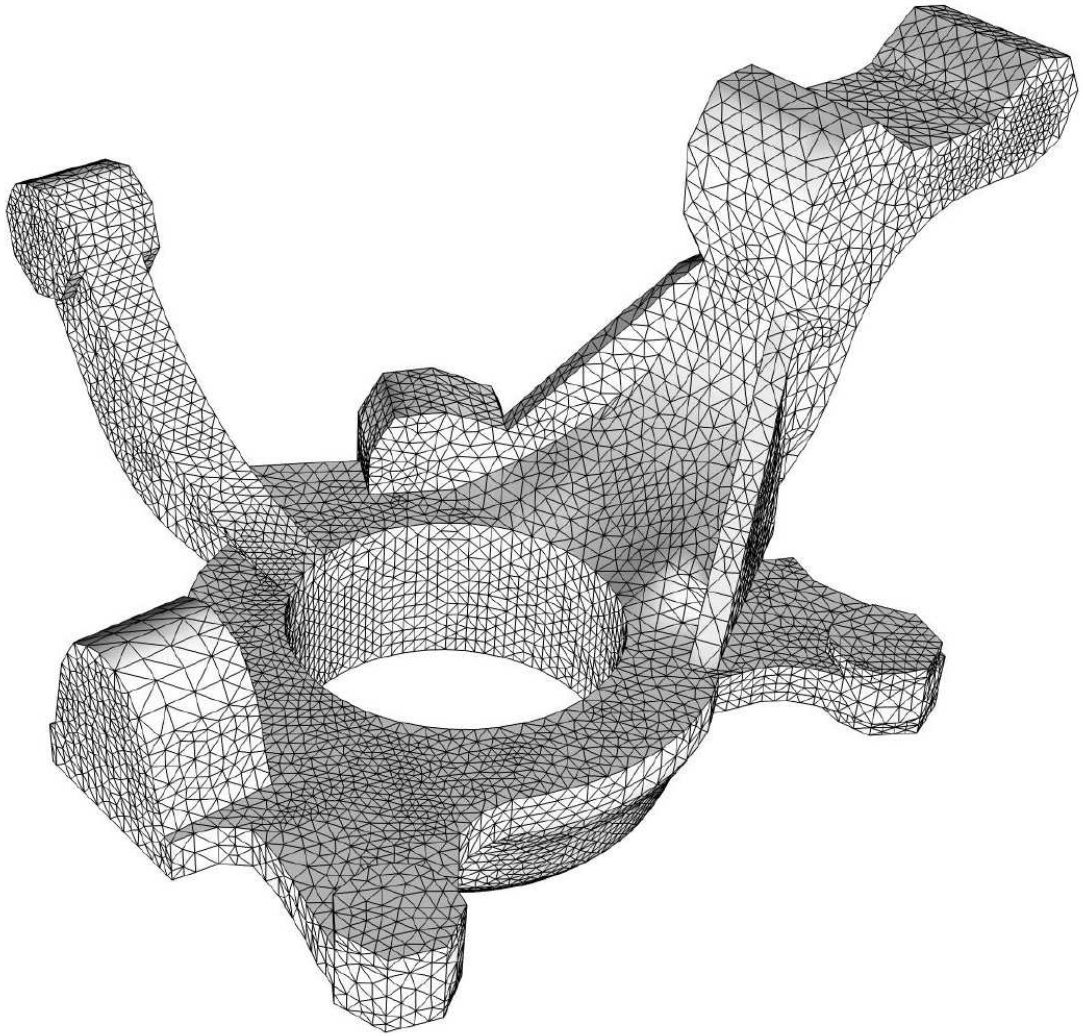

Figure 1: Finite element meshing of a mechanical object.

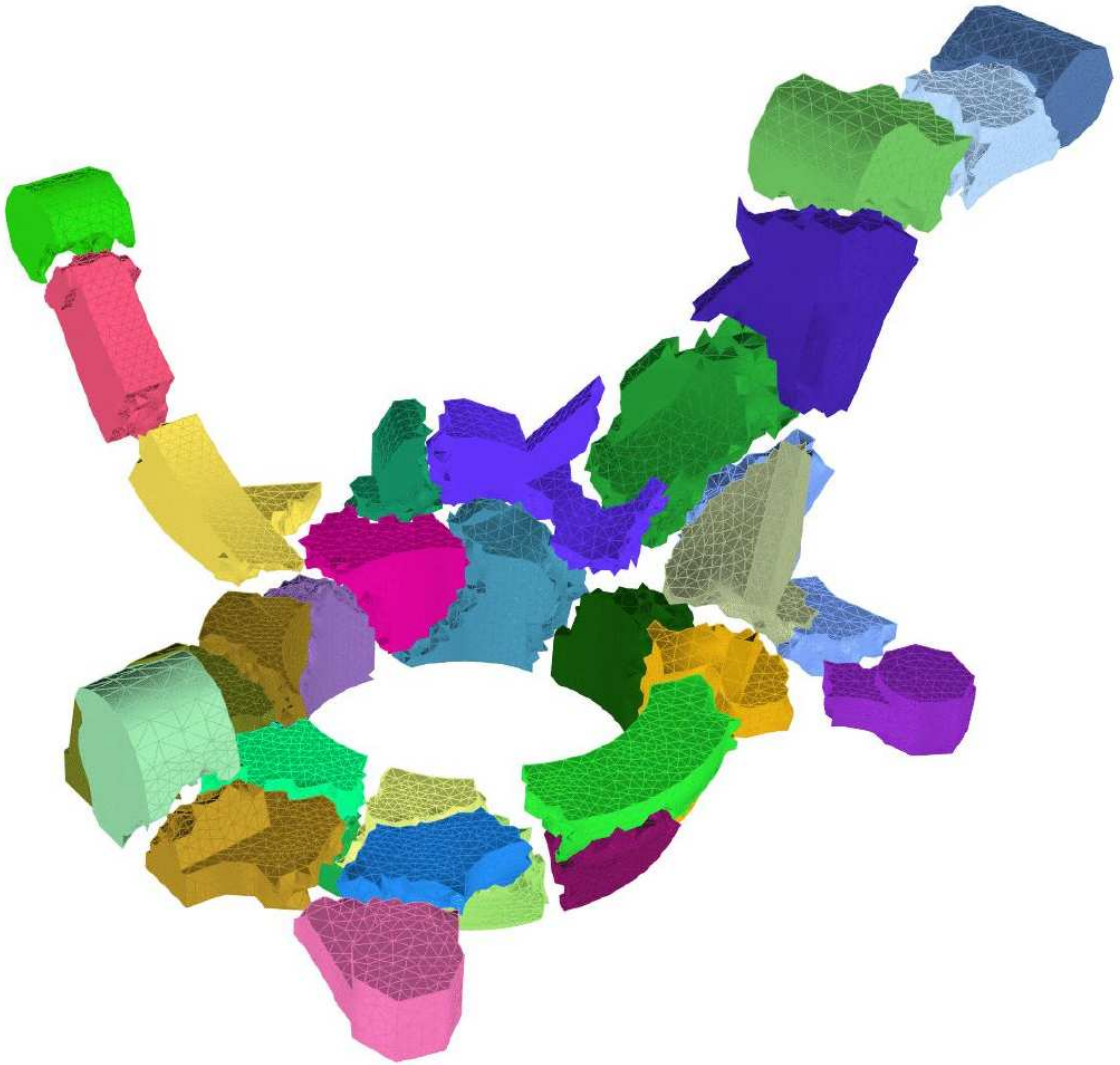

Figure 2: Partition into thirty subdomains. Courtesy Charbel Farhat.

These subdomains effectively provide our coarse mesh.

Faces, edges, and vertices of quite general subdomains can be defined in terms of certain equivalence classes of finite element nodes. These geometric objects are central in our construction of coarse problems and in the theory. These sets also highly relevant for parallel computing.

We will use face, edge, and vertex functions, providing a partition of unity on the interface. A face function $\theta_{F^{ij}}$ equals 1 at all nodes of a face common to two subdomains Ω_i and Ω_j and vanishes at all other interface nodes. They are extended as $discrete\ harmonic$ functions, i.e., with minimal elastic energy; this determines the values at interior nodes. Similarly, we have edge functions and vertex functions.

The restriction of the rigid body modes – all linear functions – to faces and edges, are used for problems of elasticity. The coarse space needs to accommodate all rigid body modes.

Overlapping Schwarz Methods

Consider a scalar elliptic problem, for $n = 3$, and a coarse space which is the range of the interpolation operator

$$I_B^h u(x) = \sum_{V^k \in \Gamma} u(V^k)\theta_{V_k}(x) + \sum_{E^\ell \subset \Gamma} \bar{u}_{E^\ell}\theta_{E^\ell}(x) + \sum_{F^{ij} \subset \Gamma} \bar{u}_{F^{ij}}\theta_{F^{ij}}(x).$$

Here $\bar{u}_{E^i}$ and $\bar{u}_{F^k}$ are averages over edges and faces of the subdomains.

$\theta_{V_k}(x)$ is essentially the standard nodal basis functions of a vertex of the subdomains, $\theta_{E^\ell}(x) = 1$ at the nodes of the edge E^i and vanishes at all other interface nodes, and $\theta_{F^{ij}}(x)$ is the similar function already defined for the face F^{ij}. These functions are extended as discrete harmonic functions into the interior of the subdomains. This interpolation operator I_B^h preserves constants; the *null space condition*.

For nice enough subregions, we have

$$|u - I_B^h u|^2_{H^1(\Omega_i)} \leq C(1 + \log(H_i/h_i))|u|^2_{H^1(\Omega_i)}.$$

We use Cauchy-Schwarz inequality, a trace theorem, bounds on face and edge functions, and finite element Sobolev inequalities; for regular subdomain these latter estimates were developed and used years ago. We can handle coefficient jumps across the interface since the bounds are local.

Recently, a bound of similar quality has been developed in 2D based on coarse vertex basis functions only. This is easy for regular subdomains; the new result is for subdomains $uniform$ in the sense of Peter Jones. The subdomains can be prefractal as in the construction of snow flake domains. These basis functions vary linearly from 1 to 0 in the direction of a vector connecting the endpoint of the edge and is constant in the perpendicular direction.

Jones Domains (or Uniform or (ϵ, δ) Domains)

A domain Ω is a Jones domain if there exists a constant C_U such that any pair of points $x_1 \in \Omega$ and $x_2 \in \Omega$ can be joined by a rectifiable curve $\gamma(t) : [0,1] \to \Omega$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$, and diameter$(\gamma) \leq C_U|x_1 - x_2|$ and $\min_{i=1,2} |x_i - \gamma(t)| \leq C_U$ distance$(\gamma(t), \partial\Omega)$ for all $t \in [0,1]$.

This is the right family for iterative substructuring methods and two dimensions. Uniformity then provides necessary and sufficient conditions for the existence of extension operators in W_p^k; Peter Jones (Acta Math, 1981). Jones uses a construction of Whitney (1934); so does Stein in his 1970 book for other results. For $n = 3$, the condition is sufficient but not necessary for bounded extensions.

Linear Elasticity

Find displacement $\mathbf{u} \in \mathbf{V}$ of the domain Ω, fixed along $\partial\Omega_D$, with a surface force of density $\mathbf{g}$, along $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$, and a body force $\mathbf{f}$:

$$2\int_\Omega \mu\ \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx + \int_\Omega \lambda\ \text{div}\ \boldsymbol{u}\ \text{div}\ \boldsymbol{v}\ dx\ = < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V}.$$

Here $\lambda(x)$ and $\mu(x)$ are the Lamé parameters and

$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

the linearized strain tensor.

Two inner products defined by

$$\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}),$$

$$< \mathbf{F}, \mathbf{v} > = \int_{\Omega} \sum_{i=1}^{3} f_i v_i \; dx + \int_{\partial\Omega_N} \sum_{i=1}^{3} g_i v_i \; dA.$$

The Lamé parameters expressed in terms of the Poisson ratio ν and Young's modulus E:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

When $\nu \rightarrow 1/2$, we go to the incompressible limit.

Almost Incompressible Elasticity

In this case, there is locking and very slow convergence of conventional finite element methods.

A well-known remedy is based on introducing the new variable $p = -\lambda \text{div } \boldsymbol{u} \in U \subset L^2(\Omega)$, called the pressure, and replacing the pure displacement problem with a mixed formulation: find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$\begin{cases} 2\int_\Omega \mu\ \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx & - & \int_\Omega \text{div } \boldsymbol{v}\ p\ dx & = & < \mathbf{F}, \mathbf{v} > & \forall \mathbf{v} \in \mathbf{V} \\ -\int_\Omega \text{div } \boldsymbol{u}\ q\ dx & - & \int_\Omega 1/\lambda\ pq\ dx & = & 0 & \forall q \in U; \end{cases}$$

Use a mixed, inf-sup stable finite element method, such as $Q_2 - P_1$, with a discontinuous approximation of $p = -\lambda \operatorname{div} \mathbf{u}$.

The term $\int_\Omega \lambda \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{u} dx$ can dominate the energy norm. Since λ is large but finite, we can eliminate the discontinuous pressure variable on the element level from the saddle point problem. The resulting matrix is then symmetric, positive definite, and very ill conditioned. Will the same domain decomposition algorithms, as for the compressible case, still be fast?

We will assume that the problems on individual subdomains Ω_i, into which Ω is subdivided, will be homogenous with close to constant Lamé parameters. We will allow arbitrarily large jumps across the interface

$$\Gamma := \cup \partial\Omega_i \setminus \partial\Omega.$$

Each subdomain is the union of elements of the triangulation of Ω.

As noted, good spaces for elasticity are obtained by multiplying the rigid body modes by the face and edge functions. The interpolation operator can then preserve all rigid body modes. The coefficients built from averages and first order moments. Results in a large coarse space.

We can also work with a smaller coarse space, as in second paper with Dohrmann, by replacing vertex, edge and face contributions by fewer terms. The new coarse basis functions are defined as linear combinations of those of the larger space and in terms of simple least squares problems. We should keep one independent face degree of freedom associated with the normal displacement. The dimension of this coarse space is about half of that of the older one.

In our proofs, we select an element $\mathbf{u}_0 \in V_0$ which has the same net flux as $\mathbf{u}$ across all individual edges or faces of the interface; for this we use the remaining coarse face degree of freedom. This makes it possible to have a divergence-free extension of the interface values of $\mathbf{u} - \mathbf{u}_0$.

Submatrices of assembled stiffness matrices can be used to compute the interior values of the basis elements of the coarse space.

The region is also covered by overlapping subregions Ω_i'. δ_i/H_i measures the relative overlap between adjacent subregions, each of which is a union of elements. The local spaces chosen for the Schwarz methods are

$$V_i = V^h \cap H_0^1(\Omega_i'), \quad i > 0.$$

The standard overlapping subdomains Ω_i' are obtained by repeatedly adding layers of elements starting with Ω_i. Another interesting choice is to work with the Ω_i and $\Omega_{i\delta}$. The latter obtained by adding layers of elements on both sides of $\Gamma_i := \partial\Omega_i \cap \Gamma$. By using a hybrid Schwarz methods, we can make all residuals interior to the Ω_i vanish in each step. And we need solvers on $\Omega)i$ to generate the coarse basis functions.

Schwarz Methods

Schwarz 1870, Pierre-Louis Lions 1987: $(I - P_2)(I - P_1)$.

Standard two-level additive 1988: $P_0 + \sum_{i \geq 1} P_i$.

Standard hybrid, e.g., as in balancing N-N: $P_0 + (I - P_0) \sum_{i \geq 1} P_i (I - P_0)$.

New hybrid: $(I - \sum_{i \geq 1} P_i)(P_0 + \sum_{i \geq 1} P_{i\delta})(I - \sum_{i \geq 1} P_i)$.

Note that $(I - \sum_{i \geq 1} P_i)$ is a projection since the subdomains Ω_i do not intersect. Therefore, after the first iteration, we need only apply this operator once per step.

Also note that the residuals vanish in the interior of the subdomains, which allow us to save storage.

Standard Theorem: (Dryja and W., 1994). *The condition number of the additive Schwarz method using a conventional coarse space V^H satisfies*

$$\kappa(T_{as}) \leq C \max_i (1 + H_i/\delta_i).$$

Under assorted assumptions, e.g., a conventional coarse space and constant coefficients, the constant C is independent of the parameters H_i, h, and δ_i.

The elements should be shape regular, but not necessarily quasi-uniform.

Susanne Brenner has shown that this result is sharp. Without a coarse space component, H/δ must be replaced by $1/(H\delta)$. Poincaré's inequality and, for elasticity, Korn's inequality play key roles in any estimate of the coarse interpolation error $w = u - u_0$.

The proof is by using a well-known decomposition lemma: find for each $u \in V^h$, $\{u_i\}$, $u_i \in V_i$, such that $u = \sum_0^N R_i^T u_i$, and a parameter C_0 such that

$$\sum_0^N a(R_i^T u_i, R_i^T u_i) \leq C_0^2 a(u,u), \quad \forall u \in V^h.$$

This, in a sense, measures the departure from having a set of orthogonal subspaces. The best possible C_0^{-2} equals the smallest eigenvalue of the preconditioned operator relevant to the iteration.

The local components can be defined by

$$u_i = R_i(I^h(\theta_i w)) \in V_i, \quad 1 \leq i \leq N.$$

Here $\{\theta_i\}$ is a piecewise linear partition of unity associated with the overlapping partition. This does not work for almost incompressible case.

Result for Overlapping Schwarz Method

Theorem *The condition number of the preconditioned operator T_{as} satisfies*

$$\kappa(T_{as}) \leq C(1 + H/\delta)(1 + \log(H/h)).$$

Here C is independent of the mesh size, the number of subdomains, the Lamé parameters, as long as the material compressible. H/δ measures the relative overlap between neighboring overlapping subregions, and H/h the maximum number of elements across any subregion.

What needs to be done in the almost incompressible case?.

For the bounds on the local components, first eliminate the interior parts of $\mathbf{w} := \mathbf{u} - \mathbf{u}_0$ vis a vis the subdomains Ω_i. They are easy to bound.

Decompose the remaining discrete, piece-wise, elasto-harmonic part: by construction, the net fluxes of $\mathbf{w}$ vanish across all interface faces of Γ. The next step involves partitioning the trace of $\mathbf{w}$ *on the interface*, at a cost of two (different) logarithmic factors, and assigning appropriate boundary values for subsets of the Ω_i' or $\Omega_{i\delta}$; each should satisfy the zero net flux condition. The elements of the local components $\mathbf{u}_i$ are then constructed as divergence-free extensions on the relevant subsets. Some of these subsets necessarily have poor aspect ratios and this is reflected in the bound, valid for the method with the richer coarse space:

$$\kappa(T_{as}) \leq C(1 + \log(H/h))(1 + \log(H/\delta))(1 + H/\delta)^3.$$

A factor $(1+H/\delta)^2$ originates from the inf-sup constants, (Dobrowolski 2003), which enters when comparing the elastic energy with the square of the norm of $(H^1)^n$. The third factor has the same origin as for the case of Poisson's equation; also related to the subsets with bad aspect ratios.

Experiments reported in the two papers with Dohrmann, previously cited. Dohrmann also has a working variant for the case of continuous pressure spaces, but no theory yet. The results has been extended to H(curl) and edge elements in two dimensions with Jones subdomains; also with Dohrmann, and to H(div) and Raviart–Thomas elements and Reissner-Mindlin plates and Falk–Tu mixed finite elements by Duk-soon Oh and Jong Ho Lee, respectively.

Oh's and Lee's bounds are of the form $C(1+\log(H/h)(1+H\delta)$ and $C(1+H/\delta)^3(1+\log(H/h))^2$, resepectively. In the latter bound, the constant C is independent of the thickness of the plate.

$H(\mathbf{curl})$ problems in 2D

Consider the variational problem: Find $\boldsymbol{u} \in H_0(\mathsf{curl};\Omega)$ such that

$$a(\boldsymbol{u},\boldsymbol{v})_\Omega = (\boldsymbol{f},\boldsymbol{v})_\Omega \quad \forall \boldsymbol{v} \in H_0(\mathsf{curl};\Omega),$$

where

$$a(\boldsymbol{u},\boldsymbol{v})_\Omega := \int_\Omega [\alpha \nabla \times \boldsymbol{u} \nabla \times \boldsymbol{v} + \beta \boldsymbol{u} \cdot \boldsymbol{v}]\, dx, \quad (\boldsymbol{f},\boldsymbol{v})_\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx. \quad (1)$$

Note that $\|\boldsymbol{u}\|^2_{H(\mathsf{curl};\Omega)} = a(\boldsymbol{u},\boldsymbol{u})$ for $\alpha = \beta = 1$. Rewrite, for piecewise constant coefficients,

$$a(\boldsymbol{u},\boldsymbol{v})_\Omega = \sum_{i=1}^{N} (\alpha_i (\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v})_{\Omega_i} + \beta_i (\boldsymbol{u},\boldsymbol{v})_{\Omega_i}).$$

There are now two relevant finite element spaces $W^{h_i}_{\text{curl}}$ of lowest order triangular edge (Nédeléc) elements and $W^{h_i}_{\text{grad}}$ of standard piecewise linear continuous elements, on the same triangulation. The Nédeléc elements are $H(\mathbf{curl})-$conforming with constant tangential components on each edge of the triangulation with values common across each edge.

A key to our work is, with Π^{h_i} the interpolant into the Nédeléc space: *For any* $\boldsymbol{u}_h \in W^{h_i}_{curl}$, *there exist* $\boldsymbol{\Psi}_h \in (W^{h_i}_{grad})^2$, $p_h \in W^{h_i}_{grad}$, *and* $q_h \in W^{h_i}_{curl}$, *such that*

$$
\begin{aligned}
\boldsymbol{u}_h &= \boldsymbol{q}_h + \Pi^{h_i}(\boldsymbol{\Psi}_h) + \nabla p_h, \\
\|\nabla p_h\|^2_{L^2(\mathcal{D})} &\leq C \left(\|\boldsymbol{u}_h\|_{L^2(\mathcal{D})} + d^2 \|\nabla \times \boldsymbol{u}_h\|^2_{L^2(\mathcal{D})} \right), \\
\|h^{-1}\boldsymbol{q}_h\|^2_{L^2(\mathcal{D})} &+ \|\boldsymbol{\Psi}_h\|^2_{H(grad,\mathcal{D})} \leq C \|\nabla \times \boldsymbol{u}_h\|^2_{L^2(\mathcal{D})}.
\end{aligned}
$$

The first term, of the last formula, comes from an interpolation error for the Scott-Zhang interpolant.

This result is borrowed from Hiptmair, Xu, and Zou. It is also central in their work on algebraic multigrid and other algorithms for $H(\mathbf{curl})$.

The second and third term in this decomposition are more important and they are decomposed separately into global and local parts. For p, we use the coarse space suggested before for the scalar elliptic problems. The tangential component of ∇p gives us the boundary data for an element in $W^{h_i}_{\text{curl}}$. The coarse subspace has only one degree of freedom for each subdomain edge and its tangential boundary value equals $\boldsymbol{d}_{\mathcal{E}} \cdot \boldsymbol{t}$ and vanishes on all other subdomain edges.

The boundary data is extended into the subdomain using minimal energy extensions. We have considered overlapping Schwarz methods as well as a variant of the first algorithm by Bramble, Pasciak, and Schatz and obtained bounds on the condition numbers of the latter method which is independent of the values of the α_i and β_i and accommodate very irregular subdomains, which are only uniform in the sense of Jones.

Our final estimate of the condition number of these domain decomposition algorithms are of the form

$$C\chi(\hat{d})^2(1+\log(H/h))^2$$

where C is independent of the α_i and β_i and $\hat{d} := \min_i \max(h_i, \sqrt{\alpha_i/\beta_i})$. $\chi(\hat{d}) = 1$ for a straight edge. In general it is related to the Hausdorff dimension of the subdomain edges; how many circles of diameter $\hat{d}$ are needed to cover the edge?