Some Algorithmic Aspects of *hp* Adaptive Finite Elements

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IMA Workshop

November 29 – December 3, 2010

Outline of Talk

1. Superconvergence and Local Error Indicators
2. Some Open Issues in h Adaptivity
3. Some Open Issues in p Adaptivity
4. Some Open Issues in hp Adaptivity

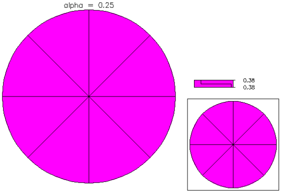
PLTMG Environment

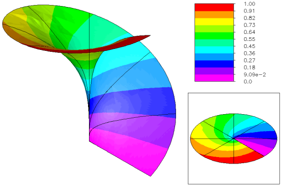
- General scalar elliptic PDE, (nonlinear, not self-adjoint)
- Problem classes: simple pde, continuation, obstacle, parameter identification, distributed optimal control.
- $\mathcal{C}^0$ (isoparametric) piecewise polynomials on unstructured triangular meshes (parametrically defined curved edges allowed)
- Options for h, p, hp, and r adaptivity.

Circle Problem

$$
\begin{aligned}
-\Delta u &= 0 && \text{in } \Omega, \\
u &= g && \text{on } \partial\Omega_1, \\
u_n &= 0 && \text{on } \partial\Omega_2,
\end{aligned}
$$

Ω is a circle of radius one, with a crack along the positive x-axis $0 \leq x \leq 1$. The boundary $\partial\Omega_2$ is the bottom edge of the crack, and $\partial\Omega_1 = \partial\Omega \setminus \partial\Omega_2$. g is chosen such that the exact solution is $u = r^{1/4} \sin(\theta/4)$.





Superconvergence

(joint with Jinchao Xu and Bin Zheng)

- Let $\mathcal{S}_p$ be $\mathcal{C}^0$ piecewise polynomials of degree $p \geq 1$. For $u_h \in \mathcal{S}_p$, $S^m Q \partial^p u_h$ superconverges to $\partial^p u$. (Q is $\mathcal{L}^2$ projection onto $\mathcal{S}_1$; S is a multigrid smoother, and $1 \leq m \leq 2$.)
- We need to recover all derivatives of order p. We then compute derivatives of order $p+1$ from $\partial(S^m Q \partial^p u_h)$.

Some Theorems

Theorem

Under appropriate hypotheses

$$\|\partial^p u - S^m Q \partial^p u_h\|_{0,\Omega} \lesssim h(mh^{1/2} + \varepsilon_m)(\|u\|_{p+2,\Omega} + |u|_{p+1,\infty,\Omega}), \quad (1)$$

ε_m *is the multigrid convergence factor for* m *smoothing steps.*

Theorem

Under appropriate hypotheses

$$\|\partial(\partial^p u - S^m Q \partial^p u_h)\|_{0,\Omega} \lesssim (mh^{1/2} + \varepsilon_m)(\|u\|_{p+2,\Omega} + |u|_{p+1,\infty,\Omega}), \quad (2)$$

Local Error Indicators

Let $\mathcal{P}_p$ denote polynomials of degree p on element t. Let $\mathcal{E}_{p+1} = span\{\psi_i\}_{i=1}^{p+2}$ be polynomials of degree $p+1$ that are zero at Lagrange points of order p. This is the hierarchical extension

$$\mathcal{P}_{p+1} = \mathcal{P}_p \oplus \mathcal{E}_{p+1}.$$

On element t we approximate

$$u - u_h \approx u - u_p \approx \bar{\varepsilon}_t \equiv \sum_{i=1}^{p+2} C_i \psi_i$$

where C_i depends on the known geometry of t and the (constant) recovered derivatives $\partial(S^m Q \partial^p u_h)$ of order $p+1$. Note $\bar{\varepsilon}_t$ is a polynomial of degree $p+1$. (variational crimes are committed on isoparametric elements.)

Scaling Factors

When $\partial^{p+1}u$ is not well defined, $\bar{\varepsilon}_t$ tend to underestimate the error. To fix this, we introduce (heuristic) scaling factors α_t such that

$$\alpha_\tau^2 \sum_{i=0}^{p} \binom{p}{i} \|\partial_x^i \partial_y^{p-i}\,\bar{\varepsilon}_t\|_t^2 = \sum_{i=0}^{p} \binom{p}{i} \|(I - S^m Q)(\partial_x^i \partial_y^{p-i} u_h)\|_t^2$$

Typically $\alpha_t \approx 1$. We then set

$$\varepsilon_t = \alpha_t\,\bar{\varepsilon}_t$$

Our local error indicator is

$$\eta_t = \|\nabla \varepsilon_t\|_t$$

Remarks

- This scheme works very well on smooth problems as predicted by theory (all degrees p).
- Higher degrees p have increasingly severe "regularity gap" near singularities. This degrades effectivity ratio in global estimates.
- It is unknown theoretically if superconvergence holds on meshes of variable p (or as a practical point, if it matters).

h-Refinement for Constant *p*

- It is desirable to have a geometric increase in size of problem in adaptive feedback (outer) loop, e.g. $N_{k+1} \approx 4N_k$.
- Good threshold strategies invariably cause erratic and non-geometric growth in problem size.
- Multiple refinement of some elements requires a scheme to compute error indicators on refined elements without resolving a global problem.
- Such strategies will certainly fail asymptotically. But some can work well in the non-asymptotic range.

Our Present Strategy

- Child elements inherit derivative information from their parents. They use their own geometric information to construct error indicators.
- For $p = 1$, it allows (empirically) $N_{k+1} \approx 4N_k$.
- For $p > 1$, it becomes increasingly less robust. e.g. $N_{k+1} \approx \theta_p N_k$ with $\theta_p \to 1$ as $p \to \infty$.
- This effect should be taken into account in determining the efficiency of higher order elements.

h-Adaptive Refinement for $p = 2$ (Slow v. Fast)

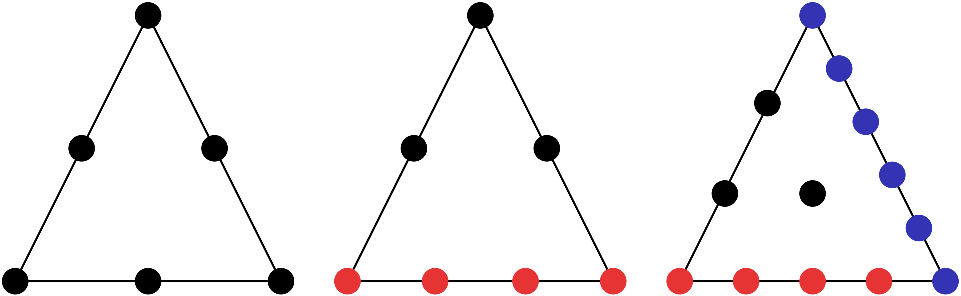
nt	exact	approx	eff	*nt*	exact	approx	eff
8	0.6	0.3	0.5	8	0.6	0.3	0.5
31	0.5	0.1	0.3	31	0.5	0.1	0.3
74	0.4	0.1	0.4				
136	0.3	0.1	0.3	133	0.3	0.1	0.3
302	0.3	0.1	0.3				
534	0.2	0.1	0.4	533	0.2	0.1	0.3
1179	0.1	4.1e-2	0.3				
2077	0.1	2.7e-2	0.3	2078	0.1	0.1	0.4
4617	5.0e-2	1.9e-2	0.4				
8204	3.0e-2	1.1e-2	0.4	8237	0.1	2.7e-2	0.4
18388	1.8e-2	6.2e-3	0.4				
32669	9.5e-3	4.2e-3	0.4	32796	3.5e-2	1.2e-2	0.3
73439	5.0e-3	1.9e-3	0.4				
130586	2.5e-3	1.0e-3	0.4	130890	1.5e-2	7.6e-3	0.5
order	1.77	1.73			1.12	0.83	

p Refinement for Constant h

- Can use this family of error indicators as the basis of p-refinement strategy (PhD of Hieu Nguyen).
- At present, the superconvergence theory for variable p meshes is incomplete.
- The issue of error indicators for refined elements is unresolved, so we are limited to at most one level of p-refinement for each element per iteration of the feedback loop.

Lagrange Basis Functions

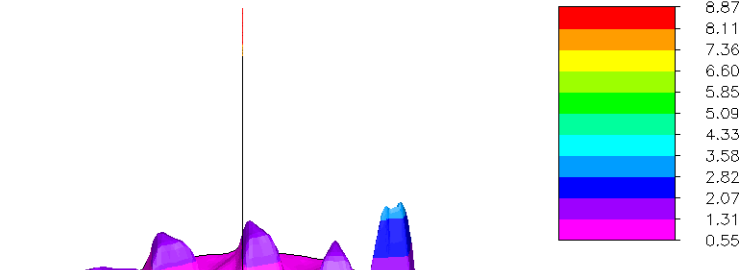
- Use nodal basis – maximum degree 10 limited by quadrature.
- Isoparametric mappings handle curved boundaries.
- We use "transition elements" on degree boundaries

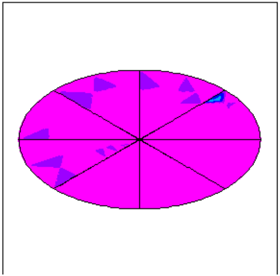


hp Refinement Strategy

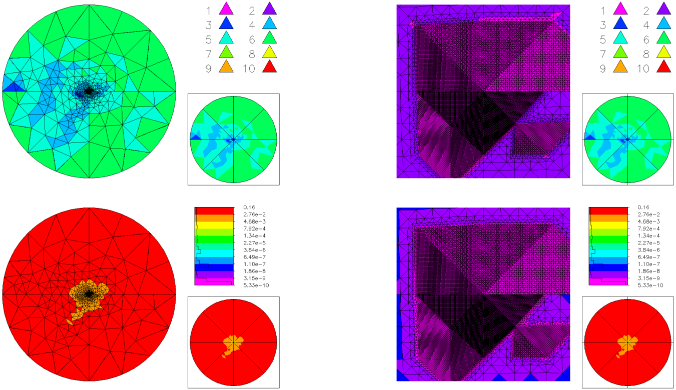
- Average degree in hp mesh is given by $q = \sqrt{ndof/nvert}$. Maximum geometric increase factor is $\theta = (1 + 1/q)^2$. $(N_{k+1} \approx \theta N_k)$.
- We compute scaling factors $\bar{\alpha}_t$ for recovered gradient of piecewise linear interpolant of u_h.
- If $\bar{\alpha}_t > 2$, use h refinement for element t; otherwise use p refinement.

Scaling Factors for Circle Problem



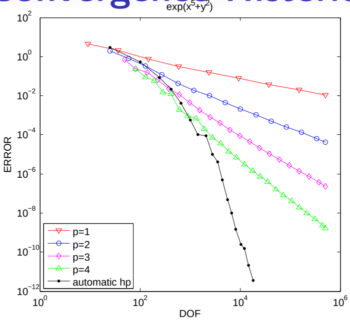


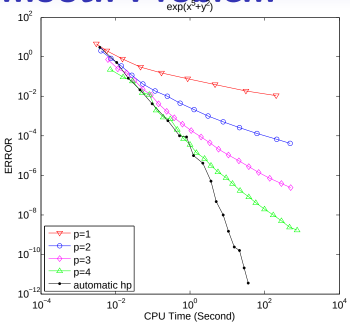
hp Meshes for Circle Problem



dof $=$ 87K, zoom$= 5 \times 10^6$.

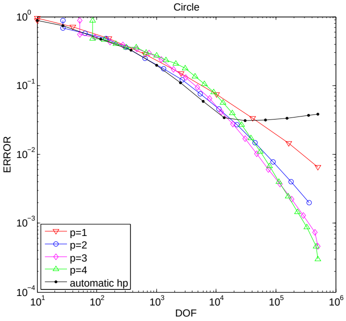
Convergence Histories a Smooth Problem

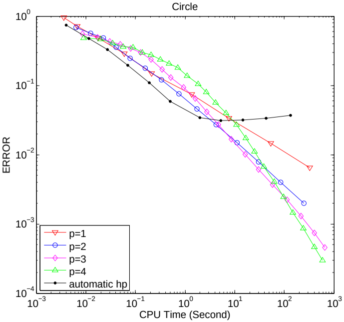




$-\Delta u = f$, on $\Omega \equiv (0,1) \times (0,1)$, with Dirichlet b.c. and exact solution $u = e^{x^5+y^2}$

Convergence Histories for Circle Problem



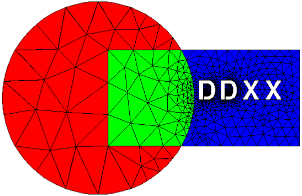


Open Issues

- We can incorporate error estimates for $p = 1$ into h-refinement strategy on coarse meshes.
- We can switch to p refinement when h-refinement would cause roundoff error problems. (We actually switch if h refinement of such an element of degree 10 would cause roundoff error problems.)
- We can use interpolant of degree p_{min} to compute $\bar{\alpha}_t$ instead of always choosing $p = 1$. Using α_t on a mesh of variable p seems risky, but it is less costly and might also be ok.

It is unknown if these are optimal or even very good strategies.

Twentieth International Conference on Domain Decomposition Methods



February 7–11, 2011
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