Exterior calculus and the finite element approximation of Maxwell’s eigenvalue problem

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1 Maxwell’s eigenvalue problem.
   • Recall the analysis for the $h$ version of edge finite elements and equivalence with mixed formulations.
   • Discrete compactness property.
   • Analysis does not extend trivially to $p$ and $hp$ version.

2 Exterior calculus.
   • Discrete compactness property in the framework of differential forms.
   • Mixed formulations
   • Recent results on Poincaré map give discrete compactness for the $p$ version of discrete differential forms.

D. Boffi
Finite element approximation of eigenvalue problems
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Maxwell eigenvalues

Ampère and Faraday’s laws: find resonance frequencies $\omega \in \mathbb{R}$ (with $\omega \neq 0$) and electromagnetic fields $(\mathbf{E}, \mathbf{H}) \neq (0, 0)$ such that

$$\text{curl } \mathbf{E} = i\omega \mu \mathbf{H} \quad \text{in } \Omega$$
$$\text{curl } \mathbf{H} = -i\omega \varepsilon \mathbf{E} \quad \text{in } \Omega$$
$$\mathbf{E} \times n = 0 \quad \text{on } \partial \Omega$$
$$\mathbf{H} \cdot n = 0 \quad \text{on } \partial \Omega$$

$\omega \neq 0$ gives divergence conditions

$$\text{div } \varepsilon \mathbf{E} = 0 \quad \text{in } \Omega$$
$$\text{div } \mu \mathbf{H} = 0 \quad \text{on } \Omega$$

It is then standard to eliminate one field and to obtain the \textbf{curl curl} problem.
Eliminate $\mathbf{H}$ and take $\mathbf{u} = \mathbf{E}$ ($\lambda = \omega^2$)

$$
\begin{align*}
\text{curl}(\mu^{-1} \text{curl} \mathbf{u}) &= \lambda \varepsilon \mathbf{u} & \text{in } \Omega \\
\text{div}(\varepsilon \mathbf{u}) &= 0 & \text{in } \Omega \\
\mathbf{u} \times \mathbf{n} &= 0 & \text{on } \partial \Omega
\end{align*}
$$

Well-known and intensively studied problem. Special (edge) finite elements required for its approximation. We review classical analysis for the $h$ version which covers basically all known families of edge finite elements. The ultimate goal of more recent work is to analyze the convergence for the $p$ and $hp$ version of FEM.

For ease of presentation, we take $\mu = \varepsilon = 1$ and simple topology from now on.
The standard variational formulation reads

\[ u \in H_0(\text{curl}) : \]
\[ (\text{curl } u, \text{curl } v) = \lambda(u, v) \quad \forall v \in H_0(\text{curl}) \]
\[ (u, \text{grad } \phi) = 0 \quad \forall \phi \in H_0^1 \]

The most commonly used variational formulation is based on the replacement of the divergence free constraint by the condition \( \lambda \neq 0 \)

\[ u \in H_0(\text{curl}) : \]
\[ (\text{curl } u, \text{curl } v) = \lambda(u, v) \quad \forall v \in H_0(\text{curl}) \]

The kernel \( \lambda = 0 \) corresponds to the infinite dimensional space \( \text{grad } H_0^1 \).
Mixed formulations

\[ <\text{Kikuchi '89}> \]

Divergence free constraint imposed via Lagrange multiplier \( \psi \)

\[
\begin{aligned}
u & \in H_0(\text{curl}), \quad \psi \in H^1_0 : \\
\{ & (\text{curl} \, u, \text{curl} \, v) + (\text{grad} \, \psi, v) = \lambda(u, v) \quad \forall v \in H_0(\text{curl}) \\
& (\text{grad} \, \phi, u) = 0 \quad \forall \phi \in H^1_0
\end{aligned}
\]

\[ <\text{B–Fernandes–Gastaldi–Perugia '99}> \]

Second mixed formulation \( (H_0(\text{div}^0) = \text{curl}(H_0(\text{curl}))) \)

\[
\begin{aligned}
\sigma & \in H_0(\text{curl}), \quad z \in H_0(\text{div}^0) : \\
\{ & (\sigma, \tau) + (\text{curl} \, \tau, z) = 0 \quad \forall \tau \in H_0(\text{curl}) \\
& (\text{curl} \, \sigma, w) = -\lambda(z, w) \quad \forall w \in H_0(\text{div}^0)
\end{aligned}
\]
Eigenvalues in mixed form

The equivalence with mixed formulations allowed us to apply general theory of eigenvalue approximation in mixed form.

\[\text{B.–Brezzi–Gastaldi '97}\]

The main tool for the analysis (exploited for the \(h\) version) is the construction of a Fortin operator that converges to the identity in norm: *Fortid* property.

\[\text{B.–Fernandes–Gastaldi–Perugia '99}\]
\[\text{B. '00–'01}\]

*Discrete Compactness Property* may also be used.

\[\text{Kikuchi '89}\]
\[\text{Monk–Demkowicz '00}\]
\[\text{Caorsi–Fernandes–Raffetto '00}\]

The two approaches are indeed equivalent

\[\text{B. '07}\]
Mixed conditions for Kikuchi formulation

[ELKER] Ellipticity in the discrete kernel
There exists $\alpha > 0$ such that

$$(\text{curl} \ v_k, \text{curl} \ v_k) \geq \alpha \|v_k\|_{L^2}^2 \quad \forall v_k \in K^d_d$$

[WA1] Weak approximability of $Q = H^{1+s}_0$
There exists $\omega_1(k)$ tending to zero such that

$$\sup_{v_k \in K^d_d} \frac{(v_k, \text{grad} \psi)}{\|v_k\|_{\text{curl}}} \leq \omega_1(k) \|\psi\|_{H^1} \quad \forall \psi \in Q$$

[SA1] Strong approximability of $V_0 = H^s_0(\text{curl}) \cap H(\text{div}^0)$
There exists $\omega_2(k)$ tending to zero such that for every $u \in V_0$ there exists $u' \in K^d_d$ such that

$$\|u - u'\|_{\text{curl}} \leq \omega_2(k) \|u\|_{V_0}$$
Kikuchi solution operators: continuous...

\[
\begin{align*}
\left\{ \begin{align*}
(curl \, u, \, curl \, v) + (grad \, p, \, v) &= (f, \, v) \quad \forall v \in H_0(curl) \\
(grad \, q, \, u) &= 0
\end{align*} \right.
\end{align*}
\]

\[T^{Ki} \in \mathcal{L}(L^2): \quad T^{Ki}(f) = u\]

... and discrete one

\[
\begin{align*}
\left\{ \begin{align*}
(curl \, u_k, \, curl \, v) + (grad \, p_k, \, v) &= (f, \, v) \quad \forall v \in V_k \\
(grad \, q, \, u_k) &= 0
\end{align*} \right.
\end{align*}
\]

\[T^{Ki}_k \in \mathcal{L}(L^2): \quad T^{Ki}_k(f) = u_k\]
Theorem

If the ellipticity in the discrete kernel [ELKER], the weak approximability of $Q$ [WA1], and the strong approximability of $V_0$ [SA1] are satisfied, then the following convergence in norm holds true

$$\| T^{K_i} - T^{K_i}_k \|_{L(L^2)} \rightarrow 0$$

Remark

Convergence in norm allows us to use the classical Babuška–Osborn theory for eigenmode convergence
Mixed conditions for second formulation

[WA2] Weak approximability of \( Z^0 = H_0^s(\text{curl}) \cap H(\text{div}^0) \)

There exists \( \omega_3(k) \) tending to zero such that

\[
(\text{curl } \tau_k, z) \leq \omega_3(k) \|
\text{curl } \tau_k \|_{L^2} \| z \|_{Z^0} \quad \forall \tau_k \in K^c_k, \forall z \in Z^0
\]

[SA2] Strong approximability of \( Z^0 = H_0^s(\text{curl}) \cap H(\text{div}^0) \)

There exists \( \omega_4(k) \) tending to zero such that for every \( z \in Z^0 \) there exists \( z^I \in K^c_k \) such that

\[
\| z - z^I \|_{L^2} \leq \omega_4(k) \| z \|_{Z^0}
\]
Fortin operator

\[ \Pi_k : V^0 \rightarrow V_k \text{ such that } \forall \sigma \in V^0 \]

\[
\begin{align*}
\left\{ 
\begin{align*}
(\text{curl}(\sigma - \Pi_k \sigma), w_k) &= 0 \quad \forall w_k \in Z_k \\
\|\Pi_k \sigma\|_{\text{curl}} &\leq C \|\sigma\|_{V^0}
\end{align*}
\right.
\]

[FORTID] Fortid property

There exists \( \omega_5(k) \) tending to zero such that

\[ \| \sigma - \Pi_k \sigma \|_{L^2} \leq \omega_5(k) \| \sigma \|_{V^0} \quad \forall \sigma \in V^0 \]
Alternative solution operators: continuous...

\[
\begin{cases}
(\sigma, \tau) + (\text{curl} \, \tau, z) = 0 & \forall \tau \in H_0(\text{curl}) \\
(\text{curl} \, \sigma, w) = -(g, w) & \forall w \in \text{curl}(H_0(\text{curl}))
\end{cases}
\]

\[T^M_2 \in \mathcal{L}(L^2): \ T^M_2(g) = z\]

... and discrete one

\[
\begin{cases}
(\sigma_k, \tau) + (\text{curl} \, \tau, z_k) = 0 & \forall \tau \in V_k \\
(\text{curl} \, \sigma_k, w) = -(g, w) & \forall w \in Z_k
\end{cases}
\]

\[\ T^M_k \in \mathcal{L}(L^2): \ T^M_k(g) = z_k\]
Theorem

If the weak approximability of $Z^0$ [WA2] and the strong approximability of $Z^0$ [SA2] are satisfied, and if there exists a Fortin operator satisfying the Fortid property [FORTID], then the following convergence in norm holds true

$$\| T^{M2} - T_k^{M2} \|_{L(L^2)} \to 0$$
Compactness properties

The space $H_0(\text{curl}) \cap H(\text{div}^0)$ is compactly embedded in $L^2$

Compactness can be rephrased as

Given a sequence $\{u_n\} \subset H_0(\text{curl})$ such that

$$ (u_n, \text{grad } \phi) = 0 \quad \forall \phi \in H_0^1, \forall n $$

If $\{u_n\}$ is uniformly bounded in $H_0(\text{curl})$, $\| \text{curl } u_n \|_{L^2} \leq 1$, then there exits a subsequence (still denoted $\{u_n\}$) and $u \in L^2$ such that

$$ \| u_n - u \|_{L^2} \to 0 $$
Discrete compactness property

Discrete analogue for the spaces $V_k \subset H_0(\text{curl})$ and $Q_k \subset H^1_0$.

Given a sequence $\{u_n\} \subset V_k$ discretely divergence free, i.e.,

$$(u_n, \text{grad} \phi_n) = 0 \quad \forall \phi_n \in Q_k, \forall n$$

If $\{u_n\}$ is uniformly bounded in $H_0(\text{curl})$, $\|\text{curl } u_n\|_{L^2} \leq 1$, then there exits a subsequence (still denoted $\{u_n\}$) and $u \in L^2$ such that

$$\|u_n - u\|_{L^2} \to 0$$

**Strong DCP**

We say that the SDCP is satisfied if $u$ is divergence free $\text{div } u = 0$. This is true, for instance, if $Q_k$ is a good approximation to $H^1_0$. 
Commuting diagram property

\[ Q \subset H^1_0, \ V \subset H_0(\text{curl}), \ U \subset H_0(\text{div}), \ S \subset L^2/\mathbb{R} \]

\[
0 \rightarrow Q \xrightarrow{\text{grad}} V \xrightarrow{\text{curl}} U \xrightarrow{\text{div}} S \rightarrow 0
\]

\[
0 \rightarrow Q_k \xrightarrow{\text{grad}} V_k \xrightarrow{\text{curl}} U_k \xrightarrow{\text{div}} S_k \rightarrow 0
\]

- Kikuchi formulation uses \( Q \) and \( V \)
- Alternative formulation uses \( V \) and \( U \)
- \( U \) and \( S \) are used for Darcy flow or mixed Laplacian

---

Douglas–Roberts ’82
Bossavit ’88
Arnold ’02

\( Q \oplus H_0, \ V \oplus H_0(\text{curl}), \ U \oplus H_0(\text{div}), \ S \oplus L^2/\mathbb{R} \)

\[
\begin{align*}
0 & \rightarrow Q \xrightarrow{\text{grad}} V \xrightarrow{\text{curl}} U \xrightarrow{\text{div}} S \rightarrow 0 \\
\downarrow \Pi^Q_k & \downarrow \Pi^V_k \downarrow \Pi^U_k \downarrow \Pi^S_k \\
0 & \rightarrow Q_k \xrightarrow{\text{grad}} V_k \xrightarrow{\text{curl}} U_k \xrightarrow{\text{div}} S_k \rightarrow 0
\end{align*}
\]
**Equivalence**

Given $V_k \subset H_0(\text{curl})$, construct $Q_k$ and $Z_k$ such that

\[ \text{grad } Q_k \subset V_k, \quad \text{curl } V_k \subset Z_k \]

- $Z_k = \text{curl } V_k$
- The kernel of $\text{curl}$ in $V_k$ consists of gradient. Take $Q_k$ as set of potentials vanishing on the boundary $\partial \Omega$

**Theorem**

The following three sets of conditions are equivalent

i) *ELKER, WA1, SA1*

ii) *WA2, SA2, FORTID*

iii) *SDCP and standard approximation property: for any $v \in V_0$ there exists $v^I_k \in V_k$ such that*

\[ \| v - v^I_k \|_{\text{curl}} \to 0 \]
Finite elements
The analysis for the $h$ version of edge elements is fairly easy in the two dimensional case.

- The two dimensional $\text{rot}$ operator is isomorphic to the $\text{div}$ operator (and $\text{curl}$ corresponds to $\text{grad}$)
- Edge elements are isomorphic to Raviart–Thomas elements
- The RT interpolant is a Fortin operator

\[
\int_K w_h \text{div}(\sigma - \Pi_h \Sigma) = - \int_K \text{grad} w_h \cdot (\sigma - \Pi_h \Sigma) + \int_{\partial K} w_h (\sigma - \Pi_h \Sigma) \cdot n = 0
\]
While the RT interpolant is still a Fortin operator, the edge interpolant is not.

Moreover, standard estimates for mixed approximations don’t help (we need uniform convergence!)

\[
\sigma \in H_0(\text{curl}), \; z \in H_0(\text{div}^0) : \\
\begin{cases}
(\sigma, \tau) + (\text{curl} \; \tau, z) = 0 & \forall \tau \in H_0(\text{curl}) \\
(\text{curl} \; \sigma, w) = -(g, w) & \forall w \in H_0(\text{div}^0)
\end{cases}
\]

\[
\|\sigma - \sigma_h\|_{\text{curl}} + \|z - z_h\| \leq C \inf_{\tau_h, w_h} \left( \|\sigma - \tau_h\|_{\text{curl}} + \|z - w_h\| \right) \text{ } _{O(1)}+_{O(h)}
\]

Estimate for \(\|z - z_h\|_{L^2}\) not involving \text{curl} \; \sigma \; \text{needed.}
A better estimate can be obtained, for instance, with the help of Fortin operator

\[ \|\sigma - \sigma_h\|_{L^2} \leq C \left( \|\sigma - \Pi_h\sigma\|_{L^2} + \left(\frac{1}{\sqrt{\alpha}}\right) \inf_{w_h} \|z - w_h\|_{L^2} \right) \]

\[ \|z - z_h\|_{L^2} \leq C \left( \inf_{w_h} \|z - w_h\|_{L^2} + \|\sigma - \sigma_h\|_{L^2} \right) \]

The result then follows from the Fortid property

\[ \|\sigma - \Pi_h\sigma\|_{L^2} \leq Ch^s \|\sigma\|_{H^s} \]
A Fortin operator can be easily constructed by using the inf-sup condition for edge elements. The uniform estimate follows from the commuting diagram and a particular bound for the edge interpolant

$$\|\sigma - \sigma^I\|_{L^2} \leq Ch^s\|\sigma\|_{H^s}$$

when \(\text{curl}\\sigma\) is discrete.

Remark

The last estimate needs to be generalized to \(p\) and \(hp\) versions.
p and hp versions

Some preliminary steps towards hp DCP

- Numerical evidence of $p$ convergence <Monk ’94>
- Convergence proof of hp DCP for 2D triangular meshes modulo a conjectured $L^2$ estimate <B.–Costabel–Demkowicz ’03>
- Rigorous proof of hp DCP for 2D rectangular meshes (allowing for 1-irregular hanging nodes) <B.–Costabel–Dauge–Demkowicz ’06>

Existing proof does not extend to more general situations (triangles or 3D)
Differential forms

<Arnold–Falk–Winther ’06–’10>

We consider a complex of differential forms, $\Omega \subset \mathbb{R}^n$

\[
0 \to \Lambda^0(\Omega) \xrightarrow{d_0} \Lambda^1(\Omega) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(\Omega) \to 0
\]

We define $V^\ell = H_0(d_\ell, \Omega)$, so that we have the complex

\[
0 \to V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} V^n(\Omega) \to 0
\]

and, given finite element approximation spaces $V_p^\ell \subset V^\ell$, we consider the discrete differential complex

\[
0 \to V_p^0 \xrightarrow{d_0} V_p^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} V_p^n(\Omega) \to 0
\]
### Identification table

<table>
<thead>
<tr>
<th>Differential form</th>
<th>Proxy representation</th>
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<td>$d = 2$</td>
<td>$d = 3$</td>
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#### $\ell = 0$

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<td>$\text{tr}_{\partial \Omega} \phi$</td>
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<td>$H_0(d_0, \Omega)$</td>
<td>$H^1_0(\Omega)$</td>
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<td>$\delta_1$</td>
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<td>$H_0(d_1, \Omega)$</td>
<td>$H_0(\text{rot})$</td>
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<td>$\rightarrow \text{rot}$</td>
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<td>$n \times (u \times n)</td>
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#### $\ell = 2$

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<td>$\rightarrow \text{grad}$</td>
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<td></td>
<td>$\text{grad}$</td>
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**Eigenvalue problem**

The abstract counterpart of Maxwell’s eigenvalue problem is the following general formulation related to Hodge–Laplace problem

\[
    u \in H_0(d_\ell, \Omega) : \\
    (d_\ell u, d_\ell v) = \lambda(u, v) \quad \forall v \in H_0(d_\ell, \Omega)
\]

The corresponding discretization reads

\[
    u_p \in V_\ell^p : \\
    (d_\ell u_p, d_\ell v) = \lambda_p(u_p, v) \quad \forall v \in V_\ell^p
\]

**Remark**

*For \( \ell = 0 \) we have the standard eigenvalue problem for Laplace operator.*
Kikuchi mixed formulation

Standard formulation

\[ u \in H_0(d_\ell, \Omega) \]
\[ (d_\ell u, d_\ell v) = \lambda(u, v) \quad \forall v \in H_0(d_\ell, \Omega) \]
\[ (d_{\ell-1} \phi, u) = 0 \quad \forall \phi \in H_0(d_{\ell-1}, \Omega) \]

Kikuchi formulation

\[ u \in H_0(d_\ell, \Omega), \quad \psi \in H_0(d_{\ell-1}, \Omega) : \]
\[
\begin{cases}
(d_\ell u, d_\ell v) + (d_{\ell-1} \psi, v) = \lambda(u, v) & \forall v \in H_0(d_\ell, \Omega) \\
(d_{\ell-1} \phi, u) = 0 & \forall \phi \in H_0(d_{\ell-1}, \Omega)
\end{cases}
\]

Inclusion \( d_{\ell-1}(H_0(d_{\ell-1}, \Omega)) \subset H_0(d_\ell, \Omega) \) implies \( d_{\ell-1} \psi = 0 \). For \( \ell \geq 2 \), \( \psi \) is not uniquely defined, but this is not an issue since we are interested in eigenfunction \( u \).
Second mixed formulation

Given the space $K_{\ell+1}^d = d_{\ell+1}(H_0(d_\ell, \Omega)) \subset H_0(d_{\ell+1}, \Omega)$, an alternative formulation reads

$$u \in H_0(d_\ell, \Omega), \ s \in K_{\ell+1}^d :$$

$$\begin{cases}
(u, v) + (d_\ell v, s) = 0 & \forall v \in H_0(d_\ell, \Omega) \\
(d_\ell u, t) = -\lambda(s, t) & \forall t \in K_{\ell+1}^d
\end{cases}$$

It is easy to show $\lambda \neq 0$ which gives $s = -d_\ell u / \lambda$
Equivalence

Proposition

If $(\lambda, u) \in \mathbb{R} \times H_0(d_\ell, \Omega)$ is a solution of the standard formulation, then it is a solution of the Kikuchi formulation as well and there exists $s \in K_{\ell+1}^d$ such that $(\lambda, s)$ is a solution of the second mixed formulation.

Conversely, if $(\lambda, u) \in \mathbb{R} \times H_0(d_\ell, \Omega)$ is a solution of the Kikuchi formulation, then $\lambda > 0$ and $(\lambda, u)$ solves the standard formulation; if $(\lambda, s) \in \mathbb{R} \times K_{\ell+1}^d$ is a solution of the second mixed formulation for some $u \in H_0(d_\ell, \Omega)$, then $(\lambda, u)$ solves the standard formulation as well.
The conditions already introduced for the approximation of mixed formulations can be rephrased in terms of differential forms

[ELKER], [WA1], [SA1] for the Kikuchi formulation

[WA2], [SA2], [FORTID] for the second formulation
DCP for differential forms

\[ H_0(d_\ell, \Omega) \cap H(\delta_0, \Omega) \text{ compact into } L^2 \]

\(<\text{Picard '84}>\)

Given a sequence \( \{u_n\} \) with \( u_n \in V_{p_n}^\ell \) and satisfying

\[ (u_n, d_{\ell-1}\phi) = 0 \quad \forall \phi \in V_{p_n}^{\ell-1} \quad \forall n, \]

if \( \{u_n\} \) is bounded uniformly in \( H_0(d_\ell, \Omega) \), \( \|d_\ell u_n\|_{L^2} \leq 1 \), then there exists a subsequence (denoted by \( \{u_n\} \)) and \( u \in L^2(\Omega, \wedge^\ell) \) such that

\[ \|u_n - u\|_{L^2} \to 0 \]

**Strong DCP**

The SDCP is satisfied if \( u \) is co-closed, i.e., \( \delta_\ell u = 0 \).

**Remark**

*Condition \( (u_n, d_{\ell-1}\phi) = 0 \) can be replaced by taking \( u_n \) discretely co-closed: \( (u_n, v) = 0 \) for all \( w \) with \( d_\ell v = 0 \).*
DCP and eigenvalue convergence

Theorem

The following three sets of conditions are equivalent

(i) SDCP and standard approximation properties: for any $v \in H_0(d_\ell, \Omega)$ with $\delta_\ell v = 0$ there exists $\{v_p\}$ with $v_p \in V^\ell_p$ such that $\|v - v_p\|_{H_0(d_\ell, \Omega)} \to 0$;

(ii) [ELKER], [WA1], [SA1];

(iii) [WA2], [SA2], FORTID.

In particular, condition (i) implies eigenvalue convergence

Remark

Compare condition (i) with theory in

<Arnold–Falk–Winther ’10>
The proof relies on a Poincaré map which respects polynomials

The main assumptions are the following ones

1. Regularity and compactness
   \[ H_0(d_\ell, \Omega) \cap H(\delta_\ell 0, \Omega) \hookrightarrow X(\Omega, \Lambda^\ell) \]

2. Locality of projectors \( \pi_p^{\ell - 1} \) and \( \pi_p^\ell \) and commuting diagram

3. Local approximation property
   \[ \| d_{\ell - 1}(\phi - \pi_p^{\ell - 1}_K \phi) \|_{L^2(K, \Lambda^{\ell - 1})} \leq \varepsilon_{\ell - 1}(p) \| \phi \|_{S(K, \Lambda^{\ell - 1})} \]

4. Poincaré map \( \kappa_j : C^\infty(K, \Lambda^j) \to C^\infty(K, \Lambda^{j - 1}) \) for \( j = \ell, \ell + 1 \) such that \( d_{\ell - 1} \circ \kappa_\ell + \kappa_{\ell + 1} \circ d_\ell = Id_\ell \) and \( \kappa_{\ell + 1} \circ d_\ell : V_p(\ell, K) \to V_p(\ell, K) \) with \( \kappa_\ell \in \mathcal{L}(X(K, \Lambda^\ell), S(K, \Lambda^{\ell - 1})) \) and \( \kappa_{\ell + 1} \in \mathcal{L}(L^2(K, \Lambda^{\ell + 1}), X(K, \Lambda^{\ell})) \)
Main theorem

Theorem (B.–Costabel–Dauge–Demkowicz–Hiptmair)

If hypotheses 1+2+3+4 are satisfied, then the Discrete Compactness Property holds true.

Basically all known edge element families of FEs for Maxwell’s equations in two and three space dimensions are covered by our theory (simplices, parallelepipeds, prisms, . . .).

Remark

Approximation properties indeed imply that the Strong Discrete Compactness Property is valid.
Error estimates

What about error estimates?

When using DCP, it is possible to use theory developed in \cite{Descloux-Nassif-Rappaz '78} for non compact operator (after shifting eigenvalues). This gives estimate for eigenfunctions (measured by the gap of Hilbert spaces) bounded by best approximation. Convergence order for eigenvalues is double.

In the case of mixed formulations, equivalent error estimates can be obtained from the theory of \cite{Mercier-Osborn-Rappaz-Raviart '81} (see also \cite{B. '10}).
Summary and additional results

- The numerical analysis of edge finite element approximation of Maxwell’s eigenvalues has been a challenging problem for more than a decade.
- The use of nodal finite element is known to produce unreliable results.
- Enforcing the divergence free condition with nodal elements and by a penalty procedure may be problematic.
- Analysis for the $h$ version of edge elements is complete.
- Exterior calculus is a powerful tool for the analysis of our problem.
- Analysis for the $p$ version of edge elements is covered by the much more general theory of DCP for differential forms.
- Extension to nonconstant coefficients and nontrivial topologies.
Conclusions

What is covered by our theory...

- Basically all known edge element families for Maxwell’s equations in two and three space dimensions (simplices, parallelepipeds, prisms,...)
- Raviart–Thomas elements for mixed Laplacian
- Standard Laplacian

...and what is not

- General quadrilateral and hexahedral elements
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...and what is not

- General quadrilateral and hexahedral elements
\[ P = L^2\text{-projection} \]

\[ \| \Pi_h \sigma - \sigma_h \|_{L^2}^2 = (\Pi_h \sigma - \sigma, \Pi_h \sigma - \sigma_h) + (\sigma - \sigma_h, \Pi_h \sigma - \sigma_h) \]
\[ = (\Pi_h \sigma - \sigma, \Pi_h \sigma - \sigma_h) - (\text{curl}(\Pi_h \sigma - \sigma_h), z - Pz) \]
\[ \leq \| \Pi_h \sigma - \sigma \| \| \Pi_h \sigma - \sigma_h \| + \| \text{curl}(\Pi_h \sigma - \sigma_h) \| \| z - Pz \| \]
\[ \leq \| \Pi_h \sigma - \sigma_h \| \left( \| \Pi_h \sigma - \sigma \| + \frac{1}{\sqrt{\alpha}} \| z - Pz \| \right) \]

\[ \| Pz - z_h \|_{L^2} \leq C \sup_{\tau_h} \frac{(Pz - z_h, \text{curl} \tau_h)}{\| \tau_h \|_{\text{curl}}} \]
\[ \leq C \sup_{\tau_h} \frac{(Pz - z, \text{curl} \tau_h) + (z - z_h, \text{curl} \tau_h)}{\| \tau_h \|_{\text{curl}}} \]
\[ \leq C \left( \| Pz - z \| + \sup_{\tau_h} -\frac{(\sigma - \sigma_h, \tau_h)}{\| \tau_h \|_{\text{curl}}} \right) \]
\[ \leq C \left( \| Pz - z \| + \| \sigma - \sigma_h \| \right) \]
Nodal finite elements

Nodal elements on unstructured meshes produce awful results

![Graph showing eigenvalue behavior for nodal elements on unstructured meshes. The graph indicates that nodal elements produce poor results.]
Nodal elements on structured meshes produce dangerous results

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Penalty formulation

Penalty formulation works on convex domains only!

\[(\text{curl } u, \text{curl } v) + s(\text{div } u, \text{div } v) = \lambda(u, v) \quad \forall v\]

Due to the fact that \(H^1 \cap H_0(\text{curl})\) is a closed subspace of \(H_0(\text{curl}) \cap H(\text{div})\)

It is possible to use a weighted formulation which weakens the constraint in the proximity of reentrant corners

<Costabel-Dauge '02>
More realistic situations

The case of variable coefficients (different materials) can be handled with the tools of <Caorsi–Fernandes–Raffetto ’01>

The case of nontrivial topologies gives rise to a de Rham complex which is no longer exact. The cohomology is however finite (a finite dimensional space of harmonic forms shows up), so that the DCP proof still remains valid
Quadrilateral finite elements have particular requirements for optimal approximation. For Maxwell’s eigenvalues, in 2D edge elements do not work: additional degrees of freedom have to be added (ABF element).

Reduced integration technique restores optimal convergence.

Mimetic techniques can also be adopted to modify standard edge elements.

Three-dimensional analysis still in progress.
We are given \( \{u_p\} \), with \( u_p \in V^\ell_p \), such that 
\[
(u_p, d_{\ell-1}\phi) = 0 \quad \forall \phi, \quad \|d_\ell u_p\|_{L^2(\Omega, \Lambda^\ell)} \leq 1.
\]

We perform the continuous Hodge decomposition of \( \{u_p\} \)

\[
\tilde{u}_p = u_p + d_{\ell-1}\tilde{\psi}_p \quad \tilde{\psi}_p \in H_0(d_{\ell-1}, \Omega)
\]

\[
(\tilde{u}_p, d_{\ell-1}\phi) = 0 \quad \forall \phi \in H_0(d_{\ell-1}, \Omega)
\]

Hence \( \tilde{u}_p \in X(\Omega, \Lambda^\ell) \). From the compactness of \( X(\Omega, \Lambda^\ell) \) in \( L^2(\Omega, \Lambda^\ell) \), \( \{\tilde{u}_p\} \) has a subsequence strongly convergent to \( u \in L^2(\Omega, \Lambda^\ell) \). We will show that the same subsequence of \( \{u_p\} \) converges to \( u \) in \( L^2(\Omega, \Lambda^\ell) \).

We use Nédélec trick

\[
\|\tilde{u}_p - u_p\|^2_{L^2(\Omega, \Lambda^\ell)} = (\tilde{u}_p - u_p, \tilde{u}_p - \pi^\ell_p \tilde{u}_p + \pi^\ell_p \tilde{u}_p - u_p)
\]

\[
= (\tilde{u}_p - u_p, \tilde{u}_p - \pi^\ell_p \tilde{u}_p + d_{\ell-1} \pi^{\ell-1}_p \tilde{\psi}_p)
\]

\[
= (\tilde{u}_p - u_p, \tilde{u}_p - \pi^\ell_p \tilde{u}_p)
\]
Sketch of the proof II

\[ \| \tilde{u}_p - u_p \|_{L^2(\Omega, \Lambda^\ell)} \leq \| \tilde{u}_p - \pi^\ell_p \tilde{u}_p \|_{L^2(\Omega, \Lambda^\ell)} \]

The final result follows from the approximation assumption and the Poincaré map

Lemma

If \( u \in X(\Omega, \Lambda^\ell) \) satisfies \( d_\ell u \in d_\ell V_p^\ell \), then

\[ \| u - \pi^\ell_p u \|_{L^2(\Omega, \Lambda^\ell)} \leq C_{\epsilon_{\ell-1}(p)} \| u \|_{X(\Omega, \Lambda^\ell)} \]

In order to prove the last estimate, we can work on a single element \( K \) (locality assumption).
Sketch of the proof III

Poincaré map gives \( u = d_{\ell - 1}\kappa_\ell u + \kappa_{\ell + 1}d_{\ell}u \) and
\[
\|\kappa_{\ell + 1}d_{\ell}u\|_{X(K,\Lambda^{\ell})} \leq C\|d_{\ell}u\|_{L^2(K,\Lambda^{\ell})}
\]
We set \( \psi = \kappa_\ell u \), so that \( \|\psi\|_{S(K,\Lambda^{\ell - 1})} \leq C\|u\|_{X(K,\Lambda^{\ell})} \)

From \( u = d_{\ell - 1}\psi + \kappa_{\ell + 1}d_{\ell}u \) we obtain
\[
(Id - \pi_{p,K}^{\ell})u = d_{\ell - 1}(Id - \pi_{p,K}^{\ell - 1})\psi + (Id - \pi_{p,K}^{\ell})\kappa_{\ell + 1}d_{\ell}u \\
= d_{\ell - 1}(Id - \pi_{p,K}^{\ell - 1})\psi
\]

Hence
\[
(Id - \pi_{p,K}^{\ell})u \leq \epsilon_{\ell - 1}(p)\|\psi\|_{S(K,\Lambda^{\ell - 1})} \leq C\epsilon_{\ell - 1}(p)\|u\|_{X(K,\Lambda^{\ell})}
\]