Maximum-principle-satisfying and positivity-preserving high order discontinuous Galerkin and finite volume schemes for conservation laws

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Joint work with Xiangxiong Zhang
Outline

• Introduction

• High order DG and finite volume schemes satisfying a strict maximum principle (scalar equations, incompressible flows and passive convection in an incompressible velocity field) or maintaining positivity of density and pressure for compressible Euler equations and water height for shallow water equations

• Numerical results

• Conclusions and future work
Maximum-principle-satisfying and positivity-preserving high order schemes

Introduction

For the scalar conservation laws

\[ u_t + \nabla \cdot \mathbf{F}(u) = 0, \quad u(x, 0) = u_0(x). \]  

(1)

An important property of the entropy solution (which may be discontinuous) is that it satisfies a strict maximum principle: If

\[ M = \max_x u_0(x), \quad m = \min_x u_0(x), \]  

(2)

then \( u(x, t) \in [m, M] \) for any \( x \) and \( t \).
First order monotone schemes can maintain the maximum principle. However, for higher order linear schemes, i.e. schemes which are linear for a linear PDE

$$u_t + au_x = 0$$

for example the second order accurate Lax-Wendroff scheme

$$u^{n+1}_j = \frac{a\lambda}{2}(1 + a\lambda)u^{n-1}_j + (1 - a^2\lambda^2)u^n_j - \frac{a\lambda}{2}(1 - a\lambda)u^n_{j+1}$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $|a|\lambda \leq 1$, the maximum principle is not satisfied. In fact, no linear schemes with order of accuracy higher than one can satisfy the maximum principle (Godunov Theorem).
Therefore, nonlinear schemes, namely schemes which are nonlinear even for linear PDEs, have been designed to overcome this difficulty. These include roughly two classes of schemes:

- **TVD schemes.** Most TVD (total variation diminishing) schemes also satisfy strict maximum principle, even in multi-dimensions. TVD schemes can be designed for any formal order of accuracy for solutions in smooth, monotone regions. However, all TVD schemes will degenerate to first order accuracy at smooth extrema.

- **TVB schemes, ENO schemes, WENO schemes.** These schemes do not insist on strict TVD properties, therefore they do not satisfy strict maximum principles, although they can be designed to be arbitrarily high order accurate for smooth solutions.
Remark: If we insist on the maximum principle interpreted as

\[ m \leq u_{j}^{n+1} \leq M, \quad \forall j \]

if

\[ m \leq u_{j}^{n} \leq M, \quad \forall j, \]

where \( u_{j}^{n} \) is either the approximation to the point value \( u(x_j, t^n) \) for a finite difference scheme, or to the cell average \( \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n) \, dx \) for a finite volume or DG scheme, then the scheme can be at most second order accurate. The proof (due to Harten) is very simple:
Think about the linear convection

\[ u_t + u_x = 0, \quad u(x, 0) = \sin(x) \]

and put the grids so that \( x = \frac{\pi}{2} \) is half way between two grid points, say \( x_{i-1} \) and \( x_i \). Then clearly

\[ u_j^0 \leq 1 - C\Delta x^2, \quad \forall j. \]

Now, let us take the CFL number \( \lambda = 0.5 \). At the next time step, the exact solution \( u(x_i, t^1) = 1 \), but the numerical solution has to obey strict maximum principle

\[ u_i^1 \leq 1 - C\Delta x^2 \]

hence the error is at least

\[ u(x_i, t^1) - u_i^1 \geq C\Delta x^2. \]
Notice that this is just for one time step! In general, schemes that satisfy the strict maximum principle are only first order accurate, regardless of whether they are linear or nonlinear schemes.
Therefore, the correct procedure to follow in designing high order schemes that satisfy a strict maximum principle is to change the definition of maximum principle. Note that a high order finite volume scheme has the following algorithm flowchart:

1. Given \( \{\bar{u}_j^n\} \)
2. reconstruct \( u^n(x) \) (piecewise polynomial with cell average \( \bar{u}_j^n \))
3. evolve by, e.g. Runge-Kutta time discretization to get \( \{\bar{u}_j^{n+1}\} \)
4. return to (1)
Therefore, instead of requiring
\[ m \leq u_{j}^{n+1} \leq M, \quad \forall j \]
if
\[ m \leq u_{j}^{n} \leq M, \quad \forall j, \]
we will require
\[ m \leq u^{n+1}(x) \leq M, \quad \forall x \]
if
\[ m \leq u^{n}(x) \leq M, \quad \forall x. \]

Similar definition and procedure can be used for discontinuous Galerkin schemes.
The flowchart for designing a high order scheme which obeys a strict maximum principle is then as follows:

1. Start with $u^n(x)$ which is high order accurate

$$|u(x, t^n) - u^n(x)| \leq C \Delta x^p$$

and satisfy

$$m \leq u^n(x) \leq M, \quad \forall x$$

therefore of course we also have

$$m \leq \bar{u}_j^n \leq M, \quad \forall j.$$
2. Evolve for one time step to get

\[ m \leq \bar{u}^{n+1}_j \leq M, \quad \forall j. \]  \hspace{1cm} (4)

3. Given (4) above, obtain the reconstruction \( u^{n+1}(x) \) which

- satisfies the maximum principle

\[ m \leq u^{n+1}(x) \leq M, \quad \forall x; \]

- is high order accurate

\[ |u(x, t^{n+1}) - u^{n+1}(x)| \leq C \Delta x^p. \]
Three major difficulties

1. The first difficulty is how to evolve in time for one time step to guarantee

\[ m \leq \bar{u}_j^{n+1} \leq M, \quad \forall j. \]  

This is very difficult to achieve. Previous works use one of the following two approaches:
Use exact time evolution. This can guarantee

\[ m \leq \bar{u}_j^{n+1} \leq M, \quad \forall j. \]

However, it can only be implemented with reasonable cost for linear PDEs, or for nonlinear PDEs in one dimension. This approach was used in, e.g., Jiang and Tadmor, SISC 1998; Liu and Osher, SINUM 1996; Sanders, Math Comp 1988; Qiu and Shu, SINUM 2008; Zhang and Shu, SINUM 2010; to obtain TVD schemes or maximum-principle-preserving schemes for linear and nonlinear PDEs in one dimension or for linear PDEs in multi-dimensions, for second or third order accurate schemes.
• Use simple time evolution such as SSP Runge-Kutta or multi-step methods. However, additional limiting will be needed on $u^n(x)$ which will destroy accuracy near smooth extrema.

We have figured out a way to obtain

$$m \leq \bar{u}_{j}^{n+1} \leq M, \quad \forall j$$

with simple Euler forward or SSP Runge-Kutta or multi-step methods without losing accuracy on the limited $u^n(x)$. 
2. **The second difficulty is:** given

\[ m \leq \bar{u}_{j}^{n+1} \leq M, \quad \forall j \]

how to obtain accurate reconstruction \( u^{n+1}(x) \) which satisfy

\[ m \leq u^{n+1}(x) \leq M, \quad \forall x. \]

Previous work was mainly for relatively lower order schemes (second or third order accurate), and would typically require an evaluation of the extrema of \( u^{n+1}(x) \), which, for a piecewise polynomial of higher degree, is quite costly.

**We have figured out a way to obtain such reconstruction with a very simple limiter,** which only requires the evaluation of \( u^{n+1}(x) \) at certain pre-determined quadrature points and does not destroy accuracy.
3. **The third difficulty is** how to generalize the algorithm and result to 2D (or higher dimensions). Algorithms which would require an evaluation of the extrema of the reconstructed polynomials $u^{n+1}(x, y)$ would not be easy to generalize at all.

Our algorithm easily generalizes to 2D or higher dimensions, with strict maximum-principle-satisfying property and provable high order accuracy.
Maximum-principle-satisfying DG and finite volume WENO schemes for scalar conservation laws and passive convection in an incompressible velocity field, and positivity-preserving (for density and pressure) DG and finite volume WENO schemes for compressible Euler equations (Zhang and Shu, SINUM 2010; JCP 2010a; JCP 2010b; Zhang, Xia and Shu, JSC submitted; Zhang and Shu, JCP submitted) and shallow water equations with mixed wet/dry regions (Xing, Zhang and Shu, Advances in Water Resources, to appear).
For the one-dimensional conservation law

\[ u_t + f(u)_x = 0, \]

the evolution of the cell average satisfies

\[ \bar{u}^{n+1}_j = \bar{u}^n_j - \lambda [h(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}) - h(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}})], \]

where \( \lambda = \frac{\Delta t}{\Delta x} \) and \( h(u^-, u^+) \) is a monotone flux \( (h(\uparrow, \downarrow)) \).
The polynomial $p_j(x)$ (either reconstructed in a finite volume method or evolved in a DG method) is of degree $k$, defined on $I_j$ such that $\bar{u}^n_j$ is its cell average on $I_j$, $u^+_j \frac{1}{2} = p_j(x_j - \frac{1}{2})$ and $u^-_j \frac{1}{2} = p_j(x_j + \frac{1}{2})$.

We take a Legendre Gauss-Lobatto quadrature rule which is exact for polynomials of degree $k$, then

$$\bar{u}^n_j = \sum_{\ell=0}^{m} \omega_{\ell} p_j(y_{\ell})$$

with $y_0 = x_j - \frac{1}{2}, y_m = x_j + \frac{1}{2}$. The scheme for the cell average is then rewritten as
\[
\bar{u}_{j}^{n+1} = \omega_m \left[ u_{j+\frac{1}{2}}^- - \frac{\lambda}{\omega_m} \left( h(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) - h(u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-) \right) \right] \\
+ \omega_0 \left[ u_{j-\frac{1}{2}}^+ - \frac{\lambda}{\omega_0} \left( h(u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-) - h(u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^-) \right) \right] \\
+ \sum_{\ell=1}^{m-1} \omega_\ell p_j(y_\ell).
\]

A simple scaling limiting can guarantee that the polynomial \( p_j(x) \), evaluated at the Legendre Gauss-Lobatto quadrature points, are within \([m, M]\). Hence, all the blue terms are within \([m, M]\), so are the terms in the square brackets (by the property of first order monotone schemes), and therefore also the convex combination \( \bar{u}_{j}^{n+1} \).
The SSP (also called TVD) Runge-Kutta or multi-step method can then be used to achieve high order accuracy in time while maintaining the strict maximum principle.
The limiter: we need to modify $p_j(x)$ such that $p_j(x) \in [m, M]$ for all $x \in S_j$ where $S_j$ is set of the Legendre Gauss-Lobatto quadrature points for $I_j$. For all $j$, assume $\bar{u}_j^n \in [m, M]$, we use the modified polynomial $\tilde{p}_j(x)$ after the limiter instead of $p_j(x)$, i.e.,

$$
\tilde{p}_j(x) = \theta(p_j(x) - \bar{u}_j^n) + \bar{u}_j^n, \quad \theta = \min \left\{ \left| \frac{M - \bar{u}_j^n}{M_j - \bar{u}_j^n} \right|, \left| \frac{m - \bar{u}_j^n}{m_j - \bar{u}_j^n} \right|, 1 \right\},
$$

with

$$
M_j = \max_{x \in S_j} p_j(x), \quad m_j = \min_{x \in S_j} p_j(x).
$$

Lemma. Assume $\bar{u}_j^n \in [m, M]$, then (6)-(7) gives a $(k + 1)$-th order accurate limiter.
We thus have a scheme which, for one dimensional scalar conservation laws,

- satisfies a strict maximum principle;
- is uniformly high order accurate.
The technique has been generalized to the following situations maintaining uniformly high order accuracy:

- 2D scalar conservation laws on rectangular or triangular meshes with strict maximum principle (Zhang and Shu, JCP 2010a; Zhang, Xia and Shu, JSC submitted).

- 2D incompressible equations in the vorticity-streamfunction formulation (with strict maximum principle for the vorticity), and 2D passive convections in a divergence-free velocity field, i.e.

  \[ \omega_t + (u\omega)_x + (v\omega)_x = 0, \]

  with a given divergence-free velocity field \((u, v)\), again with strict maximum principle (Zhang and Shu, JCP 2010a; Zhang, Xia and Shu, JSC submitted).
• One and multi-dimensional compressible Euler equations maintaining positivity of density and pressure (Zhang and Shu, JCP 2010b; Zhang, Xia and Shu, JSC submitted).

• One and two-dimensional shallow water equations maintaining non-negativity of water height and well-balancedness for problems with dry areas (Xing, Zhang and Shu, Advances in Water Resources, to appear).

• One and multi-dimensional compressible Euler equations with source terms (geometric, gravity, chemical reaction, radiative cooling) maintaining positivity of density and pressure (Zhang and Shu, JCP submitted).
Numerical results

Example 1. Accuracy check. For the incompressible Euler equation in the vorticity-streamfunction formulation, with periodic boundary condition and initial data $\omega(x, y, 0) = -2 \sin(x) \sin(y)$ on the domain $[0, 2\pi] \times [0, 2\pi]$, the exact solution is $\omega(x, y, t) = -2 \sin(x) \sin(y)$. We clearly observe the designed order of accuracy for this solution.
Table 1: Incompressible Euler equations. $P^2$ for vorticity, $t = 0.5$.

<table>
<thead>
<tr>
<th>N x N</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
</tr>
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<tr>
<td>16 x 16</td>
<td>5.12E-4</td>
<td>–</td>
<td>1.40E-3</td>
<td>–</td>
</tr>
<tr>
<td>32 x 32</td>
<td>3.75E-5</td>
<td>3.77</td>
<td>1.99E-4</td>
<td>2.81</td>
</tr>
<tr>
<td>64 x 64</td>
<td>3.16E-6</td>
<td>3.57</td>
<td>2.74E-5</td>
<td>2.86</td>
</tr>
<tr>
<td>128 x 128</td>
<td>2.76E-7</td>
<td>3.51</td>
<td>3.56E-6</td>
<td>2.94</td>
</tr>
</tbody>
</table>
Example 2. Double shear layer problem for the incompressible Euler equation. The domain is $[0, 2\pi] \times [0, 2\pi]$ with a periodic boundary condition and an initial condition

$$
\omega(x, y, 0) = \begin{cases} 
\delta \cos(x) - \frac{1}{\rho} \text{sech}^2 \left( \frac{(y - \pi/2)}{\rho} \right) & y \leq \pi \\
\delta \cos(x) + \frac{1}{\rho} \text{sech}^2 \left( \frac{(3\pi/2 - y)}{\rho} \right) & y > \pi
\end{cases},
$$

where we take $\rho = \pi/15$ and $\delta = 0.05$. For this example we cannot observe any significant difference between the two results in the contour plots.
Figure 1: Vorticity at $t = 8$, $P^2$, 30 equally spaced contours from $-4.9$ to $4.9$. $128^2$ mesh. Left: with limiter; Right: without limiter.
Example 3. The vortex patch problem. We solve the incompressible Euler equations in $[0, 2\pi] \times [0, 2\pi]$ with the initial condition

$$
\omega(x, y, 0) = \begin{cases} 
-1, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}, \frac{\pi}{4} \leq y \leq \frac{3\pi}{4}; \\
1, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}, \frac{5\pi}{4} \leq y \leq \frac{7\pi}{4}; \\
0, & \text{otherwise}
\end{cases}
$$

and periodic boundary conditions. The contour plots of the vorticity $\omega$ are given for $t = 10$. Again, we cannot observe any significant difference between the two results in the contour plots. The cut along the diagonal gives us a clearer view of the advantage in using the limiter.
Figure 2: Vorticity at $t = 10$, $P^2$. 30 equally spaced contours from $-1.1$ to $1.1$. $128^2$ mesh. Left: with limiter; Right: without limiter.
Figure 3: Vorticity at $t = 10$, $P^2$. Cut along the diagonal. $128^2$ mesh. Left: with limiter; Right: without limiter.
Example 4. Consider a two-dimensional low density problem for the compressible Euler equation. The initial condition is

\[
\rho_0(x, y) = 1 + 0.99 \sin(x + y), \quad u_0(x, y) = 1,
\]

\[
v_0(x, y) = 1, \quad p_0(x, y) = 1.
\]

The domain is \([0, 2\pi] \times [0, 2\pi]\) and the boundary condition is periodic.
Table 2: Third order RKDG scheme with the positivity limiter, for the compressible Euler equation. $\Delta x = \Delta y = \frac{2\pi}{N}$, $t=0.1$.

<table>
<thead>
<tr>
<th>$N \times N$</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
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<tbody>
<tr>
<td>20x20</td>
<td>5.58E-4</td>
<td>-</td>
<td>5.40e-3</td>
<td>-</td>
</tr>
<tr>
<td>40x40</td>
<td>7.57e-5</td>
<td>2.88</td>
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<tr>
<td>160x160</td>
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<td>2.77</td>
<td>1.97e-5</td>
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<tr>
<td>320x320</td>
<td>2.23e-7</td>
<td>2.90</td>
<td>2.57e-6</td>
<td>2.94</td>
</tr>
</tbody>
</table>
Example 5. The Sedov point-blast wave in one dimension. For the initial condition, the density is $1$, velocity is zero, total energy is $10^{-12}$ everywhere except that the energy in the center cell is the constant $\frac{E_0}{\Delta x}$ with $E_0 = 3200000$ (emulating a $\delta$-function at the center). $\gamma = 1.4$. The computational results are shown in Figure 4. We can see the shock is captured very well.
Figure 4: 1D Sedov blast. The solid line is the exact solution. Symbols are numerical solutions. \( T = 0.001 \). \( N = 800 \). \( \Delta x = \frac{4}{N} \). TVB limiter parameters \((M_1, M_2, M_3) = (15000, 20000, 15000)\). Pressure (left) and velocity (right).
Figure 5: 1D Sedov blast. The solid line is the exact solution. Symbols are numerical solutions. \( T = 0.001 \), \( N = 800 \), \( \Delta x = \frac{4}{N} \). TVB limiter parameters \((M_1, M_2, M_3) = (15000, 20000, 15000)\). Density.
Example 5. The Sedov point-blast wave in two dimensions. The computational domain is a square. For the initial condition, the density is 1, velocity is zero, total energy is $10^{-12}$ everywhere except that the energy in the lower left corner cell is the constant $0.244816 \Delta x \Delta y$. $\gamma = 1.4$. See Figure 6. The computational result is comparable to those in the literature, e.g. those computed by Lagrangian methods.
Figure 6: 2D Sedov blast, plot of density. $T = 1$. $N = 160$. $\Delta x = \Delta y = \frac{1.1}{N}$. TVB limiter parameters $(M_1, M_2, M_3, M_4) = (8000, 16000, 16000, 8000)$. 
Figure 7: 2D Sedov blast, plot of density. \( T = 1 \). \( N = 160 \).
\( \Delta x = \Delta y = \frac{1.1}{N} \). TVB limiter parameters \( (M_1, M_2, M_3, M_4) = (8000, 16000, 16000, 8000) \).
Example 6. We consider two Riemann problems. The first one is a double rarefaction. We did two tests, one is a one-dimensional double rarefaction, for which the initial condition is $\rho_L = \rho_R = 7, u_L = -1, u_R = 1, p_L = p_R = 0.2$ and $\gamma = 1.4$. The other one is a two-dimensional double rarefaction with the initial condition $\rho_L = \rho_R = 7, u_L = -1, u_R = 1, v_L = v_R = 0, p_L = p_R = 0.2$. The exact solution contains vacuum.
Figure 8: Double rarefaction problem. T=0.6. Left: 1D problem. Right: Cut at $y = 0$ for the 2D problem. Every fourth cell is plotted. The solid line is the exact solution. Symbols are numerical solutions. $\Delta x = \frac{2}{N}$, $N = 800$ with the positivity limiter. Density.

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Figure 9: Double rarefaction problem. T=0.6. Left: 1D problem. Right: Cut at $y = 0$ for the 2D problem. Every fourth cell is plotted. The solid line is the exact solution. Symbols are numerical solutions. $\Delta x = \frac{2}{N}$, $N = 800$ with the positivity limiter. Pressure.
Figure 10: Double rarefaction problem. $T=0.6$. Left: 1D problem. Right: Cut at $y = 0$ for the 2D problem. Every fourth cell is plotted. The solid line is the exact solution. Symbols are numerical solutions. $\Delta x = \frac{2}{N}, N = 800$ with the positivity limiter. Velocity.
The second one is a 1D Leblanc shock tube problem. The initial condition is $\rho_L = 2$, $\rho_R = 0.001$, $u_L = u_R = 0$, $p_L = 10^9$, $p_R = 1$, and $\gamma = 1.4$. See the next figure for the results of 800 cells and 6400 cells.
Figure 11: Leblanc problem. $T = 0.0001$. Left: $N = 800$. Right: $N = 6400$. The solid line is the exact solution. Symbols are numerical solutions. $\Delta x = \frac{20}{N}$ with the positivity limiter. Log-scale of density.
Figure 12: Leblanc problem. \( T = 0.0001 \). Left: \( N = 800 \). Right: \( N = 6400 \). The solid line is the exact solution. Symbols are numerical solutions. \( \Delta x = \frac{20}{N} \) with the positivity limiter. log-scale of pressure.
Figure 13: Leblanc problem. $T = 0.0001$. Left: $N = 800$. Right: $N = 6400$. The solid line is the exact solution. Symbols are numerical solutions. $\Delta x = \frac{20}{N}$ with the positivity limiter. Velocity.
Example 7. To simulate the gas flows and shock wave patterns which are revealed by the Hubble Space Telescope images, one can implement theoretical models in a gas dynamics simulator. The two-dimensional model without radiative cooling is governed by the compressible Euler equations. The velocity of the gas flow is extremely high, and the Mach number could be hundreds or thousands. A big challenge for computation is, even for a state-of-the-art high order scheme, negative pressure could appear since the internal energy is very small compared to the huge kinetic energy (Ha, Gardner, Gelb and Shu, JSC 2005).

First, we compute a Mach 80 (i.e. the Mach number of the jet inflow is 80 with respect to the soundspeed in the jet gas) problem without the radiative cooling.
Figure 14: Simulation of Mach 80 jet without radiative cooling. Scales are logarithmic. Density.
Second, to demonstrate the robustness of our method, we compute a Mach 2000 problem. The domain is $[0, 1] \times [0, 0.5]$. The width of the jet is 0.1. The terminal time is 0.001. The speed of the jet is 800, which is around Mach 2100 with respect to the soundspeed in the jet gas. The computation is performed on a $640 \times 320$ mesh. TVB limiter parameters are $M_1 = M_2 = M_3 = M_4 = 10000000$. 
Figure 15: Simulation of Mach 2000 jet without radiative cooling. Scales are logarithmic. Density.
Lastly, we compute a Mach 80 (i.e. the Mach number of the jet inflow is 80 with respect to the soundspeed in the jet gas) problem with the radiative cooling to test the positivity-preserving property with the radiative cooling source term.
Figure 16: Simulation of Mach 80 jet with radiative cooling. The third order positivity-preserving RKDG scheme with the TVB limiter. Scales are logarithmic. Density.
Example 8. Shock diffraction problem. Shock passing a backward facing corner. It is easy to get negative density and/or pressure below and to the right of the corner. The setup is the following: the computational domain is the union of $[0, 1] \times [6, 11]$ and $[1, 13] \times [0, 11]$; the initial condition is a pure right-moving shock of $Mach = 5.09$, initially located at $x = 0.5$ and $6 \leq y \leq 11$, moving into undisturbed air ahead of the shock with a density of $1.4$ and pressure of $1.4$. $\gamma = 1.4$ and the TVB limiter parameters $M_i = 100$ for $i = 1, 2, 3, 4$. The density and pressure at $t = 2.3$ are presented.
Figure 17: Shock diffraction problem. Density: 20 equally spaced contour lines from $\rho = 0.066227$ to $\rho = 7.0668$. Left: $\Delta x = \Delta y = 1/32$; Right: $\Delta x = \Delta y = 1/64$. 

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Concluding remarks

- We have obtained, for the first time, high order schemes for multi-dimensional nonlinear scalar conservation laws and passive convections in incompressible velocity fields that satisfy strict maximum principle, using an easy procedure involving only slight change from standard finite volume and DG schemes with SSP time discretizations.

- This technique has been generalized to 2D triangles, and to positivity preserving schemes for compressible Euler equations and shallow water equations.
The End

THANK YOU!