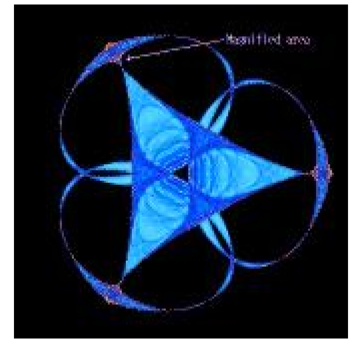
Mathematical issues in stability of viscoelastic flows



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Linear stability problem

$$\dot{x} = Ax, \quad x(0) = x_0.$$

Solution:

$$x(t) = \exp(At)x_0.$$

Finite-dimensional case (A is a matrix):
λ is an eigenvalue of A if and only if exp(λt) is an eigenvalue of exp(At).
Consequence: The system is asymptotically stable if and only if all eigenvalues of A are in the left half plane.

Infinite dimensional case: C_0 semigroups

For a class of operators A, exp(At) can be defined as

$$\exp(At) = \lim_{n\to\infty}(I - \frac{At}{n})^{-n}.$$

Spectra of infinite dimensional operators

λ is in the resolvent set of A if $(A-\lambda I)^{-1}$ is defined everywhere and bounded.
If λ is not in the resolvent set, it is in the spectrum.

For the following, we distinguish two parts of the spectrum:

1. Isolated eigenvalues of finite multiplicity.
2. Everything else, which is called the essential spectrum.

Isolated eigenvalues of A and exp(At) still correspond to each other.

But this is not necessarily true for the essential spectrum.

For Navier-Stokes in bounded domains, only isolated eigenvalues occur.

Essential spectra cannot be changed by compact perturbations.

So what does this all mean?

It means that …

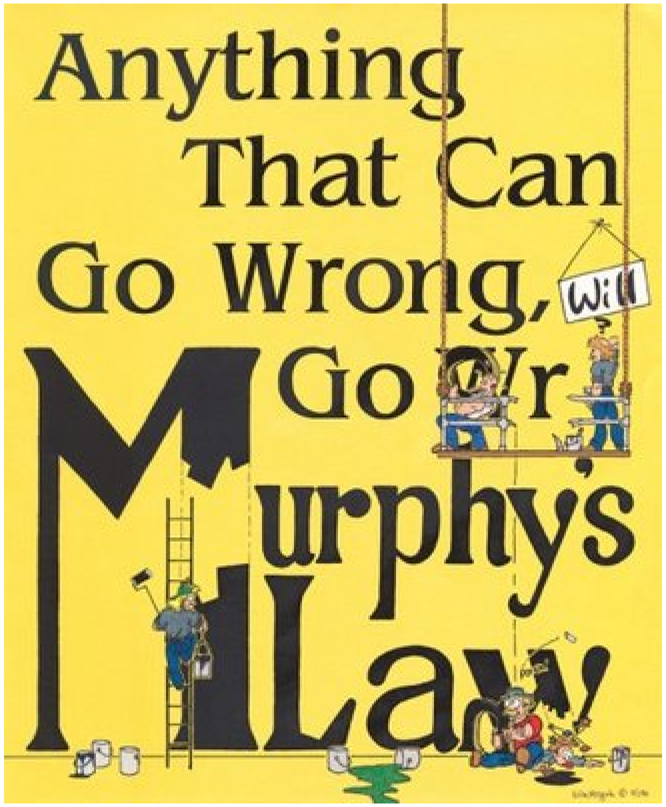
There is bad news.

There is more bad news.

The more bad news is also the good news.

The bad news: Should we really worry about this?

So essential spectra of A and exp(At) may not correspond. This is a mathematical possibility, but does this happen in “real physical problems”?



Example:

$$u_{tt} = u_{xx} + u_{yy} + e^{iy} u_x ,$$

with 2π-periodic boundary conditions. Use the function space $H^1 \times L^2$, set U=(u,v), and

$$AU = (v, u_{xx} + u_{yy} + e^{iy} u_x).$$

Then:

1. A generates a C_0-semigroup.
2. The spectrum of A consists purely of isolated eigenvalues, all of which are purely imaginary.
3. exp(At) has essential spectrum on the circle of radius exp(t/2).

More bad news: Numerical approximation

Any numerical discretization is finite dimensional. But essential spectra are immovable by finite dimensional perturbations!

Example:

$$Au(x) = e^{ix}u(x).$$

Discretize by spectral method (we know it converges fast):

$$u_N = \sum_{k=-N}^{N} u_k e^{ikx},$$

$$A_N u_N = \sum_{k=-N+1}^{N} u_{k-1} e^{ikx}.$$

A_N has precisely one eigenvalue, namely 0. This is superfast convergence to a completely wrong result!

The good news:

Lots of other operators have the same essential spectrum as the one we are interested in.

Folgers Crystal Method (FCM):

We secretly replace the fine operator which usually describes our problem with another, easier one. If the customers only care about the essential spectrum, they will not notice the difference!

Multimode Maxwell models in creeping flow

Governing equations:

$$\operatorname{div} \mathbf{T} - \nabla p = \mathbf{f}(\mathbf{x}),$$

$$\operatorname{div} \mathbf{v} = 0,$$

$$\mathbf{T} = \sum_{i=1}^{N} \mathbf{T}^i,$$

$$(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla))\mathbf{T}^i = \mathbf{G}^i(\nabla \mathbf{v}, \mathbf{T}^1, \cdots, \mathbf{T}^N).$$

We assume periodic boundary conditions. For consistency, the body force f driving the flow is assumed to have zero average. We assume there is a stationary solution

$$\mathbf{v} = \mathbf{V}(\mathbf{x}), \qquad p = P(\mathbf{x}), \qquad \mathbf{T}^i = \mathbf{S}^i(\mathbf{x}).$$

Linearized equations:

$$\mathrm{div}\,(\sum_{i=1}^{N}\mathbf{T}^i) - \nabla p = \mathbf{0},$$

$$\mathrm{div}\,\mathbf{v} = 0,$$

$$(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla))\mathbf{T}^i = -(\mathbf{v}\cdot\nabla)\mathbf{S}^i + \mathcal{A}^i(\mathbf{x}) :: \nabla\mathbf{v} + \sum_{j=1}^{N}\mathcal{B}^{ij}(\mathbf{x}) :: \mathbf{T}^j$$

$$=: \mathbf{H}^i(\mathbf{v},\mathbf{T}^1,...,\mathbf{T}^N).$$

Here $\mathcal{A}^i$ and $\mathcal{B}^{ij}$ are fourth order tensors related to derivatives of $\mathbf{G}^i$, and $\mathbf{H}^i$ is a differential operator with variable coefficients which is first order with respect to **v** and zeroth order with respect to $\mathbf{T}^i$.

Reformulation

Apply the material derivative operator to the momentum equation:

$$0 = (\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla))\left[\frac{\partial T_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i}\right] = \frac{\partial}{\partial x_j}\left[(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla))T_{ij}\right]$$

$$-\frac{\partial}{\partial x_i}\left[\frac{\partial p}{\partial t} + (\mathbf{V}\cdot\nabla)p\right] - \frac{\partial V_k}{\partial x_j}\frac{\partial T_{ij}}{\partial x_k} + \frac{\partial V_k}{\partial x_i}\frac{\partial p}{\partial x_k}.$$

Leave the last two terms alone, introduce a new variable

$$q = \frac{\partial p}{\partial t} + (\mathbf{V}\cdot\nabla)p,$$

and replace the first term by taking the divergence of the constitutive equation.

$$\mathrm{div}\left[(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla))\mathbf{T}^i\right] = \mathcal{L}^i(\mathbf{v}, \mathbf{T}^1, ..., \mathbf{T}^N).$$

Here $\mathcal{L}^i$ is a differential operator that is second order in v and first order with respect to the stresses.

Ultimately, we find an equation of the form

$$\frac{\partial}{\partial x_j}[C_{ijkl}(\mathbf{x})\frac{\partial v_l}{\partial x_k} + D_{ijl}(\mathbf{x})v_l + E_{ijklm}(\mathbf{x})T^m_{kl} + F_{ij}(\mathbf{x})p - \delta_{ij}q] = 0.$$

We combine this equation with the incompressibility condition

$$\mathrm{div}\,\mathbf{v} = 0.$$

We assume that this system (with **v** and q as unknowns) is elliptic and can be solved uniquely for **v** and q, subject to a condition of zero average.

Pseudodifferential operators

Consider periodic functions

$$\phi(\mathbf{x}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{x}).$$

A pseudodifferential operator is an operator of the form

$$A(\mathbf{x},\nabla)\phi(\mathbf{x}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}} g(\mathbf{x},\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}).$$

Here g is a smooth function of its arguments called the symbol of the operator. A pseudodifferential operator is a differential operator if g is a polynomial in **k**. A pseudodifferential operator is of order m, if, for large |**k**|,

$$g(\mathbf{x},\mathbf{k}) = g_p(\mathbf{x},\mathbf{k}) + g_r(\mathbf{x},\mathbf{k}),\ \ g_p(\mathbf{x},\lambda\mathbf{k}) = \lambda^m g_p(\mathbf{x},\mathbf{k}),\ \ g_r(\mathbf{x},\mathbf{k}) = O(|\mathbf{k}|^{m-1}).$$

g_p is called the principal part of the symbol.

The symbol of a product of pseudodifferential operators is not the product of the symbols.

However, the principal part of the symbol of a product is the product of the principal parts of the symbols.

Pseudodifferential operators of negative order arise from inverting differential operators. For instance, the symbol of Δ^{-1} is $1/|\mathbf{k}|^2$.

Our elliptic system above leads to

$$\mathbf{v} = \mathcal{M}(\mathbf{T}^1, ..., \mathbf{T}^N, p), \qquad q = \mathcal{N}(\mathbf{T}^1, ...\mathbf{T}^N, p),$$

where $\mathcal{M}$ is a pseudodifferential operator of order -1 and $\mathcal{N}$ is a pseudodifferential operator of order 0.

We finally end up with the system

$$(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla))\mathbf{T}^i = \mathbf{H}^i(\mathcal{M}(\mathbf{T}^1,...,\mathbf{T}^N,p),\mathbf{T}^1,...,\mathbf{T}^N)),$$

$$(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla))p = \mathcal{N}(\mathbf{T}^1,...,\mathbf{T}^N,p),$$

subject to the differential constraint

$$\mathrm{div}\,\mathbf{T} - \nabla p = \mathbf{0}.$$

Advective equations

These are systems of the form

$$\phi_t = \mathcal{L}\phi = -(\mathbf{V}(\mathbf{x})\cdot\nabla)\phi + A(\mathbf{x},\nabla)\phi,$$

where A is a zeroth order pseudodifferential operator with principal symbol $a_0(\mathbf{x},\mathbf{k})$, and **V** is a divergence-free vector field.

R. Shvydkoy (2006): To determine linear stability, we must investigate

1. The discrete eigenvalues of $\mathcal{L}$.
2. The growth rate associated with the following bicharacteristic amplitude system:

$$\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x}),$$

$$\dot{\mathbf{k}} = -(\nabla \mathbf{V}(\mathbf{x}))^T \mathbf{k},$$

$$\dot{b} = a_0(\mathbf{x}, \mathbf{k}) b.$$

Differential constraints like the one above translate to the requirement that b must lie in a subspace which depends on **k**.
Growth rate:

$$\mu = \lim_{t \to \infty} \frac{1}{t} \log \sup_{\mathbf{x}_0, \mathbf{k}_0, b_0} \frac{|b(\mathbf{k}_0, \mathbf{k}_0, b_0, t)|}{|b_0|}.$$

Remarks:

1. The above result is stated for stability in the function space L^2. It can be modified for other Sobolev spaces; in general with different results for μ.
2. In terms of our original variables, we can write the amplitude system as follows: Let $\mathbf{B}^i$, **w** and ψ be the amplitudes associated with $\mathbf{T}^i$, **v** and p. Then the amplitude system is

$$\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x}),$$

$$\dot{\mathbf{k}} = -(\nabla \mathbf{V}(\mathbf{x}))^T \mathbf{k},$$

$$\dot{\mathbf{B}}^i = \mathcal{A}^i(\mathbf{x}) :: (\mathbf{w}\mathbf{k}^T) + \sum_{j=1}^{N} \mathcal{B}^{ij}(\mathbf{x}) :: \mathbf{B}^j,$$

$$\mathbf{0} = \sum_{i=1}^{N} \mathbf{B}^i \mathbf{k} - \psi \mathbf{k}$$

$$0 = \mathbf{k} \cdot \mathbf{w}.$$

Where does this amplitude equation come from?

WKB approximation: Set

$$\phi(\mathbf{x}, t) = b(\mathbf{x}, t) \exp(iS(\mathbf{x}, t)/\epsilon)).$$

To leading order, you find

$$S_t + (\mathbf{V} \cdot \nabla)S = 0,$$

$$b_t + (\mathbf{V} \cdot \nabla)b = a_0(\mathbf{x}, \nabla S)b.$$

Now set $\nabla S = \mathbf{k}$.

Example: Shear flow of the Johnson-Segalman fluid

Governing equations:

$$\frac{\partial T_{11}}{\partial t} - (1+a)T_{12}\frac{\partial u}{\partial y} + \lambda T_{11} = 0,$$

$$\frac{\partial T_{12}}{\partial t} - \frac{1+a}{2}T_{22}\frac{\partial u}{\partial y} + \frac{1-a}{2}T_{11}\frac{\partial u}{\partial y} + \lambda T_{12} = \mu\frac{\partial u}{\partial y},$$

$$\frac{\partial T_{22}}{\partial t} + (1-a)T_{12}\frac{\partial u}{\partial y} + \lambda T_{22} = 0,$$

$$\frac{\partial T_{12}}{\partial y} = f(y),$$

$$\frac{\partial}{\partial y}(T_{22} - p) = 0.$$

Introduce a new variable:

$$Z = \frac{1-a}{2}T_{11} - \frac{1+a}{2}T_{22}.$$

Reduced system:

$$\frac{\partial Z}{\partial t} - (1-a^2)T_{12}\frac{\partial u}{\partial y} + \lambda Z = 0,$$

$$\frac{\partial T_{12}}{\partial t} + Z\frac{\partial u}{\partial y} + \lambda T_{12} = \mu\frac{\partial u}{\partial y},$$

$$\frac{\partial T_{12}}{\partial y} = f(y).$$

Steady flow:

$$u = U(y),$$

$$T_{12} = S_{12}(y) = \frac{\lambda\mu U'(y)}{\lambda^2 + (1-a^2)U'(y)^2},$$

$$Z = Z_0(y) = \frac{(1-a^2)\mu U'(y)^2}{\lambda^2 + (1-a^2)U'(y)^2}.$$

Linearized system:

$$\frac{\partial Z}{\partial t} - (1-a)^2(U'(y)T_{12} + S_{12}(y)\frac{\partial u}{\partial y}) + \lambda Z = 0,$$

$$\frac{\partial T_{12}}{\partial t} + U'(y)Z + Z_0(y)\frac{\partial u}{\partial y} + \lambda T_{12} = \mu\frac{\partial u}{\partial y},$$

$$\frac{\partial T_{12}}{\partial y} = 0.$$

Eliminate u:

$$\frac{\partial}{\partial y}((\mu - Z_0(y))\frac{\partial u}{\partial y} - U'(y)Z) = 0.$$

Bicharacteristic amplitude equation (for amplitude corresponding to Z):

$$b_t + (\lambda - \frac{(1-a^2)S_{12}(y)U'(y)}{\mu - Z_0(y)})b = 0.$$

This leads to instability if the steady shear stress is decreasing as a function of shear rate.

The good news:

We have a rigorous stability result covering a reasonably broad class of constitutive models.

The bad news:

1. Restricted to periodic boundary conditions: A grand challenge for experimenters.
2. Restricted to creeping flow.
3. Nonlinear stability remains open.
4. In general, we do not know if the essential spectrum of $\exp(\mathcal{L}t)$ is related to that of $\mathcal{L}$ (Shvydkoy's paper has some partial results on this).

More is known for the upper convected Maxwell fluid.

Thanks to Dan Joseph, we know where to go to get an upper convected Maxwell fluid.

Announcement of the grand opening of

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Using transformations of the equations which work only for the UCM model, the essential spectrum can be reduced to studying the following equations:

$$\frac{\partial \mathbf{T}}{\partial t} = -(\mathbf{V}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{V})\mathbf{T} - \mathbf{T}(\nabla\mathbf{V})^T - \lambda\mathbf{T},$$

$$\frac{\partial s}{\partial t} = -(\mathbf{V}\cdot\nabla)s - \lambda s,$$

$$\frac{\partial r}{\partial t} = \mathcal{B}r.$$

Here r is a function on the boundary, and the operator $\mathcal{B}$ is nonlocal, but bounded. No advection, if we assume **V** vanishes on the boundary.
The first two equations fit into the theory of “evolution semigroups” (Chicone and Latushkin). More special than “advective” equations, no pseudo stuff involved.
The essential spectrum decomposes into a “bulk” part and a “boundary” part associated with $\mathcal{B}$ (Gorodtsov-Leonov wall modes).


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Questions?