Integrals of Characteristic Polynomials of Unitary Matrices, and Applications to the Riemann Zeta Function

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$A \in U(N)$: The group of $N \times N$ unitary matrices

$\mathrm{d}A$: Haar measure on $U(N)$, normalized to have volume 1

Characteristic polynomial of A^*:

$$\Lambda_A(t) = \det(I_N - tA^*), \quad t \in \mathbb{C}$$

Conrey-Rubinstein-Snaith, Dehaye, ...

The Problem: For $k, l \in \mathbb{N}$ evaluate, exactly,

$$\int_{U(N)} |\Lambda_A(1)|^{2k} \, |\Lambda'_A(1)|^{2l} \, \mathrm{d}A$$

Determine the asymptotic behavior of the integral as $N \to \infty$

Analytic continuation to $k, l \in \mathbb{C}$?

The Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1$$

Analytic continuation of $\zeta(s)$ to other values of s

The functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\, \Gamma(1-s)\, \zeta(1-s), \qquad s \neq 0, 1$$

$\zeta(s) = 0$ for $s = -2, -4, \ldots$ because of the factor $\sin(\pi s/2)$

$s = -2, -4, \ldots$ are called the trivial zeros of $\zeta(s)$

Where are the non-trivial zeros of $\zeta(s)$?

All non-trivial zeros lie in the strip $\{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1\}$

Enormous evidence exists that all non-trivial zeros lie on the critical line $\{s \in \mathbb{C} : \mathrm{Re}(s) = 1/2\}$

The Riemann Hypothesis: All non-trivial zeros lie on the critical line $\{s \in \mathbb{C} : \mathrm{Re}(s) = 1/2\}$

X. Gordon (2004), RH "verified until the 10^{13}-th zero"

Computations carried out using a "fast Zeta multi-evaluation algorithm" developed by Odlyzko and Schönhage"

The distribution of the zeros of $\zeta(s)$ on the critical line

The asymptotic behavior of

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t, \quad T \to \infty$$

Hardy-Littlewood, Selberg, Ingham, Ramachandra, ...

Matsumoto, “Recent developments in the mean square theory of the Riemann zeta and other zeta-functions”

Relationship between the zeros of $\zeta(s)$ on the critical line and the eigenvalues of large unitary matrices

Montgomery, Dyson, Odlyzko, Berry, Hejhal, ...

Katz-Sarnak-Rudnick, Hughes-Keating-O'Connell, Hall, Conrey-Farmer-Rubinstein-Snaith, Coram-Diaconis, ...

Keating-Snaith:

> "The conclusion to be drawn is that the statistical distribution of the Riemann zeros, in the limit as one looks infinitely high up the critical line, coincides with the statistical distribution of the eigenvalues of random unitary matrices, in the limit of large matrix size."

Recall: $\Lambda_A(t) = \det(I_N - tA^*), \quad t \in \mathbb{C}$

The asymptotic behavior of

$$\int_{U(N)} |\Lambda_A(1)|^{2k} \, \mathrm{d}A, \quad N \to \infty,$$

leads to a conjecture about the asymptotic behavior of

$$\frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} \, \mathrm{d}t, \quad T \to \infty$$

And similarly if we replace Λ_A by Λ'_A and ζ by ζ'

Conrey, et al. showed that, for some constant b_k,

$$\int_{U(N)} |\Lambda'_A(1)|^{2k}\,\mathrm{d}A \sim b_k N^{k^2+2k}, \quad N \to \infty$$

This lead them to conjecture that

$$\frac{1}{T}\int_0^T |\zeta'(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \sim a_k b_k (\log T)^{k^2+2k}, \quad T \to \infty$$

where a_k is an “arithmetic factor,”

$$a_k = \prod_p \Big(1 - \frac{1}{p}\Big)^{k^2} \cdot \sum_{j=0}^{\infty} \Big(\frac{\Gamma(j+k)}{j!\,\Gamma(k)}\Big)^2 p^{-j}$$

Conrey, et al. left open the exact value of their integrals

Also open is the issue of whether $b_k \neq 0$

We have evaluated these integrals exactly

Hypergeometric functions of matrix argument

Keating-Snaith proved:

$$\int_{U(N)} |\Lambda_A(1)|^{2k} \,\mathrm{d}A = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\big(\Gamma(j+k)\big)^2}$$

An identity for the hypergeometric functions of matrix argument

We have also evaluated generalizations of this integral in which $\Lambda_A(1)$ is replaced by ${}_pF_q(A)$

Partition: $\mu = (\mu_1, \ldots, \mu_n)$, integers, where $\mu_1 \geq \cdots \geq \mu_n \geq 0$

Length of μ: The number of non-zero μ_j

Weight of μ: $|\mu| = \mu_1 + \cdots + \mu_n$

Classical rising factorial

$$(a)_m = a(a+1)\cdots(a+m-1), \quad m = 0, 1, 2, \ldots$$

Partitional rising factorial:

$$(a)_\mu = \prod_{j=1}^{n} (a - j + 1)_{\mu_j}$$

Schur function: For $t_1, \ldots, t_n \in \mathbb{C}$,

$$\chi_\mu(t_1, \ldots, t_n) = \frac{\det({t_i}^{\mu_j + n - j})}{\prod_{1 \le i < j \le n}(t_i - t_j)}$$

χ_μ is homogeneous of degree $|\mu|$, symmetric in $t_1, \ldots, t_n$

χ_μ satisfies an amazing number of properties:

Characters of irreducible representations of $U(n)$

Generating functions for weights of semi-standard Young tableaux

Spherical functions on the cone of Hermitian p.d. matrices

Extend χ_μ to matrix space:

$$\chi_\mu(T) = \chi_\mu(t_1, \ldots, t_n)$$

where $t_1, \ldots, t_n$ are the eigenvalues of T

$\{\chi_\mu(T) : |\mu| = j\}$ is a basis for the vector space of polynomials which are symmetric and homogeneous of degree j

There exists constants ω_μ such that

$$(\operatorname{tr} T)^j = \sum_{|\mu|=j} \omega_\mu \chi_\mu(T)$$

$$\omega_\mu = |\mu|! \frac{\prod_{1\le i<j\le n}(\mu_i - \mu_j - i + j)}{\prod_{j=1}^{n}(\mu_j + n - j)!}$$

$Z_\mu(T) = \omega_\mu \chi_\mu(T)$: The (complex) zonal polynomial

$$(\operatorname{tr} T)^j = \sum_{|\mu|=j} Z_\mu(T)$$

The exponential series

$$e^{\operatorname{tr} T} = \sum_{\mu} \frac{1}{|\mu|!} Z_\mu(T)$$

The binomial theorem

$$\det(I_N - T)^a = \sum_{\mu} \frac{(-a)_\mu}{|\mu|!} Z_\mu(T), \quad ||T|| < 1$$

Hypergeometric functions of Hermitian matrix argument

Warm-up exercise: The Keating-Snaith integral,

$$\int_{U(N)} \big(\Lambda_A(s)\big)^{r_1} \big(\Lambda_{A^*}(t)\big)^{r_2} \,\mathrm{d}A_N, \quad |s|,|t|<1$$

$$\big(\Lambda_A(s)\big)^{r_1} = \det(I_N - sA^*)^{r_1} = \sum_\mu \frac{(-r_1)_\mu}{|\mu|!} Z_\mu(sA^*),$$

$$\big(\Lambda_{A^*}(t)\big)^{r_2} = \det(I_N - tA)^{r_2} = \sum_\mu \frac{(-r_2)_\mu}{|\mu|!} Z_\mu(tA),$$

We need to calculate

$$\int_{U(N)} \chi_\mu(A^*)\chi_\lambda(A) \,\mathrm{d}A_N$$

χ_μ is the character of an irreducible representation of $U(N)$

Weyl orthogonality relations

$$\int_{U(N)} \chi_\mu(A^*)\chi_\lambda(A)\,\mathrm{d}A_N = \delta_{\mu,\lambda}$$

$$\int_{U(N)} \big(\Lambda_A(s)\big)^{r_1} \big(\Lambda_{A^*}(t)\big)^{r_2}\,\mathrm{d}A_N = \sum_\mu \frac{(-r_1)_\mu(-r_2)_\mu}{|\mu|!\,(N)_\mu}\, Z_\mu(stI_N)$$
$$= {}_2F_1(-r_1, -r_2; N; stI_N)$$

Let $s, t \to 1-$; apply a generalization of Gauss' formula for ${}_2F_1(1)$

$$\int_{U(N)} \left(\Lambda_A(1)\right)^{r_1} \left(\Lambda_{A^*}(1)\right)^{r_2} \mathrm{d}A_N = \prod_{j=1}^{N} \frac{\Gamma(j)\,\Gamma(j+r_1+r_2)}{\Gamma(j+r_1)\,\Gamma(j+r_2)}$$

Apply methods standard in multivariate statistical analysis to derive asymptotic expansions as $N \to \infty$

Moments of products of independent beta random variables

Central Limit Theorem

What about integrals involving Λ_A and Λ'_A ...

Theorem: For $k, l = 0, 1, 2, \ldots$,

$$\int_{U(N)} \left(\Lambda'_A(1)\right)^k \left(\Lambda'_{A^*}(1)\right)^l \, \mathrm{d}A_N$$

$$= \left[\prod_{j=1}^{l} \frac{(N+j)_k}{(j)_k}\right] (-1)^k \sum_{j=0}^{l} \binom{l}{j} N^{l-j} \sum_{|\mu|=k+j} \frac{(-N)_\mu}{(k+l)_\mu} Z_\mu(I_l).$$

Extract the highest power of N to obtain asymptotics as $N \to \infty$

Corollary: As $N \to \infty$,

$$\int_{U(N)} \left(\Lambda'_A(1)\right)^k \left(\Lambda'_{A^*}(1)\right)^l \, \mathrm{d}A_N \sim b_{k,l} N^{-(kl+k+l)}$$

where

$$b_{k,l} = \left[\prod_{j=1}^{l} \frac{1}{(j)_k}\right] \sum_{j=0}^{l} (-1)^j \binom{l}{j} \sum_{|\mu|=k+j} \frac{Z_\mu(I_k)}{(k+l)_\mu}$$

Conjecture: As $T \to \infty$,

$$\frac{1}{T}\int_0^T |\zeta'(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \sim a_k b_{k,k} (\log T)^{k^2+2k}$$

Open problem: For which values of k, l is $b_{k,l} \neq 0$?

Conjecture: $b_{k,k} \neq 0$

Laguerre polynomials of matrix argument

Similarly, we have exact results, and asymptotics as $N \to \infty$, for

$$\int_{U(N)} |\Lambda_A(1)|^{2(k-h)} \, |\Lambda'_A(1)|^{2h} \, \mathrm{d}A$$

Theorem: For $k-h, h = 0, 1, 2, \ldots$,

$$\int_{U(N)} |\Lambda_A(1)|^{2(k-h)} \, |\Lambda'_A(1)|^{2h} \, \mathrm{d}A \sim b'_{h,k} \, N^{k^2+2h}, \quad N \to \infty$$

with $b'_{h,k} > 0$ (we have an explicit, closed-form formula for $b_{h,k}$)

Conjecture: As $T \to \infty$,

$$\frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2(k-h)} \, |\zeta'(\tfrac{1}{2}+it)|^{2h} \, \mathrm{d}t$$

$$\sim a_k b_{h,k} (\log T)^{k^2+2h}$$

$e^{i\theta_1}, \ldots, e^{i\theta_N}$: The eigenvalues of $A \in U(N)$

In analogy with Hardy's Z-function, define

$$\mathcal{Z}_A(t) = e^{-\pi i N/2} \, e^{i \sum_{j=1}^{N} \theta_j/2} \, t^{-N/2} \, \Lambda_A(t), \qquad t \in \mathbb{C}$$

Conrey-Rubinstein-Snaith (2006)

For $k, l \in \mathbb{N}$ evaluate, exactly,

$$\int_{U(N)} |\mathcal{Z}_A(1)|^{2(k-h)} \, |\mathcal{Z}'_A(1)|^{2h} \, \mathrm{d}A$$

Determine the asymptotic behavior of the integral as $N \to \infty$

Analytic continuation to $k, l \in \mathbb{C}$?

By similar methods, we have also evaluated

$$\int_{U(N)} |\mathcal{Z}_A(1)|^{2(k-h)}\, |\mathcal{Z}'_A(1)|^{2h}\, \mathrm{d}A$$

exactly and deduced its asymptotic behavior as $N \to \infty$

Theorem: For $k-h, h = 0, 1, 2, \ldots$,

$$\int_{U(N)} |\mathcal{Z}_A(1)|^{2(k-h)}\, |\mathcal{Z}'_A(1)|^{2h}\, \mathrm{d}A \sim b'_{h,k}\, N^{k^2+2h}, \quad N \to \infty$$

with $b'_{h,k} > 0$ (we have an explicit, closed-form formula for $b'_{h,k}$)

Riemann-Siegel theta function:

$$\theta(t) = \arg\left(\Gamma(\tfrac{1}{2}it + \tfrac{1}{4})\right) - \tfrac{1}{2}t\log\pi, \quad t \in \mathbb{R}$$

Hardy's Z-function:

$$Z(t) = e^{i\theta(t)}\zeta(\tfrac{1}{2} + it)$$

Conjecture: As $T \to \infty$,

$$\frac{1}{T}\int_0^T \left|Z(\tfrac{1}{2} + it)\right|^{2(k-h)} \left|Z'(\tfrac{1}{2} + it)\right|^{2h} \mathrm{d}t \sim a_k b_{h,k} (\log T)^{k^2+2h}$$

with $b_{h,k} > 0$ (we have an explicit, closed-form formula for $b_{h,k}$)

The conjecture is known to be valid for $0 \le h, k \le 2$

Hardy-Littlewood, Selberg, ...

Hall (2004):

> The idea that Random Matrix Theory would predict the moment formula (4) for every k . . . is due to ... Bob Vaughan.”

How do we compute these integrals? Intuition from:

Multivariate statistical analysis

Hypergeometric functions of matrix argument

Gross and Richards, “Total positivity, spherical series, and hypergeometric functions of matrix argument,” J. Approx. Theory, 1989

Gross and Richards, “Total positivity, finite reflection groups, and a formula of Harish-Chandra,” J. Approx. Theory, 1995

RMT-based conjectures for zeta functions related to general root systems

Start with an integral (Conrey, et al)

$$\int_{U(N)} \Lambda_A(t_1)\cdots\Lambda_A(t_k)\Lambda_{A^*}(t_{k+1}^{-1})\cdots\Lambda_{A^*}(t_{k+l}^{-1})\,\mathrm{d}A_N$$

$$= \frac{\chi_{(N^l)}(t_1,\ldots,t_{k+l})}{(t_{k+1}\cdots t_{k+l})^N}$$

Write $\chi_{(N^l)}(t_1,\ldots,t_{k+l})$ as a sum of monomials (Young tableaux)

Divide the sum by $(t_{k+1}\cdots t_{k+l})^N$ to get the integral as a sum of rational terms

Differentiate w.r.t. each t_j and evaluate at all $t_j = 1$

Refuse to be daunted by the horrific-looking sum

Recognize eventually that it is a sum of derivatives of $\chi_\mu(T)$ evaluated at $T = I_l$

Apply a binomial theorem for χ_μ to evaluate those derivatives

Finally, simplify some daunting expressions

I expect that this method will work for higher derivatives of Λ_A

Conjectures for the value distribution of $\zeta^{(r)}(s)$ on the critical line