Generalized disjunctive programming:
A framework for formulation and alternative algorithms for MINLP optimization

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IMA Hot Topics Workshop:
Mixed-Integer Nonlinear Optimization: Algorithmic Advances and Applications
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Motivation

Discrete/Continuous Optimization
- Nonlinear models
- 0-1 and continuous decisions

- Optimization Models
  - Mixed-Integer Linear Programming (MILP)
  - Mixed-Integer Nonlinear Programming (MINLP)

Alternative approach:
- Logic-based: Generalized Disjunctive Programming (GDP)

Challenges
- How to develop “best” model?
- How to improve relaxation?
- How to solve nonconvex GDP problems to global optimality?
Outline

1. Overview of major relaxations for nonlinear GDP and algorithms

2. Linear GDP: hierarchy of relaxations

3. Global Optimization of nonconvex GDP

Ph.D. Students
Ramesh Raman
Metin Turkay
Sangbum Lee
Nick Sawaya
Juan Ruiz
MINLP

- Mixed-Integer Nonlinear Programming

\[ \min Z = f(x, y) \]
\[ \text{s.t. } g(x, y) \leq 0 \]
\[ x \in X, \ y \in Y \]
\[ X = \{x \mid x \in \mathbb{R}^n, x^L \leq x \leq x^U, Bx \leq b\} \]
\[ Y = \{y \mid y \in \{0,1\}^m, Ay \leq a\} \]

- \( f(x,y) \) and \( g(x,y) \) - assumed to be convex and bounded over \( X \).
- \( f(x,y) \) and \( g(x,y) \) commonly linear in \( y \)

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Mixed-integer Nonlinear Programming

Algorithms

**Branch and Bound (BB)** Ravindran and Gupta (1985),
Stubbs, Mehrotra (1999), Leyffer (2001)

**Generalized Benders Decomposition (GBD)** Geoffrion (1972)

**Outer-Approximation (OA)** Duran and Grossmann (1986),
Fletcher and Leyffer (1994)

**LP/NLP based Branch and Bound** Quesada, Grossmann (1994)

**Extended Cutting Plane (ECP)** Westerlund and Pettersson (1992)

Codes:

- **SBB** *GAMS* simple B&B
- **MINLP-BB (AMPL)** Fletcher and Leyffer (1999)
- **FilMINT** Linderot and Leyffer (2006)
- **DICOPT (GAMS)** Viswanathan and Grossman (1990)
- **AOA (AIMSS)**
- **α–ECP** Westerlund and Peterssson (1996)
- **MINOPT** Schweiger and Floudas (1998)
- **BARON** Sahinidis et al. (1998)
Generalized Disjunctive Programming

Motivation

1. Facilitate modeling of discrete/continuous optimization problems through use algebraic constraints and symbolic expressions
2. Reduce combinatorial search effort
3. Improve handling nonlinearities
Generalized Disjunctive Programming (GDP)

- Raman and Grossmann (1994) \((\text{Extension Balas, 1979})\)

\[
\begin{align*}
\min \quad & Z = \sum_k c_k + f(x) \\
\text{s.t.} \quad & r(x) \leq 0 \\
& \left[\begin{array}{c}
Y_{jk} \\
g_{jk}(x) \leq 0
\end{array}\right], k \in K \\
& c_k = \gamma_{jk} \\
& \Omega(Y) = \text{true} \\
x \in R^n, c_k \in R^1 \\
Y_{jk} \in \{\text{true, false}\}
\end{align*}
\]

Objective Function
Common Constraints
Disjunction
Constraints
Fixed Charges
Logic Propositions
Continuous Variables
Boolean Variables
Process Network with fixed charges

GDP model

Min \( Z = c_1 + c_2 + c_3 + d^T x \)

s.t.

\[
\begin{align*}
x_1 &= x_2 + x_4 \\
x_6 &= x_3 + x_5 \\
x_3 &= p_1 x_2 \\
c_1 &= r_1 \ \\
x_5 &= p_2 x_4 \\
c_2 &= r_2 \ \\
x_7 &= p_3 x_6 \\
c_3 &= r_3 \\
Y_{11} &\lor Y_{21} \\
Y_{12} &\lor Y_{22} \\
Y_{13} &\lor Y_{23} \\
Y_{11} &\lor Y_{12} \implies Y_{13} \\
Y_{13} &\implies Y_{11} \lor Y_{12} \\
Y_{21} &\lor Y_{22} \\
0 &\leq x \leq x^U \\
y_{11}, y_{21}, y_{12}, y_{22}, y_{13}, y_{23} &\in \{True, False\} \\
c_1, c_2, c_3 &\in \mathbb{R}^I
\end{align*}
\]
Generalized Disjunctive Programming (GDP)

• Raman and Grossmann (1994)

\[
\begin{align*}
\min & \quad Z = \sum_k c_k + f(x) \\
\text{s.t.} & \quad r(x) \leq 0 \\
& \quad \left[ Y_{jk} \right. \\
& \quad \left. g_{jk}(x) \leq 0 \right], k \in K \\
& \quad c_k = \gamma_{jk} \\
& \quad \Omega(Y) = \text{true} \\
& \quad x \in R^n, c_k \in R^1 \\
& \quad Y_{jk} \in \{\text{true, false}\}
\end{align*}
\]

Objective Function
Common Constraints
Disjunction
Constraints
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Continuous Variables
Boolean Variables

Relaxation?

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Big-M MINLP (BM)

• MINLP reformulation of GDP

\[
\begin{align*}
\min \ Z & = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + f(x) \\
\text{s.t.} \quad r(x) & \leq 0 \\
g_{jk}(x) & \leq M_{jk} (1 - \lambda_{jk}) \quad j \in J_k, k \in K \\
\sum_{j \in J_k} \lambda_{jk} & = 1, \quad k \in K \\
A\lambda & \leq a \\
x & \geq 0, \quad \lambda_{jk} \in \{0, 1\}
\end{align*}
\]

NLP Relaxation \[0 \leq \lambda_{jk} \leq 1\]
Convex Hull Formulation

• Consider Disjunction $k \in K$

$$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \leq 0 \\ c = \gamma_{jk} \end{bmatrix}$$

♦ Theorem: Convex Hull of Disjunction $k$  *(Lee, Grossmann, 2000)*

★ Disaggregated variables $\nu^j$

$$\{(x, c) \mid x = \sum_{j \in J_k} \nu^j \, , \, c_k = \sum_{j \in J_k} \lambda_{jk} \gamma_{jk} \, ,$$

$$0 \leq \nu^j \leq \lambda_{jk} u_{jk} \, , \, j \in J_k$$

$$\sum_{j \in J_k} \lambda_{jk} = 1, \quad 0 < \lambda_{jk} \leq 1,$$

$$\lambda_{jk} g_{jk}(\nu^j / \lambda_{jk}) \leq 0 \, , \, j \in J_k \}$$

★ $\lambda_j$ - weights for linear combination

- Generalization of Balas (1979)
- Stubbs and Mehrotra (1999)
Remarks

1. \( h(\nu, \lambda) = \lambda g(\nu / \lambda) \)

If \( g(x) \) is a bounded convex function, \( h(\nu, \lambda) \) is a bounded convex function \( \text{Hiriart-Urruty and Lemaréchal (1993)} \)

\[ h(\nu,0) = 0 \quad \text{for bounded } g(x) \]

2. Replace \( \lambda_{jk} g_{jk}(\nu_{jk} / \lambda_{jk}) \leq 0 \) where \( 0 \leq \nu_{jk} \leq U \lambda_{jk} \) by:

\[
((1-\varepsilon)\lambda_{jk} + \varepsilon)(g_{jk}(\nu_{jk} / ((1-\varepsilon)\lambda_{jk} + \varepsilon))) - \varepsilon g_{jk}(0)(1-\lambda_{jk}) \leq 0
\]

\( \text{Furman, Sawaya & Grossmann (2007)} \)

a. Exact approximation of the original constraints as \( \varepsilon \to 0 \).

b. The constraints are exact at \( \lambda_{jk} = 0 \) and at \( \lambda_{jk} = 1 \) regardless of value of \( \varepsilon \).

\( \text{if } \lambda_{jk} = 0, \Rightarrow (g_{jk}(0)) - \varepsilon g_{jk}(0)=0 \leq 0 \)

\( \text{if } \lambda_{jk} = 1, \Rightarrow ((1)g_{jk}(\nu_{jk} / (1)) - \varepsilon g_{jk}(0)(0)=(1)g_{jk}(\nu_{jk} / (1)) \leq 0 \)

c. The LHS of the new constraints are convex.
Convex Relaxation Problem (CRP)

\[
\text{CRP:} \quad \min Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + f(x)
\]
\[
\text{s.t.} \quad r(x) \leq 0
\]
\[
x = \sum_{j \in J_k} \nu_{jk}, \quad k \in K
\]
\[
0 \leq \nu_{jk} \leq \lambda_{jk} U_{jk}, \quad j \in J_k, \quad k \in K
\]
\[
\sum_{j \in J_k} \lambda_{jk} = 1, \quad k \in K
\]
\[
\lambda_{jk} g_{jk} (\nu_{jk} / \lambda_{jk}) \leq 0, \quad j \in J_k, \quad k \in K
\]
\[
A\lambda \leq a
\]
\[
x, \nu_{jk} \geq 0, \quad 0 \leq \lambda_{jk} \leq 1, \quad j \in J_k, \quad k \in K
\]

**Property:** The NLP (CRP) yields a lower bound to optimum of (GDP).

*Note: Hull relaxation as intersection of convex hull for each disjunction*
Strength Lower Bounds

Theorem: The relaxation of (CRP) yields a lower bound that is greater than or equal to the lower bound that is obtained from the relaxation of problem (BM):

RBM:

$$\min Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + f(x)$$

s.t. \( r(x) \leq 0 \)

$$g_{jk}(x) \leq M_{jk}(1 - \lambda_{jk}) \quad j \in J_k, k \in K$$

$$\sum_{j \in J_k} \lambda_{jk} = 1, \quad k \in K$$

$$A\lambda \leq a$$

$$x \geq 0, \quad 0 \leq \lambda_{jk} \leq 1$$

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MINLP Reformulation

Specify in CRP $\lambda$ as 0-1 variables

$$\min \ Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + f(x)$$

subject to

$$r(x) \leq 0$$

$$x = \sum_{j \in J_k} v^{jk}, \ k \in K$$

$$0 \leq v^{jk} \leq \lambda_{jk} U^{jk}, \ j \in J_k, \ k \in K$$

$$\sum_{j \in J_k} \lambda_{jk} = 1, \ k \in K$$

$$\lambda_{jk} g^{jk} (v^{jk} / \lambda_{jk}) \leq 0, \ j \in J_k, \ k \in K$$

$$A\lambda \leq a$$

$$x, v^{jk} \geq 0, \ \lambda_{jk} = 0,1 \ \ j \in J_k, \ k \in K$$
Methods Generalized Disjunctive Programming

Logic based methods
- Branch and bound
  (Lee & Grossmann, 2000)
- Decomposition
  Outer-Approximation
  Generalized Benders
  (Turkay & Grossmann, 1997)

Reformulation MINLP
- Outer-Approximation
- Generalized Benders
- Extended Cutting Plane
- Convex-hull
- Big-M
  Cutting plane
  (Sawaya & Grossmann, 2004)
A Branch and Bound Algorithm for GDP

- **Tree Search**
  - NLP subproblem at each node
- **Solve CRP of GDP**
  - Lower bound
- **Branching Rule**
  - Set the largest $\lambda_j$ as 1
  - Dichotomy rule
- **Logic inference**
  - CNF unit resolution (Raman & Grosmann, 1993)
- **Depth first search**
  - When all the terms are fixed
  - Upper bound
- **Repeat Branching until $Z^L > Z^U$.**

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GDP Example

- Find $x \geq 0, (x \in S_1) \lor (x \in S_2) \lor (x \in S_3)$ to minimize $Z = (x_1 - 3)^2 + (x_2 - 2)^2 + c$

- Objective Function = continuous function + fixed charge (discontinuous).

Contour of $f(x)$

Local solutions

Global Optimum
$(3.293, 1.707)$

$Z^* = 1.172$
Example: convex hull

Convex hull = \text{conv}(U S_j)
Example: CRP solution

Convex hull = \( \text{conv}(U\mathcal{S}_j) \)

Convex combination of \( z_j \)

\[ z_j = \psi / \lambda_j \]

Local solution point

Convex hull optimum, \( Z^L = 1.154 \)

\( x^L = (3.159, 1.797) \)

Infeasible to GDP

Weight

\( \lambda_1 = 0.016 \)
\( \lambda_2 = 0.955 \)
\( \lambda_3 = 0.029 \)
Example: branch and bound

First Node: $S_2$
Optimal solution: $Z^U = 1.172$

Optimal Solution
(3.293, 1.707)
$Z^* = 1.172$
Example: branch and bound

Second Node: \( \text{conv}(S_1 \cup S_3) \)
Optimal solution: \( Z^L = 3.327 \)

Lower Bound \( Z^L = 3.327 \)

Upper Bound \( Z^U = 1.172 \)
Example: Search Tree

- **Branching Rule:** $\lambda_j$ - the weight of disaggregated variable
  - Fix $Y_j$ as true: fix $\lambda_j$ as 1.
Process Network with Fixed Charges

- Türkay and Grossmann (1997)
  - Superstructure of the process

\[ Y_1 \lor Y_2 \]

\[ Y_4 \lor Y_5 \]

\[ Y_6 \lor Y_7 \]

\[ Y_i \lor Y_j \]

Specifications
Optimal solution

- Minimum Cost: $68.01M/year
Proposed BB Method

Proposed BB

\[ Z^L = 62.48 \]
\[ \lambda = [0.31, 0.69, 0.03, 1.0, 1, 0, 1, 1] \]

\[ Z^U = 68.01 = Z^* \]
\[ \lambda = [0, 1, 0.022, 1.0, 1, 0, 1] \]

Optimal Solution

Feasible Solution

\[ Z^L = 75.01 > Z^U \]
\[ \lambda = [1, 0, 0.022, 1.0, 1, 0, 1] \]

5 nodes vs. 17 nodes of Standard BB (lower bound = 15.08)

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Logic-based Outer Approximation

Main point: avoids solving MINLP in full space

Turkay, Grossmann (1997)

NLP Subproblem: (reduced)

\[
\begin{align*}
\min Z &= \sum_{k \in SD} c_k + f(x) \\
\text{s.t.} \quad g(x) &\leq 0 \\
h_{ik}(x) &\leq 0 \quad \text{for } Y_{ik} = \text{true} \quad i \in D_k, k \in SD \\
&\quad c_k = \gamma_{ik} \\
B'x &= 0 \quad \text{for } Y_{ik} = \text{false} \quad i \in D_k, i \neq \hat{i}, k \in SD \\
&\quad c_k = 0 \\
x &\in \mathbb{R}^n, c_i \in \mathbb{R}^m
\end{align*}
\] (NLPD)

Master Problem:

\[
\begin{align*}
\text{Min} \quad Z &= \sum_{i} c_i + \alpha \\
\text{s.t.} \quad \alpha &\geq f(x') + \nabla f(x')^T (x - x') \quad l = 1, \ldots, L \\
g(x') + \nabla g(x')^T (x - x') &\leq 0 \\
\bigvee_{i \in D_k} \begin{bmatrix} \gamma_{ik} \\
h_{ik}(x') + \nabla h_{ik}(x')^T (x - x') &\leq 0 \\
\end{bmatrix} &\quad k \in SD \\
\Omega(Y) &= \text{True} \\
\alpha &\in \mathbb{R}, x \in \mathbb{R}^n, c \in \mathbb{R}^m, Y \in \{\text{true, false}\}^m
\end{align*}
\] (MGDP)

Redundant constraints are eliminated with false values

Master problem solved with disjunctive branch and bound or with MILP reformulation

Proceed as OA. Requires initialization several NLPs to cover all disjunctions

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Part of GAMS Modeling System

- Disjunctions specified with IF Then ELSE statements

\[
\text{DISJUNCTION D1(I,K,J);}
\]

\[
\text{D1(I,K,J) with (L(I,K,J)) IS}
\]

\[
\text{IF Y(I,K,J) THEN}
\]

\[
\text{NOCLASH1(I,K,J);}
\]

\[
\text{ELSE}
\]

\[
\text{NOCLASH2(I,K,J);}
\]

\[
\text{ENDIF;}
\]

- Logic can be specified in symbolic form (⇒, OR, AND, NOT)
  or special operators (ATMOST, ATLEAST, EXACTLY)

- Linear case: MILP reformulation big-M, convex hull
- Nonlinear: Logic-based OA

http://www.ceride.gov.ar/logmip/

Aldo Vecchietti, INGAR
Linear Generalized Disjunctive Programming
LGDP Model


\[
\text{Min } Z = \sum_{k \in K} c_k + d^T x
\]

\[
s.t. \quad Bx \geq b
\]

\[
\begin{bmatrix}
Y_{jk} \\
A_{jk}x \geq a_{jk}
\end{bmatrix}
\]

\[
k \in K
\]

\[
\bigvee_{j \in J_k} Y_{jk}
\]

\[
\bigwedge_{j \in J_k} Y_{jk}
\]

\[
\Omega(Y) = \text{True}
\]

\[
x^L \leq x \leq x^U
\]

\[
Y_{jk} \in \{\text{True, False}\} \quad j \in J_k, k \in K
\]

\[
c_k \in \mathbb{R}^1 \quad k \in K
\]

Objective function
Common constraints
Disjunctive constraints
Logic constraints
Continuous variables
Boolean variables

Can we obtain stronger relaxations?

Logical OR operator
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Disjunctive Programming

Disjunction: A set of constraints connected to one another through the logical OR operator \( \lor \)

Conjunction: A set of constraints connected to one another through the logical AND operator \( \land \)

Constraint set of a DP can be expressed in two equivalent extreme forms

- **Disjunctive Normal Form (DNF)**
  - A disjunction whose terms do not contain further disjunctions
    \[
    F = \left\{ x \in \mathbb{R}^n : \lor_{i \in Q} (A^i x \geq a^i) \right\}
    \]

- **Conjunctive Normal Form (CNF)**
  - A conjunction whose terms do not contain further conjunctions
    \[
    F = \left\{ x \in \mathbb{R}^n : \widehat{A}x \geq \widehat{a}, \lor_{h \in Q_j} (d^h x \geq d_0^h), j = 1, \ldots, t \right\}
    \]
Linear Generalized Disjunctive Programming
LGDP Model

(LGDP)

Min $Z = \sum_{k \in K} c_k + d^T x$

s.t. $Bx \geq b$

$\bigvee_{j \in J_k} Y_{jk}$

$A^{jk} x \geq a^{jk}$

$k \in K$

$c_k = \gamma_{jk}$

$\bigvee_{j \in J_k} Y_{jk}$

$k \in K$

$\Omega(Y) = True$

$x^L \leq x \leq x^U$

$Y_{jk} \in \{True, False\}$

$j \in J_k, k \in K$

$c_k \in \mathbb{R}^l$

$k \in K$

Objective function

Common constraints

Disjunctive constraints

Logic constraints

Boolean variables

How to deal with Boolean and logic constraints in Disjunctive Programming?
Reformulating LGDP into Disjunctive Programming Formulation


\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} c_k + d^T x \\
\text{s.t. } & Bx \geq b \\
\bigvee_{j \in J_k} Y_{jk} & \quad k \in K \\
\sqrt{A^{jk} x} & \geq a^{jk} \\
\sum_{j \in J_k} Y_{jk} & \quad k \in K \\
\Omega(Y) &= \text{True} \\
X^L & \leq x \leq X^U \\
Y_{jk} & \in \{\text{True}, \text{False}\} \quad j \in J_k, k \in K \\
c_k & \in \mathbb{R}^1 \quad k \in K \\
\end{align*}
\]

\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} c_k + d^T x \\
\text{s.t. } & Bx \geq b \\
\bigvee_{j \in J_k} \lambda_{jk} & = 1 \\
\sqrt{A^{jk} x} & \geq a^{jk} \\
\sum_{j \in J_k} \lambda_{jk} & = 1 \\
\Omega(Y) &= \text{True} \\
X^L & \leq x \leq X^U \\
\lambda_{jk} & \in [0, 1] \quad j \in J_k, k \in K \\
c_k & \in \mathbb{R}^1 \quad k \in K \\
\end{align*}
\]

**Proposition.** LGDP and LDP have equivalent solutions.

**LDP** $\implies$ Integrality $\lambda$ guaranteed
There are many forms between CNF and DNF that are equivalent

**Regular Form (RF):** form represented by intersection of unions of polyhedra

Thus the RF is:

\[ F = \bigcap_{t \in T} S_t \]

where for \( t \in T \), \( S_t = \bigcup_{i \in Q_t} P_i \), \( P_i \) a polyhedron, \( i \in Q_t \).

**Proposition 1 (Theorem 2.1 in Balas (1979)).** Let \( F \) be a disjunctive set in RF. Then \( F \) can be brought to DNF by \(|T| - 1\) recursive applications of the following basic steps, which preserve regularity:

For some \( r, s \in T, r \neq s \), bring \( S_r \cap S_s \) to DNF, by replacing it with:

\[ S_{rs} = \bigcup_{i \in Q_r \cap Q_s} (P_i \cap P_i) \]
Illustrative Example: Basic Steps

\[ F = S_1 \cap S_2 \cap S_3 \]

\[ S_1 = (P_{11} \cup P_{21}) \quad S_2 = (P_{12} \cup P_{22}) \quad S_3 = (P_{13} \cup P_{23}) \]

Then \( F \) can be brought to DNF through 2 basic steps.

Apply Basic Step to:

\[ S_1 \cap S_2 = (P_{11} \cup P_{21}) \cap (P_{12} \cup P_{22}) \]

\[ S_{12} = (P_{11} \cap P_{12}) \cup (P_{11} \cap P_{22}) \cup (P_{21} \cap P_{12}) \cup (P_{21} \cap P_{22}) \]

We can then rewrite

\[ F = S_1 \cap S_2 \cap S_3 \quad \text{as} \quad F = S_{12} \cap S_3 \]

Apply Basic Step to:

\[ S_{12} \cap S_3 = ((P_{11} \cap P_{12}) \cup (P_{11} \cap P_{22}) \cup (P_{21} \cap P_{12}) \cup (P_{21} \cap P_{22})) \cap (P_{13} \cup P_{23}) \]

\[ S_{123} = \left( (P_{11} \cap P_{12} \cap P_{13}) \cup (P_{11} \cap P_{22} \cap P_{13}) \cup (P_{21} \cap P_{12} \cap P_{13}) \cup (P_{21} \cap P_{22} \cap P_{13}) \cup (P_{11} \cap P_{12} \cap P_{23}) \cup (P_{11} \cap P_{22} \cap P_{23}) \cup (P_{21} \cap P_{12} \cap P_{23}) \cup (P_{21} \cap P_{22} \cap P_{23}) \right) \]

We can then rewrite

\[ F = S_{12} \cap S_3 \quad \text{as} \quad F = S_{123} \quad \text{which is its equivalent DNF} \]
Equivalent Forms for GDP

\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} c_k + d^T x \\
\text{s.t. } & Bx \geq b \\
\forall j & \in J_k \left( A^{jk} x \geq a^{jk} \right) \quad k \in K \\
\forall j & \in J_k \left( c_k = \gamma_{jk} \right) \quad k \in K \\
\Omega(Y) &= \text{True} \\
X^L & \leq x \leq X^U \\
Y_{jk} & \in \{True, False\} \quad j \in J_k, k \in K \\
c_k & \in \mathbb{R}^i \quad k \in K
\end{align*}
\]

\(\text{LGDP}\)

\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} c_k + d^T x \\
\text{s.t. } & Bx \geq b \\
\forall j & \in J_k \left( \lambda_{jk} = 1 \right) \quad k \in K \\
\forall j & \in J_k \left( A^{jk} x \geq a^{jk} \right) \quad k \in K \\
\sum_{j \in J_k} \lambda_{jk} &= 1 \quad k \in K \\
H \lambda & \geq h \\
x^L & \leq x \leq x^U \\
0 & \leq \lambda_{jk} \leq 1 \quad j \in J_k, k \in K \\
c_k & \in \mathbb{R}^i \quad k \in K
\end{align*}
\]

\(\text{LDP}\)

\[
F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n + \sum_{k \in K} |J_k| + |K|} : \bigcap_{i \in I} \bar{b}^i z \geq \bar{b}_0^i \bigcap_{k \in K} \bigcup_{j \in J_k} (\bar{A}^{jk} z \geq \bar{a}^{jk}) \bigcup_{n \in K} \bigcup_{m \in J_n} (\bar{A}^{mn} z \geq \bar{a}^{mn}) \right\}
\]

All possible equivalent forms for GDP, obtained through any number of basic steps, are represented by:

\[
F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n + \sum_{k \in K} |J_k| + |K|} : \bigcap_{i \in I} \bar{b}^i z \geq \bar{b}_0^i \bigcap_{k \in K} \bigcup_{j \in J_k} (\bar{A}^{jk} z \geq \bar{a}^{jk}) \bigcup_{n \in K} \bigcup_{m \in J_n} (\bar{A}^{mn} z \geq \bar{a}^{mn}) \right\}
\]

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Converting LDP to MIP reformulations

**Proposition 2** (Theorem 3.3 combined with Corollary 3.5 in Balas (1979)). Let

\[ F = \bigcup_{i \in Q} P_i, \quad P_i = \{ x \in \mathbb{R}^n : \tilde{A}^i x \geq \tilde{a}_0^i \}, \quad i \in Q \]

where \( Q \) is an arbitrary set and each \( (\tilde{A}^i, \tilde{a}_0^i) \) is an \( m_i \times (n+1) \) matrix such that every \( P_i \) is a **bounded non-empty** polyhedron.

Furthermore, let \( \zeta(Q) \) be the set of all those \( x \in \mathbb{R}^n \) such that there exist vectors \( (v^i, y_i) \in \mathbb{R}^{n+1}, \quad i \in Q \), satisfying

\[
\begin{align*}
    x - \sum_{i \in Q} v^i & = 0 \\
    \tilde{A}^i v^i - \tilde{a}_0^i y_i & \geq 0 \quad i \in Q \\
    y_i & \geq 0 \quad i \in Q \\
    \sum_{i \in Q} y_i & = 1 \quad i \in Q
\end{align*}
\]

Then \( cl \ conv F = \zeta(Q) \).

**Proposition 3** (Corollary 3.7 in Balas (1979)).

Let \( \zeta'_i(Q) := \{ x \in \zeta(Q) : y_i \in \{0,1\}, \quad i \in Q \} \). \( \Rightarrow \) **MIP representation**

Then \( \zeta'_i(Q) = F \).
Family of MIP Reformulations For GDP

\[ F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n + \sum_{i \in \mathcal{T}} \sum_{j \in \mathcal{J}^i} n_{j \in \mathcal{J}^i} \mathbb{R}^+ \left\{ (\beta^i_i \geq \beta^i_0) \cap \left( \mathcal{A}^j \cap \mathcal{A}^k \right) \cap \left( \mathcal{A}^m \cap \mathcal{A}^{n} \right) \right\} \right\} \]

General template for any MILP reformulation

\[ \begin{align*}
\text{Min } Z &= \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}_k} \gamma_{jk} y_{jk} + d^T x \\
&s.t. \\
b_i^T x &\geq b_i^T \quad i \in I_{B_i} \\
h_i^T y &\geq h_i^T \quad i \in I_{H_i} \\
x_i^L &\leq x_i \leq x_i^U \quad i \in I_{X_i} \\
y_{jk} &= \sum_{m \in \mathcal{J}_n} \hat{u}_{jk}^{mn} \quad (j, k) \in \mathcal{L}_2 \cup \mathcal{K}_{S_{2a}} \cup I_{H_2}, n \in N \\
x &= \sum_{m \in \mathcal{J}_n} \hat{y}_{mn} \quad n \in N \\
b_i^T \hat{y}_{mn} &\geq b_i^T \hat{y}_{mn} \quad i \in I_{B_{2a}}, m \in \mathcal{J}_n, n \in N \\
\sum_{j \in \mathcal{J}_k} \hat{u}_{jk}^{mn} &= \hat{y}_{mn} \quad k \in \mathcal{K}_{S_{2a}}, m \in \mathcal{J}_n, n \in N \\
h_i^T \hat{u}_{mn} &\geq h_i^T \hat{y}_{mn} \quad i \in I_{H_{2a}}, m \in \mathcal{J}_n, n \in N \\
\hat{u}_{jk}^{mn} &= \hat{y}_{mn} \quad (j, k) \in \mathcal{M}_{mn}, m \in \mathcal{J}_n, n \in N \\
A^j k \hat{y}_{mn} &\geq a^j k \hat{y}_{mn} \quad (j, k) \in \mathcal{L}_3, m \in \mathcal{J}_n, n \in N \\
x^L \hat{y}_{mn} &\leq \hat{y}_{mn} \leq x^U \hat{y}_{mn} \quad m \in \mathcal{J}_n, n \in N \\
0 &\leq \hat{u}_{jk}^{mn} \leq \hat{y}_{mn} \quad (j, k) \in \mathcal{L}_3, m \in \mathcal{J}_n, n \in N \\
\sum_{m \in \mathcal{J}_n} \hat{y}_{mn} &= 1 \quad n \in N \\
\sum_{m \in \mathcal{J}_n} \hat{y}_{mn} &= 1 \quad n \in N \\
\hat{y}_{jk} &\in \{0, 1\} \quad j \in \mathcal{J}_k, k \in K \\
\hat{y}_{mn} &\geq 0 \quad m \in \mathcal{J}_n, n \in N \\
\hat{y}_{jk} &\in \{0, 1\} \quad j \in \mathcal{J}_k, k \in K
\end{align*} \]
Particular case: Convex Hull Reformulation of LGDP

Raman and Grossmann I.E. (1994) (CH)

Min \( Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} y_{jk} + d^T x \)

\[ \begin{align*} 
\text{s.t.} \quad & Bx \geq b \\
& x = \sum_{j \in J_k} v_{jk} \quad k \in K \\
& A^{jk} v_{jk} \geq a^{jk} y_{jk} \quad j \in J_k, k \in K \\
& x^L y_{jk} \leq v_{jk} \leq x^U y_{jk} \quad j \in J_k, k \in K \\
& \sum_{j \in J_k} y_{jk} = 1 \quad k \in K \\
& H y \geq h \\
y_{jk} \in \{0,1\} \quad j \in J_k, k \in K 
\end{align*} \]

While this MILP formulation has stronger relaxation than big-M, it is not strongest!!

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A Hierarchy of Relaxations for GDP

Proposition 4. For \( i \in \mathbb{N} \mid \mathbb{N} \) let \( F_{GDP_i} \) be a sequence of regular forms of the disjunctive set:

\[
F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n+\sum_{s,k} |J_s| + |K|} : \bigcap_{i \in T} \bigcap_{k \in K} \bigcap_{j \in J_k} (\tilde{A}^{i,k} z \geq \tilde{\alpha}^{i,k}) \bigcap_{n \in K} (\tilde{A}^{mn} z \geq \tilde{\alpha}^{mn}) \right\},
\]

such that

i) \( F_{GDP_0} \) corresponds to the disjunctive form:

\[
F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n+\sum_{s,k} |J_s| + |K|} : \bigcap_{i \in T} \bigcap_{k \in K} \bigcap_{j \in J_k} (\tilde{A}^{i,k} z \geq \tilde{\alpha}^{i,k}) \right\};
\]

ii) \( F_{GDP_{|T|+|K|-1}} := F_t \) is in DNF;

iii) for \( i = 1, \ldots, t \), \( F_{GDP_i} \) is obtained from \( F_{GDP_{i-1}} \) by a basic step.

Then,

\[
h \text{- rel } F_{GDP_0} \supseteq h \text{- rel } F_{GDP_1} \supseteq \cdots \supseteq h \text{- rel } F_{GDP_{|T|+|K|-1}} = \text{clconv } F_{GDP_{|T|+|K|-1}} = \text{clconv } F_t. \quad \text{(true convex hull)}
\]
Illustrative Example: Hierarchy of Relaxations

\[ x_1 - x_2 + 0.5 \geq 0 \]
\[ -x_1 - x_2 + 1 \geq 0 \]
\[
\begin{bmatrix}
  x_1 = 0 \\
  0 \leq x_2 \leq 1
\end{bmatrix}
\lor
\begin{bmatrix}
  x_1 = 1 \\
  0 \leq x_2 \leq 1
\end{bmatrix}
\]

Convex Hull of disjunction

Application of 2 Basic Steps

\[
\begin{bmatrix}
  x_1 - x_2 + 0.5 \geq 0 \\
  -x_1 - x_2 + 1 \geq 0 \\
  x_1 = 0 \\
  0 \leq x_2 \leq 1
\end{bmatrix}
\lor
\begin{bmatrix}
  x_1 - x_2 + 0.5 \geq 0 \\
  -x_1 - x_2 + 1 \geq 0 \\
  x_1 = 1 \\
  0 \leq x_2 \leq 1
\end{bmatrix}
\]

Convex Hull of disjunction

LP Relaxation

Tighter Relaxation!

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Numerical Example: Strip-packing problem

Problem statement: *Hifi (1998)*

Given a set of small rectangles with width $H_i$ and length $L_i$.
Large rectangular strip of fixed width $W$ and unknown length $L$.
Objective is to fit small rectangles onto strip without overlap
and rotation while minimizing length $L$ of the strip.
GDP/DP Model for Strip-packing problem

Objective function
Minimize length

Disjunctive constraints
No overlap between rectangles

Bounds on variables
25 Rectangle Problem  Optimal solution= 31

Original CH
1,112 0-1 variables
4,940 cont vars
7,526 constraints
LP relaxation = 9

=>

Strengthened
1,112 0-1 variables
5,783 cont vars
8,232 constraints
LP relaxation = 27!

31 Rectangle Problem  Optimal solution= 38

Original CH
2,256 0-1 variables
9,716 cont vars
14,911 constraints
LP relaxation = 10.64

=>

Strengthened
2,256 0-1 variables
11,452 cont vars
15,624 constraints
LP relaxation = 33!
**Cutting Planes for Linear Generalized Disjunctive Programming**

**GDP Model:**

\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} c_k + h^T x \\
\text{s.t. } Bx &\leq b \\
\end{align*}
\]

\[
\Omega(Y) = \text{True}
\]

\[
x \in \mathbb{R}^n, \ Y_{jk} \in \{\text{True, False}\}, \ c_k \in \mathbb{R}
\]

\[
j \in J_k, \ k \in K
\]

**Sawaya, Grossmann (2004)**

**Objective Function**

**Common Constraints**

**Disjunctive Constraints**

**Logic Constraints**

**Boolean Variables**
Motivation for Cutting Plane Method

Trade-off: Big-M fewer vars/weaker relaxation vs Convex-Hull tighter relaxation/more vars
Global Optimization Algorithms

- Most algorithms are based on spatial branch and bound method (Horst & Tuy, 1996)

**Nonconvex NLP/MINLP**

- $\alpha$BB (Adjiman, Androulakis & Floudas, 1997; 2000)
- BARON (Branch and Reduce) (Ryoo & Sahinidis, 1995, Tawarmalani and Sahinidis (2002))
- OA for nonconvex MINLP (Kesavan et al., 2004)
- Branch and Contract (Zamora & Grossmann, 1999)

**Nonconvex GDP**

- Two-level Branch and Bound (Lee & Grossmann, 2001)
Spatial Branch and Bound to obtain the Global Optimum

- Guaranteed to converge to global optimum given a certain tolerance between lower and upper bounds
Objective

Multiple minima

Lower bound

LB

UB = Upper bound

Global optimum search

Branch and bound tree

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Nonconvex GDP

\[
\begin{align*}
\text{min} & \quad Z = \sum_k c_k + f(x) \\
\text{s.t.} & \quad r(x) \leq 0 \\
& \quad \left[ \begin{array}{c} \Omega(Y) = \text{true} \\ Y_{jk} \end{array} \right], k \in K \\
& \quad Y_{jk} \in \{\text{true, false}\} \\
& \quad x \in R^n, c_k \in R^1 \\
\end{align*}
\]

Objective Function
Common Constraints
Disjunctions
Logic Propositions

OR operator

\[ j \in J_k \]

\( f, g \) and \( r \): nonconvex

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Convex Underestimator GDP (R)

- Introducing convex underestimators

\[
\begin{align*}
\min & \quad Z = \sum_k c_k + \bar{f}(x) \\
\text{s.t.} & \quad \bar{r}(x) \leq 0 \\
\forall j \in J_k & \quad \begin{bmatrix} \bar{Y}_{jk} \\ \bar{g}_{jk}(x) \leq 0 \end{bmatrix}, k \in K \\
\end{align*}
\]

Convex underestimators

Bilinear: Linear
McCormick (1976), Al-Khayyal (1992)

Linear fractional: Convex nonlinear
Quesada and Grossmann (1995)

Concave separable: Linear secant

- Problem (R) yields a valid lower bound to Problem (GDP)
Convex envelopes
Concave function

\[ g(x) = f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \]
Bilinear

\[ w = xy \]

\[ x^L \leq x \leq x^U \]
\[ y^L \leq y \leq y^U \]

McCormick convex envelopes

\[ w \geq x^L y + y^L x - x^L y^L \]
\[ w \geq x^U y + y^U x - x^U y^U \]
\[ w \leq x^L y + y^U x - x^L y^U \]
\[ w \leq x^U y + y^L x - x^U y^L \]

For other convex envelopes/underestimators see:
1. Branch and bound enumeration on disjunctions of convex GDP \((\mathcal{R})\)

2. When feasible discrete solution found switch to spatial branch and bound (NLP subproblem)
Synthesis Multiproduct Batch Plant

(Birewar & Grossmann, 1990)

Tasks 3

Equipment

Unit 1
Cast Iron w/ Agitator

Unit 2
Stainless Steel w/ Agitator

Unit 3
Cast Iron Jacketed

Unit 4
Stainless Steel Jacketed w/ Agitator

Unit 5
Tray Dryer

More than 100 alternatives: each requires nonlinear optimization
**Synthesis Multiproduct Batch Plant**

**Nonconvex GDP Model**

\[
\begin{align*}
\text{min} & \quad \text{COST} = \sum_{j=1}^{M} N_j^{EQ} C_j + \sum_{j} CS_j \\
\text{s.t.} & \quad V_r^T \geq B_i S_{it} \quad i = 1, \ldots, N_p; t = 1, \ldots, T \\
& \quad pt_{ij} = \sum_{i \in I_j} pty_{ij} \quad i = 1, \ldots, N_p; j = 1, \ldots, M \\
& \quad n_i B_i \geq Q_i \quad i = 1, \ldots, N_p \\
& \quad \sum_{i=1}^{N_t} n_i T_{Li} \leq H
\end{align*}
\]

- **Objective function**
- **Sizing**
- **Process time**
- **Demand**
- **Horizon time**

\[
\begin{align*}
\bigvee_{j \in J_t} \begin{bmatrix}
Y_{ij} \\
V_j \geq V_t^T \\
pt_{ij} = pt_{it}^T \\
pty_{ij'} = 0, j' \neq j
\end{bmatrix} & \quad t \in T
\end{align*}
\]

- **Disjunction for Task Assignments**
- **Nonconvex functions**

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GDP model (continued)

\[
\begin{align*}
YEX_j & = Y_j + \alpha V_j^{0.6} \\
C_j & = \gamma_j + \alpha V_j^{0.6} \\
V_j^L & \leq V_j \leq V_j^C \\
N_j & = 1 \\
Y_C & = \begin{cases} Y_j \text{ if } T_{i4} \geq pt_y \\ N_j \text{ otherwise} \end{cases} \\
Y_C_{1,j} & = \begin{cases} Y_j \text{ if } T_{i4} \geq pt_y \\ N_j \text{ otherwise} \end{cases} \\
N_j & = 2 \\
Y_C_{2,j} & = \begin{cases} Y_j \text{ if } T_{i4} \geq pt_y \\ N_j \text{ otherwise} \end{cases} \\
N_j & = 3 \\
Y_C_{3,j} & = \begin{cases} Y_j \text{ if } T_{i4} \geq pt_y \\ N_j \text{ otherwise} \end{cases} \\
N_j & = 4 \\
Y_C_{4,j} & = \begin{cases} Y_j \text{ if } T_{i4} \geq pt_y \\ N_j \text{ otherwise} \end{cases} \\
\neg YEX_j & = C_j = 0 \\
V_j = 0 \\
N_j^{eq} = 0 \\
pt_y = 0 \\
T_{i4} \geq 0 \\
\neg YS_j & = B_y = B_y \\
VST_j \geq S \cdot B_y \cdot NEQ_j \\
100 \leq VST_j \leq 10000 \\
CS_j & = 5000 + 80 VST_j \cdot s \\
\end{align*}
\]

\( j \in J \)

Disjunction for Equipment

Disjunction for Storage Tank

Logic Propositions

\[
\begin{align*}
YEX_1 \leftrightarrow Y_{i1}, YEX_2 \leftrightarrow Y_{i2} \lor Y_{i3}, YEX_3 \leftrightarrow Y_{i3} \\
YEX_4 \leftrightarrow Y_{i4} \lor Y_{i5} \lor Y_{i6} \lor Y_{i7} \\
W_{i4} \lor W_{i5} \lor W_{i6} \\
W_{i4} \leftrightarrow \neg Y_{i4} \land \neg Y_{i5} \land \neg Y_{i6} \land \neg Y_{i7} \\
W_{i4} \leftrightarrow (Y_{i4} \land \neg Y_{i5} \land \neg Y_{i6} \land \neg Y_{i7}) \lor (\neg Y_{i4} \land Y_{i5} \land \neg Y_{i6} \land \neg Y_{i7}) \lor (\neg Y_{i4} \land \neg Y_{i5} \land Y_{i6} \land \neg Y_{i7}) \lor (\neg Y_{i4} \land \neg Y_{i5} \land \neg Y_{i6} \land Y_{i7}) \\
W_{i5} \leftrightarrow \neg Y_{i4} \land \neg Y_{i6} \land \neg Y_{i7} \\
W_{i6} \leftrightarrow \neg Y_{i4} \land \neg Y_{i5} \land \neg Y_{i7} \\
0 \leq C_j, V_j, V^{eq}_j, n_j, B_j, T_{i4}, pt_y, N_j^{eq}, \neg pt_y, YEX_j, Y_j, YC_j; W \in \{\text{true, false}\}
\end{align*}
\]
Proposed Algorithm for Nonconvex GDP

Step 0
Nonconvex MINLP

\[ Z^U \]

OA \text{ (Viswanathan and Grossmann, 1990)}

Step 1
Bound Contraction

New Bound

Step 2
BB with Y’s

Update \( Z^L \)

Stop when \( Z^L \geq Z^U \)

(Lee and Grossmann, 2000)

When solution is Integral

Add Integer Cut

Step 3
Spatial BB

Update \( Z^U \)

Fixed Y’s

(Quesada and Grossmann, 1995)

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Upper Bound Solution

Cost = $ 277,928 (by GAMS/DICOPT++)

- Use 4 Stages (6 units) without Storage Tank

\[ j = 1 \]
\[ V_1 = 4,842 \text{ L} \]
\[ j = 2 \]
\[ V_2 = 2,881 \text{ L} \]
\[ j = 4 \]
\[ V_4 = 2,469 \text{ L} \]
\[ j = 5 \]
\[ V_5 = 8,071 \text{ L} \]

A 243 batches, 4.5hrs
B 260 batches, 6hrs
C 372 batches, 9hrs

6000 hrs

1093 hrs
1562 hrs
3345 hrs

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Optimal Solution: Multiproduct Batch Plant

- Global optimal cost = $264,887 (5% improvement)
- 3 Stages + 1 storage tank (5 units) (43 nodes, 48 sec)

Mixing | Reaction | Storage | Crystallization | Drying

Tank

$V_2 = 4,309 \text{ L} \quad VST_2 = 4,800 \text{ L} \quad V_3 = 3,600 \text{ L} \quad V_5 = 11,753 \text{ L}$

A 250 batches, 5hrs
B 293 batches, 3hrs
C 418 batches, 5.5hrs

Storage

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Global Optimization of Bilinear Generalized Disjunctive Programs

Juan Ruiz

Min \[ Z = f(x) + \sum_{k \in K} c_k \]  \rightarrow \text{Objective Function}

s.t. \[ g(x) \leq 0 \]  \rightarrow \text{Global Constraints}

\[ \bigvee_{i \in D_k} \begin{bmatrix} Y_{ik} \\ r_{ik}(x) \leq 0 \\ c_k = \gamma_{ik} \end{bmatrix} \]  \rightarrow \text{Disjunctions}

\[ \Omega(Y) = \text{True} \]  \rightarrow \text{Logic Propositions}

\[ x \in \mathbb{R}^n, c_k \in \mathbb{R}, Y_{ik} \in \{\text{True, False}\} \quad i \in D_k, k \in K \]

Bilinearities may lead to multiple local minima \rightarrow \text{Global Optimization techniques are required}

Relaxation of Bilinear terms using McCormick envelopes leads to a LGDP \rightarrow \text{Improved relaxations for Linear GDP has recently been obtained (Sawaya & Grossmann, 2007)}
Guidelines for applying basic steps in Bilinear GDP

- Replace bilinear terms in GDP by McCormick convex envelopes (LGDP)
- Apply basic steps between those disjunctions with at least one variable in common.
  
  The more variables in common two disjunctions have the more the tightening can be expected
- If bilinearities are outside the disjunctions apply basic steps by introducing them in the disjunctions previous to the relaxation.
  
  If bilinearities are inside the disjunctions a smaller tightening effect is expected.
- A smaller increase in the size of the formulation is expected when basic steps are applied between improper disjunctions and proper disjunctions.
Methodology

Step 1: GDP reformulation (Apply basic steps following the rules presented)

Step 2: Bound Contraction (Zamora & Grossmann, 1999)

Step 3: Branch and Bound Procedure (Lee & Grossmann, 2001)
Case Study I: Water treatment network design

Process superstructure

Generalized Disjunctive Program

Min \( Z = \sum_{k \in PU} CP_k \)

s.t.

\( f_k^j = \sum_{i \in M_i} f_i^j \quad \forall j \quad k \in MU \)
\( \sum_{i \in S_i} f_i^j = f_k^j \quad \forall j \quad k \in SU \)
\( \sum_{i \in S_i} \zeta_i^k = 1 \quad k \in SU \)
\( f_i^j = \zeta_i^k f_k^j \quad \forall j \quad i \in S_k \quad k \in SU \)
\( F_k = \sum_{j} f_i^j, i \in OPU_k \quad k \in PU \)
\( CP_k = \delta_{ik} F_k \)
\( 0 \leq \zeta_i^k \leq 1 \quad \forall j, k \)
\( 0 \leq f_i^j, f_k^j \quad \forall i, j, k \)
\( 0 \leq CP_k \quad \forall k \)
\( YP_k^h \in \{true, false\} \quad \forall h \in D_k \quad \forall k \in PU \)

Optimal structure

N of cont. vars.: 114
N of disc. vars.: 9
N of bilinear terms: 36

\( Z^* = 1.214 \)
Case Study II: Pooling network design

**Process superstructure**

Stream i  Pool j  Product k

- S1  P1  1
- S2  P2  2
- S3  P3  3
- S4  P3  3
- S5  P4  3

N of cont. vars.: 76
N of disc. vars.: 9
N of bilinear terms: 24

**Optimal structure**

Stream i  Pool j  Product k

- S1  P1  1
- S2  P2  2
- S5  P3  3

Z* = -4.640

**Generalized Disjunctive Program**

Min \( Z = \sum_{j \in J} CP_j + \sum_{i \in I} CST_i + \sum_{j \in J} \sum_{i \in I} c_{ij} \sum_{w \in W} f_{ijw} - \sum_{k \in K} \sum_{j \in J} \sum_{w \in W} f_{jkw} \)

s.t.

- \( \sum_{i \in I} \sum_{w \in W} f_{ijw} = \sum_{k \in K} \sum_{j \in J} f_{jkw} \quad \forall j \in J \)
- \( \sum_{j \in J} f_{jkw} - S_k = 0 \quad \forall k \in K \)
- \( f_{ijw} = \lambda_{iw} \sum_{w \in W} f_{ijw} \quad \forall i \in I, \forall j \in J, \forall w \in W \)
- \( \sum_{j \in J} f_{jkw} - Z_{kw} \sum_{j \in J} \sum_{w \in W} f_{jkw} = 0 \quad \forall k \in K, \forall w \in W \)

\[
\begin{bmatrix}
YST_i \\
\end{bmatrix} 
\begin{bmatrix}
\bar{f}^{lo} \\
\bar{YST}_i \\
\end{bmatrix} \quad \begin{cases}
\bar{YST}_i, \\
\bar{f}^{lo}, \\
\end{cases} \\
\begin{cases}
\bar{f}^{lo} \leq \sum_{j \in J} \sum_{w \in W} f_{ijw} \\
\sum_{k \in K} f_{jkw} = \sum_{i \in I} f_{ijw}, \forall w \in W \\
f_{jkw} = \zeta_j \sum_{i \in I} f_{ijw}, \forall w \in W, k \in K \\
\sum_{k \in K} \zeta_j = 1 \\
CP_j = \gamma_j \\
0 \leq \zeta_j \leq 1; 0 \leq f_{jkw}, f_{ijw} \leq f^{up} \\
0 \leq CST_i, CP_j, YST_i, YP_j \in \{true, false\}\end{cases}
\]

- \( \bar{f}^{lo} \leq \sum_{i \in I} \sum_{w \in W} f_{ijw} \)
- \( \sum_{k \in K} f_{jkw} = \sum_{i \in I} f_{ijw}, \forall w \in W \)
- \( f_{jkw} = \zeta_j \sum_{i \in I} f_{ijw}, \forall w \in W, k \in K \)
- \( \sum_{k \in K} \zeta_j = 1 \)
- \( CP_j = \gamma_j \)
- \( 0 \leq \zeta_j \leq 1; 0 \leq f_{jkw}, f_{ijw} \leq f^{up} \)
- \( 0 \leq CST_i, CP_j, YST_i, YP_j \in \{true, false\}\)
## Performance

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<tr>
<th>Example</th>
<th>Initial Lower Bound</th>
<th>Global Optimization Technique using Lee &amp; Grossmann relaxation</th>
<th>Global Optimization Technique using proposed relaxation</th>
<th>Relative Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>Initial Lower Bound</td>
<td>400.66</td>
<td>499.86</td>
<td>24.90%</td>
</tr>
<tr>
<td></td>
<td>Bound contraction</td>
<td></td>
<td></td>
<td>99.7%</td>
</tr>
<tr>
<td></td>
<td>Nodes</td>
<td>399</td>
<td>204</td>
<td>51%</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>Example 2</th>
<th>Initial Lower Bound</th>
<th>Global Optimization Technique using Lee &amp; Grossmann relaxation</th>
<th>Global Optimization Technique using proposed relaxation</th>
<th>Relative Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial Lower Bound</td>
<td>-5515</td>
<td>-5468</td>
<td>0.90%</td>
</tr>
<tr>
<td></td>
<td>Bound contraction</td>
<td></td>
<td></td>
<td>8%</td>
</tr>
<tr>
<td></td>
<td>Nodes</td>
<td>748</td>
<td>683</td>
<td>9%</td>
</tr>
</tbody>
</table>
Conclusions

GDP modeling framework
- Provides a logic-based framework for discrete-continuous optimization
- Big-M and convex hull alternative formulations for GDP
- Solution methods: reformulation, branch and bound, decomposition

Unified Linear GDP with Disjunctive Programming
- Developed DP equivalent formulation for GDP
- Developed a family of MIP reformulations for GDP
- Developed a hierarchy of relaxations for GDP
- Numerical results have shown great improvement in lower bound for strip packing problem

Nonconvex GDPs
- Spatial branch and bound methods can be developed
- Tighter lower bounds can be obtained in bilinear problems by applying basic steps
Open Cyberinfrastructure for Mixed-integer Nonlinear Programming: Collaboration and Deployment via Virtual Environments

CMU: Grossmann, Biegler, Belotti, Cornuejols, Margot, Ruiz, Sahinidis
IBM: Lee, Wächter

General Goals

(a) Create a library of optimization problems in different application areas in which one or several alternative models are presented with their derivation. In addition, each model has one or several instances that can serve to test various algorithms.
(b) Provide a mechanism for researchers and users to contribute towards the creation of the library of optimization problems.
(c) Provide a forum of discussion for algorithm developers and application users where alternative formulations can be discussed as well as performance and comparison of algorithms.
(d) Provide information on MINLP tutorials and bibliography to disseminate this information.

Major emphasis

Collect optimization problems in which alternative model formulations are documented with corresponding computational results (engineering, finance, operations management, biology)