Preprocessing Techniques for Discrete Optimization Problems

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IMA Workshop on Mixed-Integer Nonlinear Optimization
University of Minnesota
November 17–21, 2008

MIMPEC

Mixed-Integer Mathematical Programs with Equilibrium Constraints

Class of nonconvex optimization problems

$$
\begin{array}{lll}
\min_{x,y} & f(x,y) & \\
& 0 \le g(x,y) & \\
& 0 \le h(x,y) \quad \perp & y \ge 0 \\
& x_I \text{ integer} & y_J \text{ integer}
\end{array}
$$

Applications in electricity markets, network design, etc.

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Nonlinear programming reformulation

$$\begin{array}{lll} \min_{x,y\ge 0,s\ge 0} & f(x,y) & \\ & 0 \le g(x,y) & \\ & s = h(x,y) & \\ & s^T y \le 0 & \\ & x_I \text{ integer} & y_J \text{ integer} \end{array}$$

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Constraint qualification not satisfied when integrality is relaxed

Preprocessing

- Reduce size and complexity and strengthen formulation
- Start simple and fast, then add more rules to lexicon
  - Standard linear reductions
  - Quadratic constraint reductions
  - General nonlinear constraints

Preprocessing

- Reduce size and complexity and strengthen formulation
- Start simple and fast, then add more rules to lexicon
  - Standard linear reductions
  - Quadratic constraint reductions
  - General nonlinear constraints
- More information leads to better preprocessing
  - Require user to provide function properties
  - Derive function properties from expression tree
- More work can lead to more reductions
  - Preprocessing must not dominate time to solve

Simple Linear Constraint Reductions

- Singleton rows generate bounds
- Forcing conditions fix variables
- Imply variable bounds
- Detect duplicate rows
- Improve coefficient in constraints
- Identify special structure

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Example: given the constraint

$$a^T x + by + c = 0$$

where a and x are integer, b is a noninteger scalar, and y is a single binary variable.

- If c is integer, then $y^* = 0$.
- If c is noninteger and $b + c$ is integer, then $y^* = 1$.
- If c is noninteger and $b + c$ is noninteger, then infeasible.

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More useful if b is a function and variable bounds imply noninteger.

Forcing Conditions in Linear Programming

Idea: if the feasible region collapses to a point, then fix all referenced variables and eliminate the constraint.

- Given the linear constraint

$$a^T x + b \leq 0$$

- Compute function bounds

$$fl \leq a^T x + b \leq fu \quad \forall x \in [xl, xu]$$

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  - If $fl = 0$, then

$$0 = fl \leq a^T x + b \leq 0$$

    implying $a^T x + b = 0$, which is satisfied only when

$$\begin{array}{ll} x_i^* = xl_i & \forall a_i > 0 \\ x_i^* = xu_i & \forall a_i < 0 \end{array}$$

  - If $fl > 0$, then the constraint is infeasible.
  - If $fu \leq 0$, then the constraint is redundant.

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- Dual reductions evaluate columns of constraint matrix.

Linear Dual Forcing Constraints

Structure

- Given a convex or nonconvex optimization problem with or without a constraint qualification

$$\begin{array}{ll} \min\limits_{x \in X, y \geq 0} & f(x) + cy \\ \text{subject to} & g(x) + y \geq 0 \\ & h(x) + y \geq 0 \end{array}$$

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- First-order optimality conditions

$$\begin{array}{lcl} X \ni x & \perp & \nabla f(x) - \nabla g(x)\lambda_1 - \nabla h(x)\lambda_2 \\ 0 \leq y & \perp & c - \lambda_1 - \lambda_2 \geq 0 \\ 0 \leq \lambda_1 & \perp & g(x) + y \geq 0 \\ 0 \leq \lambda_2 & \perp & h(x) + y \geq 0 \end{array}$$

- Compute implied function bounds

$$-\infty = fl \leq c - \lambda_1 - \lambda_2 \leq fu = c$$

Linear Dual Forcing Constraints

Implications

- If $fu \leq 0$, then form and solve the reduced problem[1]

$$\min_{x \in X \cap dom\, g \cap dom\, h} \quad f(x)$$

  - If infeasible, then the original problem is infeasible.
  - If unbounded, then the original problem is unbounded.
  - If optimal solution x^* and $fu < 0$, then the original problem is unbounded.[2] In particular, $y = \infty$ satisfies the constraints and objective is negative infinity.
  - If optimal solution x^* and $fu = 0$, then the original problem has an optimal solution. In particular, any

$$y^* \geq \max\{0, -g(x^*), -h(x^*)\}$$

    satisfies the constraints without changing the objective value.

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[1] $dom\, g = \{x | g(x) > -\infty\}$

[2] An optimization problem is grievous if feasibility implies unboundedness.

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If y is integer, then similar implications hold.

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Linear Dual Redundant Constraints

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- Given the optimization problem

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Nonconvex Dual Forcing Constraints

Separable Structure

- Given the optimization problem

$$\begin{array}{ll} \min\limits_{x \in X, y \geq 0} & f(x) + k(y) \\ \text{subject to} & g(x) + c(y) \geq 0 \\ & h(x) + d(y) \geq 0 \end{array}$$

where[3]

$$\begin{array}{lllll} \liminf\limits_{y \to \infty} k(y) & = & \inf\limits_{y \geq 0} k(y) & = & \bar{k} \\ \lim\limits_{y \to \infty} c(y) & = & \sup\limits_{y \geq 0} c(y) & = & \bar{c} \\ \lim\limits_{y \to \infty} d(y) & = & \sup\limits_{y \geq 0} d(y) & = & \bar{d} \end{array}$$

[3] Generalizations are possible.

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- Examples
  - k is nonincreasing, c and d are nondecreasing
  - $k(y) = y \sin(y)$, c and d are nondecreasing

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If y is integer, then similar implications hold.

Nonconvex Dual Constraints Summary

- If y has finite lower and upper bounds and

$$\emptyset \neq Y = \arg \min_{yl \leq y \leq yu} k(y) \cap \arg \max_{yl \leq y \leq yu} c(y) \cap \arg \max_{yl \leq y \leq yu} d(y),$$

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- Generalizations include
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  - Union of intervals and disjunctions
- Nonseparable functions
  - Bilinear terms in objective and constraints

$$\begin{array}{ll} \min\limits_{x \geq 1, 0 \leq y \leq 1} & f(x) - xy \\ \text{subject to} & g(x) + xy \geq 0 \\ & h(x) + y \geq 0 \end{array}$$

  - Reduced problem is

$$\begin{array}{ll} \min\limits_{x \geq 1} & f(x) - x \\ \text{subject to} & g(x) + x \geq 0 \\ & h(x) + 1 \geq 0 \end{array}$$

  - More general results can be proved

Quadratic Constraints

- Quadratic singleton

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$$x^T Ax + c^T x + d \leq 0$$

  - Compute eigenvalue decomposition and scaling

$$A = QRERQ^T$$

    - Q is orthogonal
    - R is positive diagonal
    - E is a diagonal with +1, -1, or 0 entries
  - Define the sets

$$\begin{array}{rcl} I_+ & = & \{i | E_{i,i} = 1\} \\ I_- & = & \{i | E_{i,i} = -1\} \\ I_0 & = & \{i | E_{i,i} = 0\} \end{array}$$

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  - If $I_- = \emptyset$, then constraint is convex
  - If $I_+ = \emptyset$, then constraint is reverse convex

Second-Order Cone Constraints

- Otherwise, let

$$\begin{aligned} y &= RQ^T x \\ b &= R^{-1} Q c \\ z &= d + \sum_{k \in I_0} b_k y_k \end{aligned}$$

- Rewrite quadratic constraint

$$\sum_{i \in I_+} \left(y_i^2 + b_i y_i \right) + z \leq \sum_{i \in I_-} \left(y_j^2 - b_j y_j \right)$$

- If $card(I_-) = 1$ and z is constant, then factor

$$\sum_{i \in I_+} \left(y_i + \frac{b_i}{2} \right)^2 + z + \frac{b_j^2 - \sum_{i \in I_+} b_i^2}{4} \leq \left(y_j - \frac{b_j}{2} \right)^2$$

- Otherwise, constraint is not a second-order cone

Second-Order Cone Constraints

- Let $\tilde{z} = z + \frac{b_j^2 - \sum_{i \in I_+} b_i^2}{4}$
- If $\tilde{z}$ is a nonnegative constant, then

$$\left\| \begin{matrix} E_+ \left(y + \frac{b}{2}\right) \\ \sqrt{\tilde{z}} \end{matrix} \right\|_2 \leq \left| y_j - \frac{b_j}{2} \right|$$

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- If $y_j - \frac{b_j}{2} \geq 0$, then

$$\left\| \begin{array}{c} E_+ \left(y + \frac{b}{2}\right) \\ \sqrt{\tilde{z}} \end{array} \right\|_2 \leq y_j - \frac{b_j}{2}$$

- If $y_j - \frac{b_j}{2} \leq 0$, then

$$\left\| \begin{array}{c} E_+ \left(y + \frac{b}{2}\right) \\ \sqrt{\tilde{z}} \end{array} \right\|_2 \leq \frac{b_j}{2} - y_j$$

- Otherwise
    - Keep absolute value function
    - Model it with binary variables

Complementarity Constraints

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  - Function bounds and forcing conditions
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  - Represent solution set as union of convex sets
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  - Check existence of solution for all possible right-hand sides
  - Computing solution during postsolve by enumeration
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- Possible reformulations
  - Bilinear constraints with slacks
  - Disjunctions

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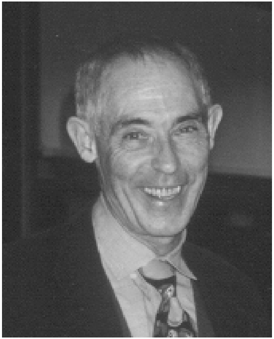

Mixed-Integer Nonconvex Optimization Toolbox

- Algorithms
- Underestimators
- Relaxations

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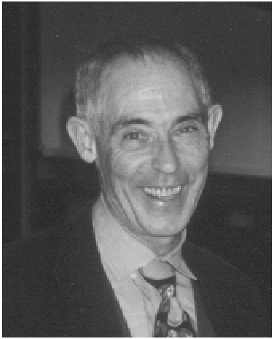

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MINOTAUR: It's only half bull!