An open problem concerning breakup of fluid jets

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Governing equation

\[ s_t = \lambda(t)s^2 - s^{3/2}. \]

Describes breakup of a jet of Newtonian fluid in Stokes flow. \( s(X,t) \) is the stretch factor, and \( \lambda \) is the force in the jet. This force is not given, but is determined by the constraint

\[ \int_0^1 s(X, t) \, dX = 1. \]

Problem:

Link the asymptotic behavior at jet breakup, i.e. near the point where \( s \to \infty \) to the behavior of the initial condition near the point where \( s(X,0) \) has its maximum.
Similarity solutions (Papageorgiou 1995, Eggers 1997)

\[ s(X, t) = (-t)^{-\alpha} \tilde{s}(\frac{X}{(-t)^{\beta}}), \]

\[ u(X, t) = (-t)^{\beta-\alpha-1} \tilde{u}(\frac{X}{(-t)^{\beta}}), \]

\[ \lambda(t) = k(-t)^{\gamma}. \]

Here we have put the breakup time at \( t=0 \) and the breakup point at \( X=0 \). If we want all terms in the equations to have the same order of magnitude, we need \( \alpha=2, \gamma=1 \). The similarity ansatz leads to the following set of ODEs:

\[ 3(2\tilde{s}(\xi) + \beta \xi \tilde{s}'(\xi)) = k\tilde{s}(\xi)^2 - \tilde{s}(\xi)^{3/2}, \]

\[ k = \frac{\int_{-\infty}^{\infty} \tilde{s}(\xi)^{3/2} \, d\xi}{\int_{-\infty}^{\infty} \tilde{s}(\xi)^2 \, d\xi}. \]
Regular behavior at $\xi=0$:  
\[ \tilde{s}(\xi) \sim \tilde{s}(0) - a\xi^{2n}. \]

The simplest case $n=1$ leads to

\[ \tilde{s}(0) = 144(\beta - 1)^2, \quad k = \frac{2\beta - 1}{24(\beta - 1)^2}. \]

Next, we substitute

\[ \tilde{s}(\xi) = \frac{\tilde{s}(0)}{(1 + 2u(\xi)^2)^2}. \]

This leads to the implicit solution

\[ \chi(u) \equiv u(\beta + u^2)^{\beta - 1/2} = C\xi. \]
There is a one-parameter family of solutions which differ by rescaling. W.l.o.g. set C=1. We must satisfy the equation for $k$:

\[
\frac{2\beta - 1}{2(\beta - 1)} = \frac{\int_0^\infty \chi'(u)(1 + 2u^2)^{-3} \, du}{\int_0^\infty \chi'(u)(1 + 2u^2)^{-4} \, du}.
\]

This can be rewritten as

\[
2(\beta - 1)\left(\frac{7}{2} - \beta\right) {}_2F_1\left(2, \frac{1}{2}, \frac{7}{2} - \beta, 1 - 2\beta\right)
\]

\[
= (2\beta - 1)(3 - \beta) {}_2F_1\left(3, \frac{1}{2}, \frac{9}{2} - \beta, 1 - 2\beta\right).
\]

The smallest positive root is at \(\beta=2.17487\).
Finite time blowup

Suppose $s(X,0)$ has a unique maximum at $X=0$, and moreover,

$$-s_X(X,0)X \geq CX^{2n}$$

for $X$ close to zero where $C$ is positive and $n$ is some integer. Then $s(0,t)$ blows up in finite time.


Expected result:

If $s(X,0) \sim A-Bx^{2n}$, then the asymptotics of blowup is described by the similarity solution corresponding to that value of $n$.

If we know a priori that $\lambda(t) \sim k(t_c-t)$, where $t_c$ is the blowup time, then this is true.
Consider
\[ s_t = k(t_c - t)s^2 - s^{3/2}. \]

Substitute
\[ s = (t_c - t)^{-2}r. \]

Leads to
\[ (t_c - t)r_t = kr^2 - r^{3/2} - 2r. \]

If we set \( t_c - t = e^{-\tau} \), we obtain
\[ r_\tau = kr^2 - r^{3/2} - 2r. \]

Self-similar behavior follows from the closed form solution of this ODE. A perturbation argument can be used if we know that
\[ \lim_{t \to t_c} (t_c - t)^{-1} \lambda(t) = k. \]
