Talk Materials

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21 Sept. 2007 F. Sottile

New Fewnomial upper bounds from Gale dual polynomial systems

The real numbers are fundamentally different from the complex numbers

Ex A univariate polynomial f has up to degree(f) $\mathbb{C}$omplex roots, but at most #terms in f -1 positive roots.

Multivariate Polynomials

Consider a system

(⊛) $$f_1(x_1,\ldots,x_n) = \ldots = f_n(x_1,\ldots,x_n) = 0$$

of n (Laurent) polynomials in n variables having $n+k+1$ monomials among the polynomials.

A monomial $x_1^{w_1}x_2^{w_2}\cdots x_n^{w_n}$ corresponds to its exponent vector
$$w = (w_1,\ldots,w_n) \in \mathbb{Z}^n.$$

Let $\mathcal{W}$ = Set of exponent vectors of monomials in ⊛

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Kouchnirenko (1976) The number of non degenerate solutions to ⊛ in $(\mathbb{C}^x)^n$ is at most $n! \cdot \mathrm{vol}(\mathrm{conv}(W))$.

The real numbers are quite different.

Khovanskii (1980) If $|W| = n+k+1$, then ⊛ has at most $2^{\binom{n+k}{2}}(n+1)^{n+k}$ non degenerate positive solutions.

This fundamental finiteness result bounds the number of solutions by the description complexity of the system.

It is astronomical $n=k=2 \rightsquigarrow 2^6 \cdot 3^4 = 5184$.

Everyone believes that it is unrealistically large.
It is meaningful only for k small & $\mathrm{conv}(W)$ large.

First concrete evidence by Li, Rojas & Wang:
A system of 2 trinomials in 2 variables has at most 5 positive solutions.

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How to relate to Khovanskii?

→ We can multiply any polynomial by a monomial so that it has a constant term, w/o changing its solutions.

e.g. 2 trinomials have $2+2+1 = 5$ terms & so $n = k = 2$ & $5 < 5184$

However, 2 trinomials is not a general system with $n = k = 2$ & the problem of realistic fewnomial bounds remained open, even in this case.

In 2004 Bihan built on work with Bertrand & I to show

Thm A system with $k=1$ ($n+2$ monomials) has at most $n+1$ positive solutions. This is sharp.

→ Relied upon analyzing a peculiar univariate polynomial & bounding its solns.

Today, I'll discuss joint work with Bihan improving Khovanskii's bounds.

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~~Math.~~ AG/0609544

<u>Theorem</u> A system of n polynomials in n variables w/ $n+k+1$ monomials has at most

$$\frac{e^2+3}{4} \cdot 2^{\binom{k}{2}} n^k$$

positive solutions, and this is not sharp.

(When $n=k=2$, we can improve this to $15 < 5184$)

- Significantly smaller than Khovanskii's bound, but both have asymptotics

$$2^{O(k^2)} \quad n^{O(k)}$$

- Our method of proof is completely different. Papers

Gale dual systems

Suppose that the exponent vectors are

$\mathcal{W} = \{0, \omega_1, \omega_2, \ldots, \omega_{n+k}\} \subset \mathbb{R}^n$ & they span $\mathbb{R}^n$. & last n are linearly indep.

Given the system, perturb its coefficients if necessary & use Gauß elimination to put it in diagonal form.

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$$x^{w_i} = g_i(x) = h_i(x^{w_{n+1}}, \dots, x^{w_{n+k}})$$

linear

$i = 1, \dots, n.$

If (A) $0 = w_1 a_1 + w_2 a_2 + \cdots + w_{n+k} a_{n+k}$

Then $1 = (x^{w_1})^{a_1} \cdot (x^{w_2})^{a_2} \cdots (x^{w_{n+k}})^{a_{n+k}}$

So

$$(\oplus) \quad 1 = (g_1(x))^{a_1} \cdots g_n(x)^{a_n} \cdot (x^{w_{n+1}})^{a_{n+1}} \cdots (x^{w_{n+k}})^{a_{n+k}}.$$

Set $y_i = x^{w_{n+i}}$ $i = 1, \dots, k$ & $h_{n+i} = y_i$

Then $\oplus$ becomes

$$1 = \prod_{i=1}^{n+k} (h_i(y))^{a_i} =: \varphi(y)$$

A basis of linear relations (A) is a Gale dual vector configuration for w.

This gives a system of k equations in k unknowns, called a Gale system associated to our original system.

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The Gale System makes sense in the polyhedron

$$\Delta := \{ y \mid h_i(y) > 0 \ \forall i \} \subseteq \mathbb{R}^k_>$$

<u>Theorem</u> Gale duality for polynomial systems

The association

$$\mathbb{R}^n_> \longrightarrow \mathbb{R}^k_>$$

$$x \longmapsto (x^{w_{n+1}}, \dots, x^{w_{n+k}}) =: y$$

is a bijection between positive solutions to the original system & solutions in Δ to the Gale dual system.

! We could have $k > n$ | Papers

$$\psi_j = \log \varphi_j = \sum a_{ij} \log h_i(y) = 0$$

For $j = k, k-1, \dots, 1$ set

$$\Gamma_j := \mathrm{Jac}(\psi_1, \dots, \psi_j ; \Gamma_{j+1}, \dots, \Gamma_k)$$

$$C_j = \{ \psi_1 = \dots = \psi_{j-1} = 0 = \Gamma_{j+1} = \dots = \Gamma_k \} \subset \Delta$$

$\left(\prod h_i\right)^{2^{k-j}} \Gamma_j$ is a polynomial

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Set $u(C_j) = \#$ unbounded components of C_j

Khovanskii-Rolle gives

$$\#\{\psi_1 = \cdots = \psi_k = 0\} \leq u(C_k) + \#\{\psi_1 = \cdots = \psi_{k-1} = \Gamma_k = 0\}$$

$$\leq \underbrace{u(C_k) + \cdots + u(C_1)}_{②} + \underbrace{\#\{\Gamma_1 = \cdots = \Gamma_k = 0\}}_{①}$$

① $\leq 2^{\binom{k}{2}} n^k$ Bézout

② $u(C_j) \leq \frac{1}{2} 2^{\binom{k-j}{2}} n^{k-j} \binom{n+k+1}{j}$

by Bézout & Polyhedral combinatorics.

Bounding this & estimating by a sum, get

$$\leq \left(\frac{e^2-1}{4} + 1\right) \cdot 2^{\binom{k}{2}} n^k.$$

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