The numerical computation of the multiplicity of a component of an algebraic set

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Motivation

Basic problem: Compute the regularity and multiplicity of a 0-scheme supported at a single point.

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Why we got involved:

To develop algorithms for computing numerical free resolutions, we needed to know which degrees could appear among the syzygies.

In other words, we needed a way to compute regularity. Multiplicity came for free!

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1. Like I said, regularity bounds the degrees that syzygies could have, so regularity is a stopping criterion.

2. Multiplicity is intrinsic to the polynomial system. In some settings, it is the number of paths leading to the point during homotopy continuation.

Motivation

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The witness sets of numerical algebraic geometry provide a kind of numerical primary decomposition *of the radical*. These witness points are found by slicing the algebraic set with hyperplane sections.

One step towards the numerical primary decomposition of the *original* ideal is to retain the scheme structure when slicing.

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A: Here are two reasons:

1. Coefficient blowup (and other computational difficulties).
2. Inexact input (since the polynomial system may contain random complex numbers and the point is almost surely inexact).

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- Can compute the multiplicity and the regularity of a 0-scheme supported at a single point numerically
- No coefficient blowup
- Works under small perturbations

Motivation

There are actually two related methods:

- B., Peterson, and Sommese (see [BPS]): Inspired by known modern symbolic methods; uses a regularity criterion of Bayer and Stillman (see [BS]) to know when to stop
- Dayton and Zeng (see [DZ]): Inspired by the duality approach of Macaulay; relies on the use of structured matrices

The two methods each give the multiplicity, but the other output differs….

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It can be extracted from the Hilbert polynomial.

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The regularity tells us a point at which the Hilbert function stabilizes.

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- $R = \mathbb{C}[x_0, x_1, \ldots, x_n]$.

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- $(I)_k$ is the degree k homogeneous part of I.
- I_p is the prime ideal at the point p.
- (I:F) = {G in R | GF is in I} is the *ideal quotient* of I by F (where F is in R).
- The *saturation* of I is the intersection of all primary ideals in a reduced primary decomposition of I that are not m-primary (i.e., those that have geometric content).

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3. (something else)

Background

Corollary (to Bayer and Stillman, 1987):

Suppose I (homogeneous) is generated in degree at most k, vanishing only at the point p in $\mathbb{P}^n$. Let L be a linear form not contained in I_p. Then $\mathrm{reg}(I) \leq k$ if and only if $(I:L)_k=(I)_k$ and $(I, L)_k=(R)_k$.

The Algorithm (roughly)

A 0-scheme consists of a point and some multiplicity structure around the point. Our idea is to look at successively larger infinitessimal neighborhoods around the point until all multiplicity information is captured.

This is similar to computing the multiplicity of a monomial ideal using the "staircase" method.

Background

Two other facts:

1. Suppose I_p is an associated, non-embedded prime of an ideal I (p a point in $\mathbb{P}^n$). Let $J_k = (I, I_p^k)$. Then the multiplicity of I at I_p is equal to the multiplicity of J_k at I_p for $k >> 0$.

2. If the multiplicity of J_k at I_p is equal to the multiplicity of J_{k+1} at I_p, then the multiplicity of I at I_p is also equal to the same.

Background

As mentioned before, if regularity is known, then computing the multiplicity is trivial:

$\mu = \dim(R/I)_k$ for $k >> 0$.

Which k suffices? $k = \mathrm{reg}(I)$.

The Algorithm (detailed)

INPUT: $I=\{F_1, \ldots, F_r\}$ homogeneous in R (n+1 variables) and the point p at which I is supported (with the final coordinate of p being nonzero, WLOG).

OUTPUT: Multiplicity and regularity.

Let $I_p = \{p_i z_j - p_j z_i \mid 0 \leq i,j \leq n\}$.

Let $m = (z_0, \ldots, z_n)$.

Let k = maximal degree in which I is generated, $\mu(k-1) = -2$, and $\mu(k) = -1$.

While $\mu(k) \neq \mu(k-1)$ do:

- Form I_p^k, $J_k=(I, I_p^k)$, $z_n m_k$. Let A=0, B=1.
- While $A \neq B$ do:
  - Form $(J_k)_{k+1}$.
  - Compute $P = (J_k)_{k+1} \cap z_n m_k$.
  - Compute the preimage P' of P.
  - Compute $A = \text{rank}((J_k)_k)$, $B = \text{rank}(P')$.
  - If A = B, $\mu(k) = \text{rank}((m)_k) - A$; else let $J_k = P'$.
- If $\mu(k) = \mu(k-1)$, then $\mu = \mu(k)$ and $\text{reg}(I) \leq k$; else increment k.

Complete examples

Example: $I = (x^2, y)$ in the variables (x, y, z) at the point $(0, 0, 1)$.

$I_p = (x, y)$.

$m = (x, y, z)$.

$k = 2$, $\mu(2) = -2$, and $\mu(3) = -1$

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After some work at the board: $\mu = 2$ and $reg(I) = 3$.

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$k = 5$: $A=15$, $B=15$, so $\mu(5) = 21-15 = 6$.

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$k = 4$: $A=9$, $B=9$, so $\mu(4) = 15-9 = 6$.

$k = 5$: $A=15$, $B=15$, so $\mu(5) = 21-15 = 6$.

So $\mu = 6$ and $\mathrm{reg}(I) = 4$.

Implementation Details

After fixing a term order, homogeneous polynomials may easily be represented as vectors of coefficients.

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Most steps of the algorithm are very straightforward symbolic maneuvers.

While $\mu(k) \neq \mu(k\text{-}1)$ do:

- Form I_p^k, $J_k=(I, I_p^k)$, $z_n m_k$. Let A=0, B=1.
- While $A \neq B$ do:
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- If $\mu(k) = \mu(k\text{-}1)$, then $\mu = \mu(k)$ and $\text{reg}(I) \leq k$; else increment k.

Implementation Details

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However, the matrices are inexact, so the exact rank cannot be computed. Therefore, we use the SVD to detect the rank of a matrix.

Implementation Details

The singular value decomposition (SVD) of a matrix A has the form $A=U\Sigma V^*$ with:

- U, V unitary;
- Σ is diagonal with the "singular values" along the diagonal;
- the number of "nonzero" singular values is the numerical rank; and
- it is easy to find a basis for the nullspace of A.

Implementation Details

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$V = (J_k)_{k+1}$ and $W = z_n m_k$ are just vector spaces, so we use the SVD to compute bases for the nullspaces of V and W, concatenate them, and finally find the nullspace of that matrix..

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NOTE: This was implemented in Bertini. It will not be released with the first release of Bertini.

Other examples

$I = \{x^4+2x^2y^2+y^4+3x^2yz-y^3z,$
$\quad x^6+3x^4y^2+3x^2y^4+y^6-4x^2y^2z^2\}$

$p = (0, 0, 1)$

Multiplicity is known to be 14.

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- $p = (0.000000001, 0.000000001, 1)$

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- $p = (0, 0, 1)$
- $p = (0.000000001, 0.000000001, 1)$
- $p = (0.0001, 0.0001, 1)$

Other examples

$$I = \{3x^2 + \ldots + (383678/77763)w^2,$$
$$x^3 + \ldots + (2600452/699867)w^3,$$
$$z^3 + \ldots + (12496664/766521)w^3,$$
$$x^2 + \ldots + (64/9)w^2\}$$

$p = (1, 8/3, -2/7, 34/23)$

Multiplicity = 2.

Other examples

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$p = (1, 8/3, -2/7, 34/23)$

Multiplicity = 2.

Key remark: We used truncated floating point numbers in place of all rational numbers, and it still worked.

A similar method

Another method for computing the multiplicity of a 0-scheme supported at a single point may be found in [DZ].

Each method has its own benefits aside from producing the multiplicity.

The method of [DZ] provides a more elaborate "multiplicity structure" as well as a better bound on the number of deflations needed, while the method of [BPS] produces the regularity.

A similar method

A little about deflation:

Given a singular root x of a polynomial system, F, one can use deflation to produce a system containing F that has (x,u) as a nonsingular solution (where the u variables are added during deflation).

[OWM] developed the method, [LVZ] proved that it terminates after no more than μ steps, and [DZ] sharpened the bound. Also see Anton Leykin's talk on September 27.

References

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