Real monoid surfaces

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"Non-linear Computational Geometry"
IMA, May 31, 2007

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Outline

Introduction

A monoid surface is a surface of degree d which has a singular point of multiplicity $d-1$.

Any monoid surface admits a rational parameterization, hence is of potential interest in computer aided geometric design.

Monoid surfaces (complex and real) of degree ≤ 4 can be described, in terms of their singularities.

The space of quartic monoid surfaces has a stratification according to the singularities of the surface; the dimension and number of components of the strata can be described.

The first part of this work is joint with M. Løberg and P. H. Johansen, and complements the work of Rohn (1884) and Takahashi–Watanabe–Higuchi (1982); the last part is due to Johansen. The figures are made using SURF.

Monoid surfaces

Consider a surface $X = Z(F) \subset \mathbb{P}^3$ of degree d, such that the point $O := (1 : 0 : 0 : 0)$ is a point of multiplicity $d - 1$. It is "almost" a cone!

Then

$$F(x_0, x_1, x_2, x_3) = x_0 f_{d-1}(x_1, x_2, x_3) + f_d(x_1, x_2, x_3),$$

where f_{d-1} and f_d are homogeneous polynomials of degrees $d - 1$ and d.

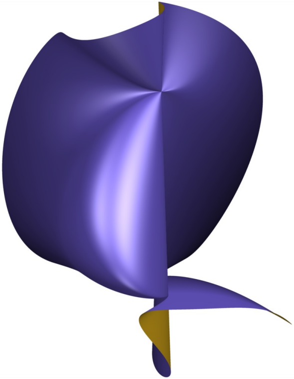
The (projectivized) tangent cone to X at O is the plane curve $Z(f_{d-1}) \subset \mathbb{P}^2$.

The curve $Z(f_d) \subset \mathbb{P}^2$ is the intersection of X with the plane at infinity $Z(x_0)$.

Example. The surface $X \subset \mathbb{P}^3$ defined by

$$F = x_0(x_1x_2^2 + x_3^3) + x_1^4$$

is a quartic monoid. Its singular points are O and $(0:0:1:0)$.



The natural parameterization

The natural parameterization of the monoid X is the map

$$\theta_F \colon \mathbb{P}^2 \to X \subset \mathbb{P}^3$$

given by

$$\theta_F(a) = (-f_d(a) : f_{d-1}(a)a_1 : f_{d-1}(a)a_2 : f_{d-1}(a)a_3),$$

for $a = (a_1 : a_2 : a_3) \in \mathbb{P}^2 \setminus Z(f_{d-1}(a), f_d(a))$.

For every $a = (a_1 : a_2 : a_3) \in \mathbb{P}^2$, the line

$$L_a := \{(s : ta_1 : ta_2 : ta_3) | (s : t) \in \mathbb{P}^1\}$$

intersects $X = Z(F)$ with multiplicity at least $d - 1$ in O.

If $f_{d-1}(a) \neq 0$ or $f_d(a) \neq 0$, then the line L_a also intersects X in the point $\theta_F(a)$.

The natural parameterization is the inverse of the projection of X to $\mathbb{P}^2$ from the point O.

Note that θ_F maps $Z(f_{d-1}) \setminus Z(f_d)$ to O.

For each base point $b \in Z(f_{d-1}, f_d)$, the line L_b is contained in the monoid surface. Conversely, every line of type L_b contained in the monoid surface corresponds to a base point b.

If $P \in X$ is singular point on the monoid X, then the line L through P and O has intersection multiplicity at least $d - 1 + 2 = d + 1$ with X. Hence, by Bezout's theorem, L is contained in X.

Lemma

(i) All singular points of X are on lines L_b, where $b \in Z(f_{d-1}, f_d)$ is a base point.

(ii) Both $Z(f_{d-1})$ and $Z(f_d)$ are singular in a point $b \in \mathbb{P}^2$ if and only if all points on L_b are singular on X.

(iii) If not all points on L_b are singular, then at most one point other than O on L_b is singular.

If $Z(f_{d-1})$ and $Z(f_d)$ have no common singular points, then each line L_b contains at most one singular point in addition to O.

Hence in this case the surface has only finitely many singular points. In what follows, we shall only consider monoid surfaces of this kind.

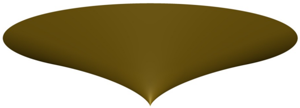
The singular point on L_b is of type A_{m-1}, where m is the intersection multiplicity of $Z(f_{d-1})$ and $Z(f_d)$ at b.

The maximal number of singular points that a monoid surface of degree d can have is $\frac{d(d-1)}{2} + 1$.

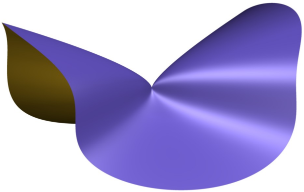
Real surfaces and real singularities

In the case that the singular point is *real*, it is of type A^-_{m-1}.

The two real versions of the A_2 singularity:



$A^+_2 : x^2 + y^2 - z^3 = 0$



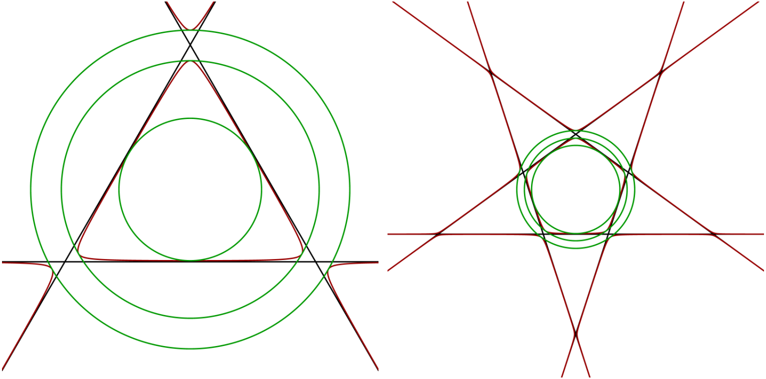
$A^-_2 : x^2 - y^2 - z^3 = 0$

Real monoid with max number of singularities

To construct a monoid with the maximal number of *real* singularities, it is sufficient to construct two *affine* real curves in the xy-plane defined by equations f_{d-1} and f_d of degrees $d-1$ and d such that the curves intersect in $d(d-1)/2$ points with multiplicity 2. Assume $d-1$ is odd. Set

$$f_{d-1} = \varepsilon - \prod_{i=1}^{d-1} \left(x \sin\left(\frac{2i\pi}{d-1}\right) + y \cos\left(\frac{2i\pi}{d-1}\right) + 1 \right).$$

For $\varepsilon > 0$ sufficiently small there exist at least $\frac{d}{2}$ radii $r > 0$, one for each (positive real) root of the univariate polynomial $f_{d-1}|_{x=0}$, such that the circle $x^2 + y^2 - r^2$ intersects f_{d-1} in $d-1$ points with multiplicity 2. Let f_d be a product of such circles. The homogenizations of f_{d-1} and f_d define a monoid surface with $1 + \frac{1}{2}d(d-1)$ singularities.



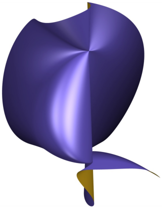
The curves f_{d-1} for $d-1=3,5$ and corresponding circles.

Real monoid with max A_m-singularity

The maximal Milnor number of a singularity other than O is $d(d-1)-1$. The following example shows that this bound can be achieved on a *real* monoid surface:

Example. The surface $X \subset \mathbb{P}^3$ defined by $F = x_0(x_1x_2^{d-2} + x_3^{d-1}) + x_1^d$ has precisely two singular points. The point O is a singularity of multiplicity 3 with Milnor number $\mu = (d^2-3d+1)(d-2)$, while the point $(0:0:1:0)$ is an $A_{d(d-1)-1}$ singularity.

For $d = 4$:



Quartic monoid surfaces

Theorem

On a quartic monoid surface, all singularities other than the monoid point O can occur as given in the following table. Moreover, all possibilities are realizable on *real* quartic monoids with a real monoid point, and with the additional singularities being real and of type A^-.

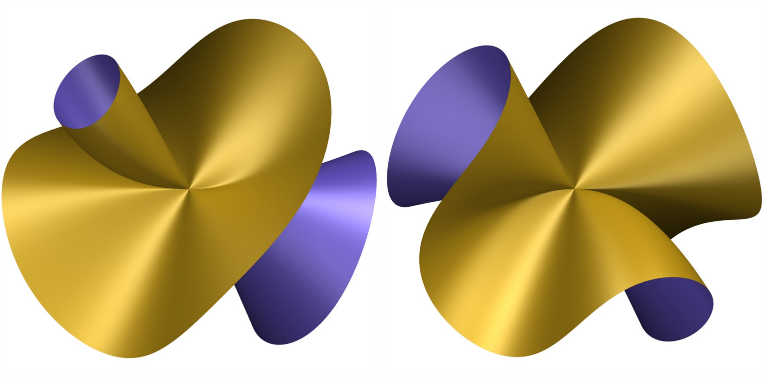
(In the table, the first column gives the type of the tangent cone: nonsingular cubic, nodal cubic, cuspidal cubic, . . . , triple line.

The notation in the second column refers to Arnold's notation for singularity types.

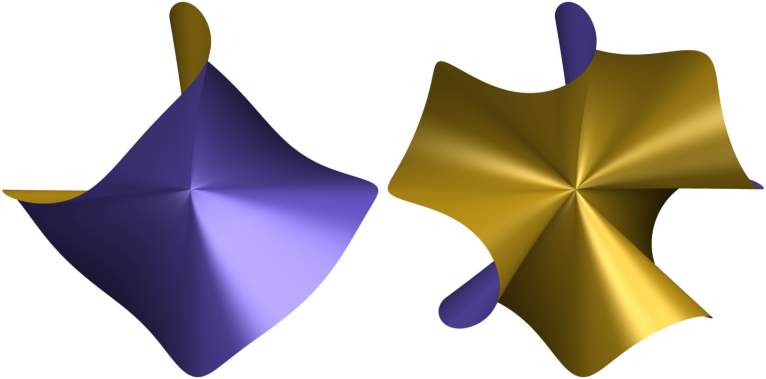
The third column refers to the possible intersections of the curves $Z(f_3)$ and $Z(f_4)$; their total number of intersections is 12.)

Case	Triple point	Invariants and constraints	Other singularities
0	$P_8 = T_{3,3,3}$		A_{m_i-1}, $\sum m_i = 12$
1	$T_{3,3,4}$		A_{m_i-1}, $\sum m_i = 12$
	$T_{3,3,3+m}$	$m = 2, \ldots, 12$	A_{m_i-1}, $\sum m_i = 12 - m$
2	Q_{10}		A_{m_i-1}, $\sum m_i = 12$
	T_{9+m}	$m = 2, 3$	A_{m_i-1}, $\sum m_i = 12 - m$
3	$T_{3,4+r_0,4+r_1}$	$r_0 = \max(j_0, k_0)$, $r_1 = \max(j_1, k_1)$, $j_0 > 0 \leftrightarrow k_0 > 0$, $\min(j_0, k_0) \leq 1$, $j_1 > 0 \leftrightarrow k_1 > 0$, $\min(j_1, k_1) \leq 1$	A_{m_i-1}, $\sum m_i = 4 - k_0 - k_1$, $A_{m'_i-1}$, $\sum m'_i = 8 - j_0 - j_1$
4	S series	$j_0 \leq 8$, $k_0 \leq 4$, $\min(j_0, k_0) \leq 2$, $j_0 > 0 \leftrightarrow k_0 > 0$, $j_1 > 0 \leftrightarrow k_0 > 1$	A_{m_i-1}, $\sum m_i = 4 - k_0$, $A_{m'_i-1}$, $\sum m'_i = 8 - j_0$
5	$T_{4+j_k,4+j_l,4+j_m}$	$m_1 + l_1 \leq 4$, $k_2 + m_2 \leq 4$, $k_3 + l_3 \leq 4$, $k_2 > 0 \leftrightarrow k_3 > 0$, $l_1 > 0 \leftrightarrow l_3 > 0$, $m_1 > 0 \leftrightarrow m_2 > 0$, $\min(k_2, k_3) \leq 1$, $\min(l_1, l_3) \leq 1$, $\min(m_1, m_2) \leq 1$, $j_k = \max(k_2, k_3)$, $j_l = \max(l_1, l_3)$, $j_m = \max(m_1, m_2)$	A_{m_i-1}, $\sum m_i = 4 - m_1 - l_1$, $A_{m'_i-1}$, $\sum m'_i = 4 - k_2 - m_2$, $A_{m''_i-1}$, $\sum m''_i = 4 - k_3 - l_3$
6	U series	$j_1 > 0 \leftrightarrow j_2 > 0 \leftrightarrow j_3 > 0$, at most one of $j_1, j_2, j_3 > 1$, $j_1, j_2, j_3 \leq 4$	A_{m_i-1}, $\sum m_i = 4 - j_1$, $A_{m'_i-1}$, $\sum m'_i = 4 - j_2$, $A_{m''_i-1}$, $\sum m''_i = 4 - j_3$
7	V series	$j_0 > 0 \leftrightarrow k_0 > 0$, $\min(j_0, k_0) \leq 1$, $j_0 \leq 4$, $k_0 \leq 4$	A_{m_i-1}, $\sum m_i = 4 - j_0$,
8	V' series		None

Real types



The monoids $Z(x^3 + y^3 + 5xyz - z^3(x + y))$ and $Z(x^3 + y^3 + 5xyz - z^3(x - y))$ both have a $T_{3,3,5}$ singularity.



The monoids $Z(z^3 + xy^3 + x^3y)$ and $Z(z^3 + xy^3 - x^3y)$ are of the same type over $\mathbb{C}$, but are different over $\mathbb{R}$.

Stratification of the space of quartic monoids

The space of quartic surfaces with a triple point O has dimension 24. The space of quartic monoid surfaces (with only isolated singularities) is an open subset of this space and is contained in

$$(\mathbb{A}^{10} \setminus \{0\}) \times (\mathbb{A}^{15} \setminus \{0\}) / \sim \subset \mathbb{P}^{24}.$$

The *stratum* of a given $X = Z(F)$ is the set of quartic monoid surfaces that have the same type of tangent cone $Z(f_3)$, and the same kind of intersections between the tangent cone and the curve at infinity $Z(f_4)$.

Each stratum S has a (not necessarily rational) parameterization $B_S \times G \to S$, where B_S is an open in (a hypersurface of) an affine space and G is the group of projective transformations fixing O.

For each tangent cone type, compute (use Singular) a certain matrix group, which is used to compute the components of dimension of B_S, and the dimension of S.

Type	Invariants	$\dim S$	Comp
0	$m_1 + \ldots + m_r = 12$	$12 + r$	?
1	$m = 0, m_1 + \ldots + m_r = 12$ $2^{e_1}3^{e_2} := \gcd(m_1, \ldots, m_r)$	$11 + r$	$1 + e_1$
	$m = 2, \ldots, 12, m_1 + \ldots + m_r = 12 - m$	$12 + r$	1
2	$m = 0, m_1 + \ldots m_r = 12$	$10 + r$	1
	$m = 2, 3, m_1 + \ldots + m_r = 12 - m$	$11 + r$	1
$\ldots$	$\ldots$	$\ldots$	$\ldots$
7	$j_0 = k_0 = 0$ $m_1 + \ldots + m_r = m'_1 + \ldots + m'_{r'} = 4$	$r + r' + 11$	1
	$j_0, k_0 > 0,\ m_1 + \ldots + m_r = 4 - j_0$ $m'_1 + \ldots + m'_{r'} 4 - j_0$	$r + r' + r'' + 11$	1
8	$m_1 + \ldots + m_r = 4$	$r + 13$	1

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