

Real monoid surfaces

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Outline

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Introduction

A monoid surface is a surface of degree d which has a singular point of multiplicity $d - 1$.

Any monoid surface admits a rational parameterization, hence is of potential interest in computer aided geometric design.

Monoid surfaces (complex and real) of degree ≤ 4 can be described, in terms of their singularities.

The space of quartic monoid surfaces has a stratification according to the singularities of the surface; the dimension and number of components of the strata can be described.

The first part of this work is joint with [M. Løberg](#) and [P. H. Johansen](#), and complements the work of Rohn (1884) and Takahashi–Watanabe–Higuchi (1982); the last part is due to [Johansen](#). The figures are made using [SURF](#).

Monoid surfaces

Consider a surface $X = Z(F) \subset \mathbb{P}^3$ of degree d , such that the point $O := (1 : 0 : 0 : 0)$ is a point of multiplicity $d - 1$. It is “almost” a cone!

Then

$$F(x_0, x_1, x_2, x_3) = x_0 f_{d-1}(x_1, x_2, x_3) + f_d(x_1, x_2, x_3),$$

where f_{d-1} and f_d are homogeneous polynomials of degrees $d - 1$ and d .

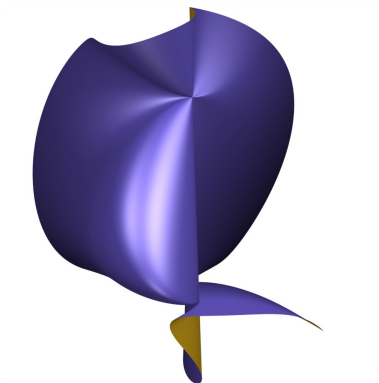
The (projectivized) tangent cone to X at O is the plane curve $Z(f_{d-1}) \subset \mathbb{P}^2$.

The curve $Z(f_d) \subset \mathbb{P}^2$ is the intersection of X with the plane at infinity $Z(x_0)$.

Example. The surface $X \subset \mathbb{P}^3$ defined by

$$F = x_0(x_1x_2^2 + x_3^3) + x_1^4$$

is a quartic monoid. Its singular points are O and $(0 : 0 : 1 : 0)$.



The natural parameterization

The natural parameterization of the monoid X is the map

$$\theta_F: \mathbb{P}^2 \rightarrow X \subset \mathbb{P}^3$$

given by

$$\theta_F(a) = (-f_d(a) : f_{d-1}(a)a_1 : f_{d-1}(a)a_2 : f_{d-1}(a)a_3),$$

for $a = (a_1 : a_2 : a_3) \in \mathbb{P}^2 \setminus Z(f_{d-1}(a), f_d(a))$.

For every $a = (a_1 : a_2 : a_3) \in \mathbb{P}^2$, the line

$$L_a := \{(s : ta_1 : ta_2 : ta_3) \mid (s : t) \in \mathbb{P}^1\}$$

intersects $X = Z(F)$ with multiplicity at least $d - 1$ in O .

If $f_{d-1}(a) \neq 0$ or $f_d(a) \neq 0$, then the line L_a also intersects X in the point $\theta_F(a)$.

The natural parameterization is the inverse of the projection of X to \mathbb{P}^2 from the point O .

Note that θ_F maps $Z(f_{d-1}) \setminus Z(f_d)$ to O .

For each base point $b \in Z(f_{d-1}, f_d)$, the line L_b is contained in the monoid surface. Conversely, every line of type L_b contained in the monoid surface corresponds to a base point b .

If $P \in X$ is singular point on the monoid X , then the line L through P and O has intersection multiplicity at least $d - 1 + 2 = d + 1$ with X . Hence, by Bezout's theorem, L is contained in X .

Lemma

- (i) All singular points of X are on lines L_b , where $b \in Z(f_{d-1}, f_d)$ is a base point.
- (ii) Both $Z(f_{d-1})$ and $Z(f_d)$ are singular in a point $b \in \mathbb{P}^2$ if and only if all points on L_b are singular on X .
- (iii) If not all points on L_b are singular, then at most one point other than O on L_b is singular.



If $Z(f_{d-1})$ and $Z(f_d)$ have no common singular points, then each line L_b contains at most one singular point in addition to O .

Hence in this case the surface has only finitely many singular points. In what follows, we shall only consider monoid surfaces of this kind.

The singular point on L_b is of type A_{m-1} , where m is the intersection multiplicity of $Z(f_{d-1})$ and $Z(f_d)$ at b .

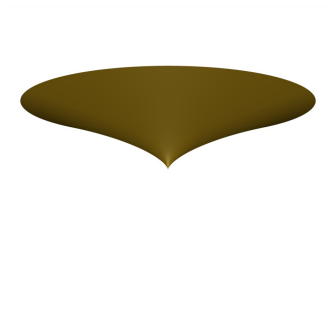
The maximal number of singular points that a monoid surface of degree d can have is $\frac{d(d-1)}{2} + 1$.



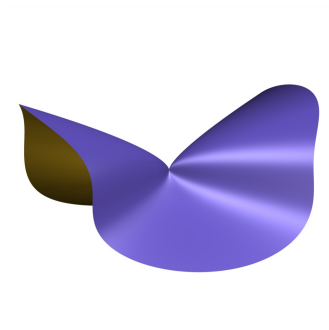
Real surfaces and real singularities

In the case that the singular point is *real*, it is of type A_{m-1}^- .

The two real versions of the A_2 singularity:



$$A_2^+ : x^2 + y^2 - z^3 = 0$$



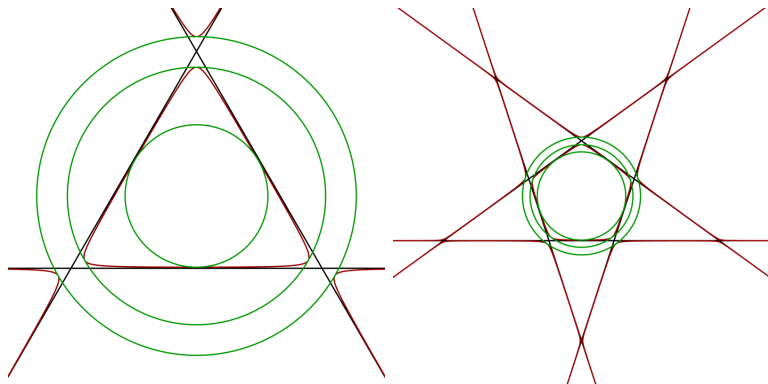
$$A_2^- : x^2 - y^2 - z^3 = 0$$

Real monoid with max number of singularities

To construct a monoid with the maximal number of *real* singularities, it is sufficient to construct two *affine* real curves in the xy -plane defined by equations f_{d-1} and f_d of degrees $d-1$ and d such that the curves intersect in $d(d-1)/2$ points with multiplicity 2. Assume $d-1$ is odd. Set

$$f_{d-1} = \varepsilon - \prod_{i=1}^{d-1} \left(x \sin \left(\frac{2i\pi}{d-1} \right) + y \cos \left(\frac{2i\pi}{d-1} \right) + 1 \right).$$

For $\varepsilon > 0$ sufficiently small there exist at least $\frac{d}{2}$ radii $r > 0$, one for each (positive real) root of the univariate polynomial $f_{d-1}|_{x=0}$, such that the circle $x^2 + y^2 - r^2$ intersects f_{d-1} in $d-1$ points with multiplicity 2. Let f_d be a product of such circles. The homogenizations of f_{d-1} and f_d define a monoid surface with $1 + \frac{1}{2}d(d-1)$ singularities.



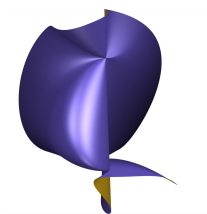
The curves f_{d-1} for $d - 1 = 3, 5$ and corresponding circles.

Real monoid with max A_m -singularity

The maximal Milnor number of a singularity other than O is $d(d-1) - 1$. The following example shows that this bound can be achieved on a *real* monoid surface:

Example. The surface $X \subset \mathbb{P}^3$ defined by $F = x_0(x_1x_2^{d-2} + x_3^{d-1}) + x_1^d$ has precisely two singular points. The point O is a singularity of multiplicity 3 with Milnor number $\mu = (d^2 - 3d + 1)(d - 2)$, while the point $(0 : 0 : 1 : 0)$ is an $A_{d(d-1)-1}$ singularity.

For $d = 4$:



Quartic monoid surfaces

Theorem

On a quartic monoid surface, all singularities other than the monoid point O can occur as given in the following table. Moreover, all possibilities are realizable on *real* quartic monoids with a real monoid point, and with the additional singularities being real and of type A^- .

(In the table, the first column gives the type of the tangent cone: nonsingular cubic, nodal cubic, cuspidal cubic, . . . , triple line.

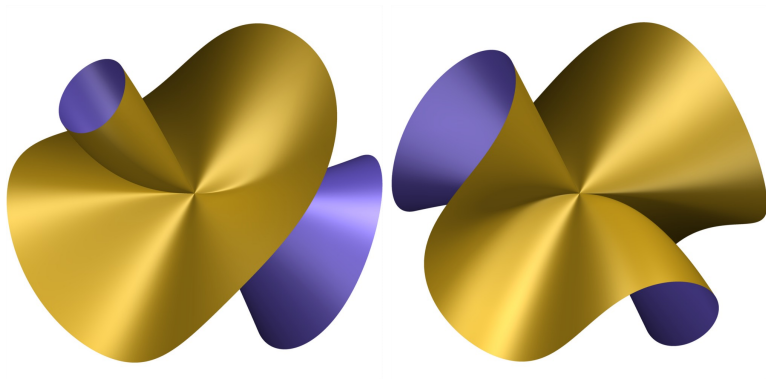
The notation in the second column refers to Arnold's notation for singularity types.

The third column refers to the possible intersections of the curves $Z(f_3)$ and $Z(f_4)$; their total number of intersections is 12.)

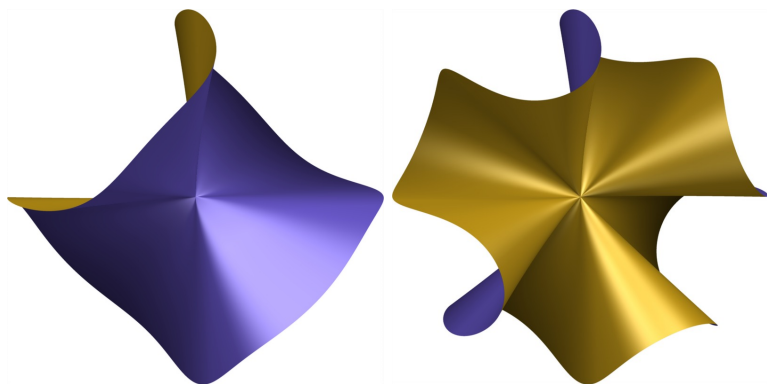


Case	Triple point	Invariants and constraints	Other singularities
0	$P_8 = T_{3,3,3}$		$A_{m_i-1}, \sum m_i = 12$
1	$T_{3,3,4}$		$A_{m_i-1}, \sum m_i = 12$
	$T_{3,3,3+m}$	$m = 2, \dots, 12$	$A_{m_i-1}, \sum m_i = 12 - m$
2	Q_{10}		$A_{m_i-1}, \sum m_i = 12$
	T_{9+m}	$m = 2, 3$	$A_{m_i-1}, \sum m_i = 12 - m$
3	$T_{3,4+r_0,4+r_1}$	$r_0 = \max(j_0, k_0), r_1 = \max(j_1, k_1),$ $j_0 > 0 \leftrightarrow k_0 > 0, \min(j_0, k_0) \leq 1,$ $j_1 > 0 \leftrightarrow k_1 > 0, \min(j_1, k_1) \leq 1$	$A_{m_i-1}, \sum m_i = 4 - k_0 - k_1,$ $A_{m'_i-1}, \sum m'_i = 8 - j_0 - j_1$
4	S series	$j_0 \leq 8, k_0 \leq 4, \min(j_0, k_0) \leq 2,$ $j_0 > 0 \leftrightarrow k_0 > 0, j_1 > 0 \leftrightarrow k_0 > 1$	$A_{m_i-1}, \sum m_i = 4 - k_0,$ $A_{m'_i-1}, \sum m'_i = 8 - j_0$
5	$T_{4+j_k,4+j_l,4+j_m}$	$m_1 + l_1 \leq 4, k_2 + m_2 \leq 4,$ $k_3 + l_3 \leq 4, k_2 > 0 \leftrightarrow k_3 > 0,$ $l_1 > 0 \leftrightarrow l_3 > 0, m_1 > 0 \leftrightarrow m_2 > 0,$ $\min(k_2, k_3) \leq 1, \min(l_1, l_3) \leq 1,$ $\min(m_1, m_2) \leq 1, j_k = \max(k_2, k_3),$ $j_l = \max(l_1, l_3), j_m = \max(m_1, m_2)$	$A_{m_i-1}, \sum m_i = 4 - m_1 - l_1,$ $A_{m'_i-1}, \sum m'_i = 4 - k_2 - m_2,$ $A_{m''_i-1}, \sum m''_i = 4 - k_3 - l_3$
6	U series	$j_1 > 0 \leftrightarrow j_2 > 0 \leftrightarrow j_3 > 0,$ at most one of $j_1, j_2, j_3 > 1,$ $j_1, j_2, j_3 \leq 4$	$A_{m_i-1}, \sum m_i = 4 - j_1,$ $A_{m'_i-1}, \sum m'_i = 4 - j_2,$ $A_{m''_i-1}, \sum m''_i = 4 - j_3$
7	V series	$j_0 > 0 \leftrightarrow k_0 > 0, \min(j_0, k_0) \leq 1,$ $j_0 \leq 4, k_0 \leq 4$	$A_{m_i-1}, \sum m_i = 4 - j_0,$
8	V' series		None

Real types



The monoids $Z(x^3 + y^3 + 5xyz - z^3(x + y))$ and $Z(x^3 + y^3 + 5xyz - z^3(x - y))$ both have a $T_{3,3,5}$ singularity.



The monoids $Z(z^3 + xy^3 + x^3y)$ and $Z(z^3 + xy^3 - x^3y)$ are of the same type over \mathbb{C} , but are different over \mathbb{R} .

Stratification of the space of quartic monoids

The space of quartic surfaces with a triple point O has dimension 24. The space of quartic monoid surfaces (with only isolated singularities) is an open subset of this space and is contained in

$$(\mathbb{A}^{10} \setminus \{0\}) \times (\mathbb{A}^{15} \setminus \{0\}) / \sim \subset \mathbb{P}^{24}.$$

The *stratum* of a given $X = Z(F)$ is the set of quartic monoid surfaces that have the same type of tangent cone $Z(f_3)$, and the same kind of intersections between the tangent cone and the curve at infinity $Z(f_4)$.

Each stratum S has a (not necessarily rational) parameterization $B_S \times G \rightarrow S$, where B_S is an open in (a hypersurface of) an affine space and G is the group of projective transformations fixing O .

For each tangent cone type, compute (use Singular) a certain matrix group, which is used to compute the components of dimension of B_S , and the dimension of S .

Type	Invariants	$\dim S$	Comp
0	$m_1 + \dots + m_r = 12$	$12 + r$?
1	$m = 0, m_1 + \dots + m_r = 12$ $2^{e_1} 3^{e_2} := \gcd(m_1, \dots, m_r)$	$11 + r$	$1 + e_1$
	$m = 2, \dots, 12, m_1 + \dots + m_r = 12 - m$	$12 + r$	1
2	$m = 0, m_1 + \dots + m_r = 12$	$10 + r$	1
	$m = 2, 3, m_1 + \dots + m_r = 12 - m$	$11 + r$	1
...
7	$j_0 = k_0 = 0$ $m_1 + \dots + m_r = m'_1 + \dots + m'_{r'} = 4$	$r + r' + 11$	1
	$j_0, k_0 > 0, m_1 + \dots + m_r = 4 - j_0$ $m'_1 + \dots + m'_{r'} = 4 - j_0$	$r + r' + r'' + 11$	1
8	$m_1 + \dots + m_r = 4$	$r + 13$	1

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