

Algebra & Algorithms for Differential Elimination & Completion.

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Abstract

Differential algebra provides an algebraic viewpoint on nonlinear differential systems. The motivating questions for this talk are:

- How do we define the general solution of a nonlinear equations
- What are the conditions for a differential system to have a solution
- How do we measure the *degrees of freedom* for the solution set of a differential system

Theory and algorithms for those are extensions of commutative algebra (prime ideal decomposition, Hilbert polynomials) and Groebner bases techniques.

The library `difalg` in Maple supports this introduction to constructive differential algebra. It has been developed by F. Boulier (1996) and the speaker afterwards. A recent extension of differential algebra to non-commutative derivations, and its implementation in `difalg`, allow to treat systems bearing on differential invariants.

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1 Sample problems

Envelope

Consider a family of curves

$$y = cx + c^2$$

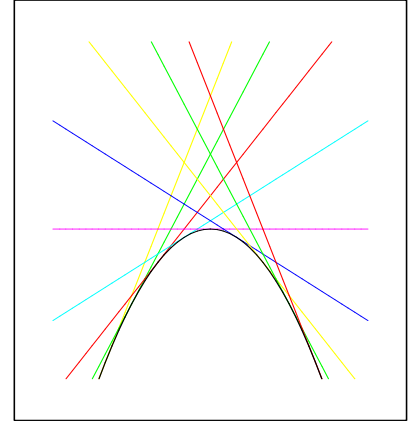
What is their envelope?

The differential equation satisfied by this family is:

$$\frac{dy^2}{dx} + x \frac{dy}{dx} = y$$

General solution: $y = cx + c^2$

Singular solution: $y = -1/4x^2$



Kepler \Rightarrow Newton

Kepler's observational laws:

K_1 the planets move along ellipses with the sun as focus

$$r = p + ex$$

$$\text{where } r^2 = x^2 + y^2$$

$$\text{and } \dot{p}, \dot{e} = 0$$

K_2 the vector from the sun to the planet sweeps equal area in equal times

$$x \dot{y} - \dot{x} y = s$$

$$\text{with } \dot{s} = 0$$

Newton's gravitational laws:

N_1 the acceleration is inversely proportional to the square of the distance to the sun.

$$\frac{d}{dt} (r^2 a) = 0$$

$$\text{where } a^2 = (\ddot{x}^2 + \ddot{y}^2)$$

N_2 the acceleration vector is directed towards the sun

$$x \ddot{y} - \ddot{x} y = 0$$

$$K_2 \Rightarrow N_2, \quad K_1, K_2 \Rightarrow N_1 ?$$

Membership [Wu 91]

Orthogonal waves

[G. Metivier]

$$s(\phi_{xx} + \phi_{yy}) + s_x \phi_x + s_y \phi_y + \phi = 0$$

$$s(\psi_{xx} + \psi_{yy}) + s_x \psi_x + s_y \psi_y + \psi = 0$$

$$\psi_x \phi_x + \psi_y \phi_y = 0$$

- Is there a solution?

consistency

- What are the *degrees of freedom*

completion

$$\begin{aligned}
s &= f_1(y) + f_2(y)x + s_{20}\frac{x^2}{2} + s_{30}\frac{x^3}{6} \dots \\
\psi &= f_3(y) + c_1x + \psi_{20}\frac{x^2}{2} + \psi_{11}xy + \psi_{20}\frac{x^2}{2} \dots \\
\phi &= f_4(y) + \phi_{10}x + \phi_{01}y + \phi_{20}\frac{x^2}{2} + \phi_{11}xy + \dots
\end{aligned}$$

4 functions of 1 variable, 1 constant.

- Conditions on s

elimination

Equivalence

[Neut 03]

$$\begin{array}{ccc}
y'' = f(x, y, y') & \xrightarrow{\exists? \xi, \eta} & Y'' = 0 \\
& & X = \xi(x, y), Y = \eta(x, y)
\end{array}$$

$$\begin{array}{ccc}
y_2 = f(x, y_0, y_1) & \xrightarrow{\exists? \xi, \eta} & Y_2 = 0 \\
& & X = \xi(x, y_0), Y_0 = \eta(x, y_0)
\end{array}$$

$$\left\{ \begin{array}{l}
\xi_{y_1} = 0, \quad \eta_{y_1} = 0, \\
(\eta_{y_0}\xi_x - \xi_{y_0}\eta_x) f + (\eta_{y_0y_0}\xi_{y_0} - \eta_{y_0}\xi_{y_0y_0}) y_1^3 \\
+ (2\eta_{xy_0}\xi_{y_0} - 2\eta_{y_0}\xi_{xy_0} - \eta_x\xi_{y_0y_0} + \eta_{y_0y_0}\xi_x) y_1^2 \\
+ (2\eta_{xy_0}\xi_x - 2\eta_x\xi_{xy_0} - \eta_{y_0}\xi_{xx} + \eta_{x,x}\xi_{y_0}) y_1 - \eta_x\xi_{xx} + \eta_{xx}\xi_x = 0
\end{array} \right.$$

Differential indeterminates : ξ, η, f , functions of x, y_0, y_1

Derivations : $\partial_x = \frac{\partial}{\partial x}$, $\partial_{y_0} = \frac{\partial}{\partial y_0}$, $\partial_{y_1} = \frac{\partial}{\partial y_1}$

Differential indeterminates : ξ, η, f

Consider ∂_{y_0} , ∂_{y_1} , and $D_x = \partial_x + y_1\partial_{y_0} + f\partial_{y_1}$

$$\partial_{y_0} D_x - D_x \partial_{y_0} = f_{y_0} \partial_{y_1}, \quad \partial_{y_1} D_x - D_x \partial_{y_1} = \partial_{y_0} + f_{y_1} \partial_{y_1}$$

$$\left\{ \begin{array}{l}
\xi_{y_1} = 0, \quad \eta_{y_1} = 0, \\
\xi_x \eta_{xx} - \xi_{xx} \eta_x = 0
\end{array} \right.$$

General case :

$$f_{y_1y_1y_1y_1} = 0, \quad f_{xy_1y_1} = 4f_{xy_0y_1} + f_{y_1}f_{xy_1y_1} - 4f_{y_1}f_{y_0y_1} - 6f_{y_0y_0} + 3f_{y_1y_1}f_{y_0}$$

Fiber preserving case ($\xi_{y_0} = 0$)

$$f_{y_1y_1y_1} = 0, \quad f_{xy_1y_1} = f_{y_0y_1}, \quad f_{xy_0y_1} = 2f_{y_0y_0} - f_{y_1y_1}f_{y_0} + f_{y_1}f_{y_0y_1}$$

2 Software

Software for nonlinear differential systems

diffalg: Differential Algebra MapleV

1996 Rosenfeld-Gröbner algorithm [BLOP 1997]
by F. Boulier (1996) then at SCG, U. of Waterloo. Maple V.5

1998 Singular solutions [H 99]; Efficiency improvement [H 00]
by E. Hubert then at SCG, UWaterloo. Maple 6

1999 Redesign of help pages. Maple 7

2004 Non-commuting derivations [H 2005]
<http://www.inria.fr/cafe/Evelyne.Hubert/diffalg>

BLAD: Bibliothèques Lilloises d'Algèbre Différentielle GLGPL

C libraries distributed under Gnu Lesser General Public License

<http://www2.lifl.fr/~boulier/BLAD> by F. Boulier

RegularChains for polynomial systems Maple10

by F. Lemaire & M. Moreno Maza *et al.* U.WO Maple 10

RIF: Reduced Involutive Forms Maple6

A. Wittkopf, G. Reid Maple 6, DEtools

CRACK: PDE solver Reduce

T. Wolf (<http://lie.math.brocku.ca/crack>)

Software for linear functional systems

kan/sm1 by N. Takayama (<http://www.math.kobe-u.ac.jp/KAN>)

D-Macaulay by A. Leykin & H. Tsai (<http://www.ima.umn.edu/~leykin/Dmodules>)

Cocoa, <http://cocoa.dima.unige.it/>

Plural:Singular by V. Levandovskyy & H. Schönemann (<http://www.singular.uni-kl.de/plural>)

In Maple: Groebner by F. Chyzak; OreModule with A. Quadrat & D. Robertz.
(<http://wwwb.math.rwth-aachen.de/OreModules>)

3 Ring of differential polynomials

Classical construction [Ritt 51, Kolchin 73]

$$\mathbb{F} = \mathbb{Q} \text{ or } \mathbb{Q}(x, y) \quad \begin{cases} s(\phi_{xx} + \phi_{yy}) + s_x \phi_x + s_y \phi_y + \phi = 0 \\ s(\psi_{xx} + \psi_{yy}) + s_x \psi_x + s_y \psi_y + \psi = 0 \\ \psi_x \phi_x + \psi_y \phi_y = 0 \end{cases} \quad \mathbb{F} \text{ a field}$$

$$\delta_1 = \frac{\partial}{\partial x}, \delta_2 = \frac{\partial}{\partial y} \quad \Delta = \{\delta_1, \dots, \delta_m\} \text{ derivations on } \mathbb{F}$$

$$\delta_i(a+b) = \delta_i(a) + \delta_i(b) \quad \delta_i(ab) = a\delta_i(b) + \delta_i(a)b$$

$$\mathcal{Y} = \{s, \phi, \psi\} \quad \mathcal{Y} = \{y_1, \dots, y_n\}$$

$$\mathbb{F}[s, s_x, s_y, s_{xx} \dots] = \mathbb{F}[[s, \phi, \psi]] \quad \mathbb{F}[[\mathcal{Y}]] = \mathbb{F}[y_\alpha \mid \alpha \in \mathbb{N}^m, y \in \mathcal{Y}]$$

$$s_{xy} \rightsquigarrow s_{x^2y} \rightsquigarrow s_{(2,1)}$$

$$\frac{\partial}{\partial x}(s_{xy}) = s_{xxy} \quad \delta_i(y_\alpha) = y_{\alpha + \epsilon_i}$$

$$\rightsquigarrow \delta_1(s_{(1,1)}) = s_{(2,1)} \quad \epsilon_i = (0, \dots, \underset{i^{\text{th}}}{1}, \dots, 0)$$

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} = \frac{\partial}{\partial y} \frac{\partial}{\partial x} \quad \delta_i \delta_j = \delta_j \delta_i$$

Declare it in Maple

> with(difalg);

[Rosenfeld-Groebner, characters, delta_polynomial, denote, derivatives, differential_ring, differentiate, equations, essential_components, field_extension, greater, inequations, initial, initial_conditions, is_prime, leader, power_series_solution, preparation_polynomial, print_ranking, rank, reduce, reduced_form, rewrite_rules, separant]

> R := differential_ring(ranking=[[s,phi,psi]], derivations=[x,y], notation=jet);

R := differential_polynomial_ring

> differentiate(s[x,y], x, R);

s_{x,x,y}

> denote(s[x,y], vjet, R);

s_{1,1}

> denote(s[x,y], diff, R);

$\frac{\partial^2}{\partial x \partial y} s(x, y)$

Rankings < on $\mathbb{F}[[\mathcal{Y}]]$

A total order on $\mathcal{DY} = \{y_\alpha \mid \alpha \in \mathbb{N}^m, y \in \mathcal{Y}\}$ s.t.

- $y_\alpha < y_{\alpha+\gamma}$
- $y_\alpha < z_\beta \Rightarrow y_{\alpha+\gamma} < z_{\beta+\gamma}$

$$|\alpha| = \alpha_1 + \dots + \alpha_m$$

Orderly ranking: $|\alpha| < |\beta| \Rightarrow y_\alpha < z_\beta, \quad \forall y, z$

Semi-orderly ranking: $|\alpha| < |\beta| \Rightarrow y_\alpha < y_\beta$

Elimination ranking: $y_\alpha < z_\beta, \quad \forall \alpha, \beta$

then $\delta_i \delta_j(p) = \delta_j \delta_i(p) + \sum_{l=1}^m c_{ijl} \delta_l(p) \quad \forall p \in \mathbb{K}[y_\alpha \mid \alpha \in \mathbb{N}^m] = \mathbb{K}[\mathcal{Y}]$

With *diffalg*

$$\partial_{y_0} D_x - D_x \partial_{y_0} = f_{y_0} \partial_{y_1}, \quad \partial_{y_1} D_x - D_x \partial_{y_1} = \partial_{y_0} + f_{y_1} \partial_{y_1}$$

> $R := \text{differential_ring}(\text{ranking}=[[x_i, \text{eta}], f], \text{derivations}=[x, y_0, y_1], \text{commutations}=[[y_0, x]=[0, 0, f[y_0]], [y_1, x]=[0, 1, f[y_1]]]):$

> $\text{differentiate}(\text{differentiate}(\text{eta}, y_0, R), x, R);$

$$\eta_{x, y_0}$$

> $\text{differentiate}(\text{eta}, x, y_0, R);$

$$\eta_{x, y_0}$$

> $\text{differentiate}(\text{eta}, y_0, x, R);$

$$\eta_{x, y_0} + f_{y_0} \eta_{y_1}$$

Leader, initial, separant

- $\mathcal{Y} = \{y_1, \dots, y_n\}, \quad \mathcal{D} = \{\delta_1, \dots, \delta_m\}, \quad \mathbb{F}[\mathcal{Y}], \quad \prec$

- Nota^o: $\delta^\alpha = \delta_1^{\alpha_1} \dots \delta_m^{\alpha_m}$

$$\delta^\alpha \delta^\beta \neq \delta^{\alpha+\beta} \text{ but } \delta^\alpha \delta^\beta = \delta^{\alpha+\beta} + \sum_{|\gamma| < |\alpha+\beta|} a_\gamma \delta^\gamma, \quad a_\gamma \in \mathbb{K}[\mathcal{Y}]$$

- $p \in \mathbb{K}[\mathcal{Y}] \setminus \mathbb{K}$

$$p = i_p y_\alpha^d + c_1 y_\alpha^{d-1} \dots + c_d \quad c_i \prec y_\alpha$$

y_α leader
lead(p)

i_p initial
init(p)

s_p separant
sep(p)

- $s_p = \frac{\partial p}{\partial y_\alpha} = d i_p y_\alpha^{d-1} + (d-1) c_1 y_\alpha^{d-2} + \dots + c_{d-1}$

Prop:

$$\text{lead}(p) = y_\alpha \quad \Rightarrow \quad \delta^\beta(p) = \text{sep}(p) y_{\alpha+\beta} + \underbrace{\dots}_{\prec y_{\alpha+\beta}}$$

4 Differential ideals

Differential Ideals

$$\mathcal{Y} = \{y_1, \dots, y_n\},$$

$$\mathcal{D} = \{\delta_1, \dots, \delta_m\},$$

$$\mathbb{F}[[\mathcal{Y}]]$$

$$\{p_1, \dots, p_k\} \subset \mathbb{F}[[\mathcal{Y}]]$$

I , a **differential ideal** of $\mathbb{F}[[\mathcal{Y}]]$:

- $a \in I \Rightarrow \delta a \in I, \forall \delta \in \Delta$
- I is an ideal

$$[p_1, \dots, p_k]$$

$$[p] = (\delta^\alpha p \mid \alpha \in \mathbb{N}^m)$$

J , **radical** differential ideal of $\mathbb{F}[[\mathcal{Y}]]$:

- $a^k \in J \Rightarrow a \in J$
- J is a differential ideal

$$[[p_1, \dots, p_k]]$$

$$[[p]] = \{q \mid q^r \in [p]\}$$

P , **prime** differential ideal of $\mathbb{F}[[\mathcal{Y}]]$:

- $ab \in P \Rightarrow a \in P$ or $b \in P$
- P is a differential ideal

P radical and irreducible.

Differential Nullstellensatz

$$p_1, \dots, p_r \in \mathbb{F}[[\mathcal{Y}]]$$

- NOTE:
- if $q \in [[p_1, \dots, p_r]]$ then $q^e = \sum c_{i\alpha} \delta^\alpha p_i$ and q vanishes on all the common zeros of p_1, \dots, p_r .
 - if $1 \in [[p_1, \dots, p_r]]$ then $1 = \sum c_{i\alpha} \delta^\alpha p_i$ so that p_1, \dots, p_r have no common zero.

- THEO:
- p_1, \dots, p_k admit a common zero iff $1 \notin [[p_1, \dots, p_k]]$
 - $q \in \mathbb{F}[[\mathcal{Y}]]$ vanishes on all the common zeros of p_1, \dots, p_k iff $q \in [[p_1, \dots, p_k]]$.

Ritt-Raudenbush Theorem

THEO: A radical differential ideal J of $\mathbb{F}[[\mathcal{Y}]]$ is

- finitely generated: $J = [[p_1, \dots, p_r]]$
for some $p_1, \dots, p_r \in \mathbb{F}[[\mathcal{Y}]]$.

Differential ideals need not be finitely generated.

- the intersection of finitely many prime differential ideals.

The irredundant decomposition $J = \bigcap_{i=1}^r P_i$ is unique.

$$\text{EX: } [[y_x^2 + x y_x - y]] = [[y_x^2 + x y_x - y, y_{xx}]] \cap [[4y + x^2]].$$

Saturation ideals

- $h \in \mathbb{F}[\mathcal{Y}]$ $I:h^\infty = \{q \mid \exists k \text{ s.t. } h^k q \in I\}$
- $H = \{h_1, \dots, h_s\}$ $I:H^\infty = I:h^\infty$ where $h = h_1 \dots h_s$

NOTE: The zeros of $\llbracket p \rrbracket : h^\infty$ are the zeros of p that don't vanish on h , except for some *adherent* piece.

EX: $\llbracket y_x^2 + x y_x - y \rrbracket = \llbracket y_x^2 + x y_x - y \rrbracket : (2y_x + x)^\infty \cap \llbracket 4y + x^2 \rrbracket$.

5 Representation of radical differential ideals

Purpose

Given p_1, \dots, p_r in $\mathbb{F}[\mathcal{Y}]$ we want to compute a representation of $\llbracket p_1, \dots, p_r \rrbracket$ that allows to

- test membership to $\llbracket p_1, \dots, p_r \rrbracket$
- *measure* the zero set (completion)
- compute $\llbracket p_1, \dots, p_r \rrbracket \cap \mathbb{F}[\mathcal{Z}]$ (elimination)

In a factorisation free way.

There is no strict analogue of Gröbner bases.

Differential ideals do not admit in general

- a finite set
- that is generating for the differential ideal
- and *reduces* to zero the elements of the differential ideal

Characterisable differential ideals are defined by

- a finite set
- that *reduces* to zero the elements of the differential ideal
- generate the ideal *outside of some hypersurface*

Prime differential ideals are characterizable

Radical diff. ideal are \cap° of characterisable diff. ideals

Characteristic Decomposition

ALGO: (Rosenfeld-Groebner in *diffalg*)

In: $\{p_1, \dots, p_k\} \subset \mathbb{F}[\mathcal{Y}]$, \prec

Out: C_1, \dots, C_r s.t.

$$\llbracket p_1, \dots, p_k \rrbracket = [C_1]:S_1^\infty \cap \dots \cap [C_r]:S_r^\infty$$

Membership: $p \in [C_i]:S_i \Leftrightarrow p \xrightarrow{C_i} 0$

Completion: If \prec orderly then

$$[C_i]:S_i^\infty \cap \mathbb{F}[\mathcal{D}_\kappa \mathcal{Y}] = (\mathcal{D}_\kappa C_i):S_i^\infty$$

Elimination: If \prec eliminates $\mathcal{Y} \setminus \mathcal{Z}$ and $C'_i = C_i \cap \mathbb{F}[\mathcal{Z}]$ then

$$[C_i]:S_i^\infty \cap \mathbb{F}[\mathcal{Z}] = [C'_i]:S_i'^\infty$$

Output characteristic sets C

or pseudo-Gröbner bases

- C is coherent:

$$a, b \in C, \quad \text{lead}(a) = y_\alpha, \text{lead}(b) = y_\beta$$

$$\gamma = \alpha + \bar{\alpha} = \beta + \bar{\beta} \quad \Rightarrow \quad s_a \delta^{\bar{\beta}} b - s_b \delta^{\bar{\alpha}} a \in (\mathcal{D}C_{< y_\gamma}):S^\infty$$

Recall:

$$\begin{aligned} \delta^{\bar{\beta}} b &= s_b y_\gamma + \dots \\ \delta^{\bar{\alpha}} a &= s_a y_\gamma + \dots \end{aligned}$$

A generalisation of

$$\left. \begin{aligned} u_x &= f(x, y) \\ u_y &= g(x, y) \end{aligned} \right\} \Rightarrow f_y - g_x = 0$$

- C is a differential triangular set:

$$a, b \in C, \quad \text{lead}(a) = y_\alpha, \text{lead}(b) = y_\beta \quad \Rightarrow \quad \beta \neq \alpha + \gamma$$

- $[C]:S^\infty$ is a radical differential ideal.

- $p \xrightarrow{c} q$ means:

– q is free of proper derivatives of $\text{lead}(c)$

$$s_c^e p = q_1 \pmod{(\delta^\beta c, \dots, \delta^\gamma c)}$$

– the degree of q in $\text{lead}(p)$ is lower than that of p

$$i_p^f q_1 = q \pmod{(p)}$$

Envelope

> R:= differential_ring(ranking=[c,y],derivations=[x]):
 > C := Rosenfeld_Groebner([y - c x - c², c_x], R);

$$C := [\textit{characterisable}]$$

> equations(G);

$$[[-y_x + c, y_x^2 + x y_x - y]]$$

> p := equations(G[1])[-1];

$$p := y_x^2 + x y_x - y$$

> C:= Rosenfeld_Groebner([p],R);

$$C := [\textit{characterisable}, \textit{characterisable}]$$

> equations(C), inequations(C)

$$[[y_x^2 + x y_x - y], [4y + x^2]], [[2y_x + x], []]$$

That is: $\llbracket p \rrbracket = [p] : (2y_x + x)^\infty \cap [4y + x^2]$

Kepler \Rightarrow Newton

> R:= differential_ring(ranking=[[x,y],[a,r],[p,e,s]],derivations=[t], parameters=[p,e,s]):
 > K := {a² - x_{t,t}² - y_{t,t}², r² - x² - y²}, r - p + e x, x y_t - y x_t - s}:
 > C := Rosenfeld_Groebner(K, {p, e, s}, R);

$$C := [\textit{characterisable}]$$

> N := differentiate(r² a,t,R):

$$N := r (a_t r + 2 r_t a)$$

> reduce(N, C[1]);

$$0$$

With $\llbracket K \rrbracket : (pes)^\infty = [e \mathbf{x} + r - p, es \mathbf{y} - prr_t, r^2 p^2 \mathbf{r}_t^2 - s^2 (e^2 r^2 + r^2 - 2rp + p^2), r^4 p^2 \mathbf{a}^2 - s^4] : (epsarr_t)^\infty$

Orthogonal Waves

$$\begin{aligned} s (\phi_{xx} + \phi_{yy}) + s_x \phi_x + s_y \phi_y + \phi &= 0 \\ s (\psi_{xx} + \psi_{yy}) + s_x \psi_x + s_y \psi_y + \psi &= 0 \\ \psi_x \phi_x + \psi_y \phi_y &= 0 \\ s_x, s_y, \phi_x, \phi_y, \psi_x, \psi_y &\neq 0 \end{aligned}$$

> with(diffalg)

> R:=differential_ring(ranking=[[s,phi,psi]],derivations=[x,y]):

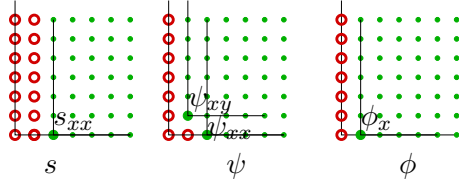
> S:=[s*(phi[x,x]+phi[y,y])+s[x]*phi[x]+...]

> H :={s[x],s[y],phi[x],...}

> C :=Rosenfeld_Groebner(S,H,R);

$$C := [\textit{characterisable}]$$

$$C = \{s_{xx} = \dots, \psi_{xx} = \dots, \psi_{xy} = \dots, \phi_x = \dots\}$$



$$s = s_{00} + s_{10}x + s_{01}y + s_{20}\frac{x^2}{2} + s_{11}xy + s_{02}\frac{y^2}{2} + s_{30}\frac{x^3}{6} \dots$$

$$\psi = \psi_{00} + \psi_{10}x + \psi_{01}y + \psi_{20}\frac{y^2}{2} + \psi_{11}xy + \psi_{02}\frac{x^2}{2} \dots$$

$$\phi = \phi_{00} + \phi_{10}x + \phi_{01}y + \phi_{20}\frac{x^2}{2} + \phi_{11}xy + \phi_{02}\frac{y^2}{2} \dots$$

$$s = f_1(y) + f_2(y)x + s_{20}\frac{x^2}{2} + s_{30}\frac{x^3}{6} \dots$$

$$\psi = f_3(y) + c_1x + \psi_{20}\frac{x^2}{2} + \psi_{11}xy + \psi_{20}\frac{x^2}{2} \dots$$

$$\phi = f_4(y) + \phi_{10}x + \phi_{01}y + \phi_{20}\frac{x^2}{2} + \phi_{11}xy + \dots$$

Challenge Computational Problem

$$\begin{aligned} s(\phi_{xx} + \phi_{yy}) + s_x\phi_x + s_y\phi_y + \phi &= 0 \\ s(\psi_{xx} + \psi_{yy}) + s_x\psi_x + s_y\psi_y + \psi &= 0 \\ \psi_x\phi_x + \psi_y\phi_y &= 0 \\ s_x, s_y, \phi_x, \phi_y, \psi_x, \psi_y &\neq 0 \end{aligned}$$

> `R:=differential_ring(ranking=[[phi,psi],s],derivations=[x,y]):`

6 Symmetric Systems

$$S \begin{cases} s(\phi_{xx} + \phi_{yy}) + s_x\phi_x + s_y\phi_y + \phi = 0 \\ s(\psi_{xx} + \psi_{yy}) + s_x\psi_x + s_y\psi_y + \psi = 0 \\ \psi_x\phi_x + \psi_y\phi_y = 0 \end{cases}$$

Idea : factor out the symmetry [Mansfield 01], [Lisle & Reid 06] [15pt]

Tool : moving frame construction [Fels & Olver 99]

[Desolv, Vessiot]

$$S \begin{cases} s(\phi_{xx} + \phi_{yy}) + s_x\phi_x + s_y\phi_y + \phi = 0 \\ s(\psi_{xx} + \psi_{yy}) + s_x\psi_x + s_y\psi_y + \psi = 0 \\ \psi_x\phi_x + \psi_y\phi_y = 0 \end{cases}$$

$$(x, y, s, \phi, \psi) \xrightarrow{g_{\alpha, \beta, \rho, \tau, \mu, \nu, a, b}} (X, Y, S, \Phi, \Psi) \quad \beta^2 + \alpha^2 = 1$$

$$X = \frac{\alpha}{\rho}x - \frac{\beta}{\rho}y + \frac{a}{\rho} \quad Y = \frac{\alpha}{\rho}x + \frac{\beta}{\rho}y + \frac{b}{\rho}$$

$$S = \frac{s}{\rho^2\tau} \quad \Phi = \frac{\phi}{\mu} \quad \Psi = \frac{\psi}{\nu}$$

$$\begin{aligned} S_X &= \frac{\beta}{\rho}s_x - \frac{\alpha}{\rho}s_y & S_Y &= \frac{\alpha}{\rho}s_y + \frac{\beta}{\rho}s_x \\ \Phi_X &= \frac{\rho\alpha}{\mu}\phi_x - \frac{\rho\beta}{\mu}\phi_y & \Phi_Y &= \frac{\rho\alpha}{\mu}\phi_y + \frac{\rho\beta}{\mu}\phi_x \\ & & & \dots \end{aligned}$$

All differential invariants can be written in terms of the **fundamental invariants** [Vesiot, Groebner]

$$\begin{aligned}
s_1^2 &:= \frac{s_x^2 + s_y^2}{4s}, \\
s_2 &:= \frac{s_{xy}(s_y^2 - s_x^2) + s_x s_y (s_{xx} - s_{yy})}{8s s_1^3}, & s_3 &:= \frac{s_x^2 s_{yy} + s_y^2 s_{xx} - 2s_x s_y s_{xy}}{8s s_1^3} - s_1, \\
\psi_1 &:= \frac{s_y \psi_x - s_x \psi_y}{2s_1 \psi}, & \psi_2 &:= \frac{s_x \psi_x + s_y \psi_y}{2s_1 \psi}, \\
\phi_1 &:= \frac{s_y \phi_x - s_x \phi_y}{2s_1 \phi}, & \phi_2 &:= \frac{s_x \phi_x + s_y \phi_y}{2s_1 \phi}.
\end{aligned}$$

(s_1, s_2, s_3 depend only on s and derivatives)

and their derivatives with respect to the **invariant derivations**:

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \frac{\pm \sqrt{s(s_y^2 + s_x^2)}}{s_x^2 - s_y^2} \begin{pmatrix} -s_y & s_x \\ s_x & -s_y \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\mathcal{P} = (x, y, s-1, \phi-1, \psi-1, s_x)$$

Invariants: $s_1 = \iota(s_y)/2$, $s_2 = \frac{\iota(s_{xy})}{\iota(s_y)}$, $s_3 = \frac{\iota(s_{xx})}{\iota(s_y)} - \iota(s_y)/2$, $\psi_1 = \iota(\psi_x)$, $\psi_2 = \iota(\psi_y)$, $\phi_1 = \iota(\phi_x)$ and $\phi_2 = \iota(\phi_y)$.

We can write \mathcal{S} in terms of $\mathcal{Y} = \{s_1, s_2, s_3, \phi_1, \phi_2, \psi_1, \psi_2\}$ and $\Delta = \{\delta_1, \delta_2\}$: expressions of $s_1, s_2, s_3, \phi_1, \phi_2, \psi_1, \psi_2$ not needed

$$\mathcal{S} \begin{cases} \delta_1(\phi_1) + \delta_2(\phi_2) + \phi_1^2 + \phi_2^2 - s_2 \phi_1 + (2s_1 + s_3)\phi_2 + 1 = 0, \\ \delta_1(\psi_1) + \delta_2(\psi_2) + \psi_1^2 + \psi_2^2 - s_2 \psi_1 + (2s_1 + s_3)\psi_2 + 1 = 0, \\ \phi_1 \psi_1 + \phi_2 \psi_2 = 0. \end{cases}$$

but now

$$\delta_1 \delta_2 - \delta_2 \delta_1 = s_3 \delta_1 + s_2 \delta_2$$

and the fundamental invariants are not differentially independent.
The *syzygies* are

$$\mathcal{Z} \begin{cases} \delta_1(s_1) &= s_1 s_2 \\ \delta_1(s_2) - \delta_2(s_3) &= s_3^2 + s_2^2 + s_1(s_2 + s_3) \\ \delta_1(\phi_2) - \delta_2(\phi_1) &= \phi_1 s_3 + \phi_2 s_2, \\ \delta_1(\psi_2) - \delta_2(\psi_1) &= \psi_1 s_3 + \psi_2 s_2. \end{cases}$$

Project status

Rational Invariants of a Group action. Construction and Rewriting, with I. Kogan, JSC, in press.

Smooth and Algebraic Invariants. Local and Global construction, with I. Kogan, submitted.

Algebraic Algorithms for Computing Differential Invariants, with I. Kogan, in preparation.

Algebra for differential invariants, in preparation.

Differential polynomial algebra with non commuting derivations. Journal of Pure and Applied Algebra, 200:1-2, p 163-190 (2005).

Software: *aida*, *diffalg*
inria.fr/cafe/Evelyne.Hubert/aida
inria.fr/cafe/Evelyne.Hubert/diffalg

Thanks.