Computing Tropical Varieties in Gfan

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IMA Workshop: Software for Algebraic Geometry,
October 2006

Joint work with Bogart, Fukuda, Speyer, Sturmfels, Thomas

Gfan

- Gfan is a C++ program for computing **Gröbner fans** and **tropical varieties**.
- It is a command line tool consisting of many small programs.
- Uses gmp and cddlib.
- Works on Linux (and MacOS X).
- It is installed on the IMA Linux computers.

A simple Gröbner fan example

Command `gfan`

Input `{a-b-ab, a^2+ab}`

Output
```
{{b^3-2*b^2,
a-b+b^2}
,
{b^2-b+a,
a*b+b-a,
a^2-b+a}
,
{b-a-a^2,
a^3+2*a^2}}
```

All *marked reduced Gröbner bases* of the input polynomial ideal $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ are computed.

An algorithmic "definition" of Gröbner fans

Buchberger's algorithm:

- **Input 1** A list of generators for an ideal $I \subseteq k[x_1, \ldots, x_n]$
- **Input 2** A term order $\prec$ (represented by a vector in $\mathbb{R}^n_{\geq 0}$)
- **Output** A reduced Gröbner basis for I w.r.t. $\prec$

Observe:

- Varying Input 2 we get different Gröbner bases.
- Two vectors are *equivalent* if they produce the same Gröbner basis.
- The equivalence classes are the maximal cones in the *Gröbner fan of I*.

Initial forms and initial ideals

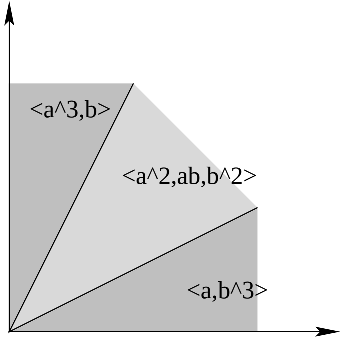
Consider the polynomial ring $k[x_1, \ldots, x_n]$. Let $\omega \in \mathbb{R}^n$.

- The *weight* of a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $a \in \mathbb{N}^n$ is $\langle \omega, a \rangle$.
- The *initial form* $in_\omega(f)$ of a polynomial $f \in k[x_1, \ldots, x_n]$ is the sum of terms with maximal weights.
  Example:
  $$in_{(1,2)}(x_1^4 + 2x_2^2 + x_1x_2 + 1) = x_1^4 + 2x_2^2$$
- The *initial ideal* of an ideal $I \subseteq k[x_1, \ldots, x_n]$ is defined as
  $$in_\omega(I) = \langle in_\omega(f) \rangle_{f \in I}$$

The following things are in bijection

- The marked reduced Gröbner bases of *I*
- The full-dimensional Gröbner cones
- The monomial initial ideals of *I*

The previous example



From a marked reduced Gröbner basis it is easy to read off the initial ideal and the defining inequalities of the Gröbner cone.

How to read off the cone

Consider the marked reduced Gröbner basis:

$$\{\underline{b^3} - 2b^2, \\ \underline{a} - b + b^2\}$$

For which termorders $\prec_\omega$ is this a Gröbner basis?

- Observation: Buchberger's S-pair criterion only depends on the marked terms - not on the term order.
- For $\omega \in \mathbb{R}^2$ to pick out the marked terms, it must satisfy:

$$\begin{aligned} 3\omega_2 &> 2\omega_2 \\ \omega_1 &> \omega_2 \\ \omega_1 &> 2\omega_2 \end{aligned}$$

These inequalities defines the (open) Gröbner cone.
The inequality $\omega_1 > \omega_2$ is redundant.

The Gröbner fan of an ideal

Definition (Mora, Robbiano)

- Let $I \subseteq k[x_1, \ldots, x_n]$ be a homogeneous ideal.
- Define an equivalence relation $\sim$ on $\mathbb{R}^n$.

$$u \sim v \Leftrightarrow \text{in}_u(I) = \text{in}_v(I)$$

- The closure of each equivalence class is called a *Gröbner cone*.
- The set of all these cones is a polyhedral complex.
- We call this the *Gröbner fan* of I.

Properties and algorithms of Gröbner fans

Algorithm (Collart, Kalkbrener, Mall)

Input *A reduced Gröbner basis and a facet normal*

Output *A Gröbner basis on the other side of the facet*

Theorem (Bayer, Morrison, Sturmfels)

The Gröbner fan of a homogeneous ideal I is the normal fan of a polytope - the state polytope of I.

Algorithm (Avis, Fukuda)

“Reverse search” is a memory-less method for enumerating the vertices of a polytope.

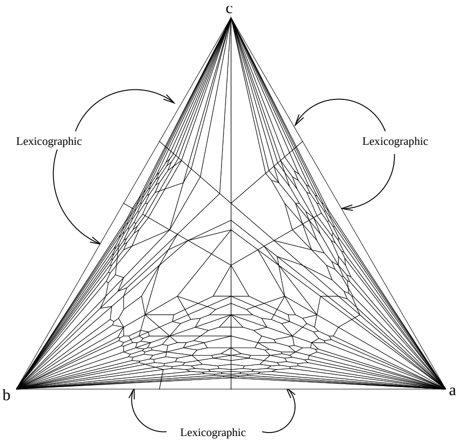
Theorem (Fukuda, Jensen, Thomas)

“Reverse search” works for Gröbner fans - even if the ideal is not homogeneous.

A bigger Gröbner fan example

Example

$I = \langle a^5 + b^3 + c^2 - 1, a^2 + b^2 + c - 1, a^6 + b^5 + c^3 - 1 \rangle \subseteq \mathbb{Q}[a, b, c]$has 360 reduced Gröbner bases and 360 full-dimensional cones in its fan. (Not homogeneous!) Intersection of fan and 2-simplex:



The tropical semi-ring

In the tropical semi-ring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ the operations are

$\oplus$ maximum

$\odot$ addition

Two examples:

$$5 \odot (3 \oplus 2) = 8$$

$$5 \odot 3 \oplus 5 \odot 2 = 8$$

This explains the word "tropical".

- Tropical polynomial functions are piecewise linear.
- The tropical semi-ring gives rise to tropical varieties.
- We will define them using initial ideals.

Tropical varieties

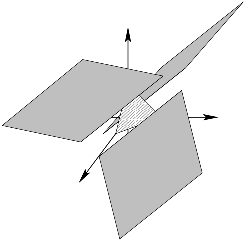
Definition

If $I \subseteq k[x_1, \ldots, x_n]$ is an ideal then we define

$$T(I) := \{\omega \in \mathbb{R}^n : \text{in}_\omega(I) \text{ is monomial-free}\}.$$

Example

The tropical variety of a principal ideal is called a *tropical hypersurface*. $T(\langle x_1 + x_2 + x_3 \rangle) \subseteq \mathbb{R}^3$ is the union of three 2-dimensional cones:



Lemma

Any tropical variety is an intersection of hypersurfaces:

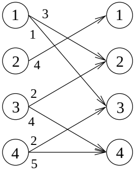
$$T(I) = \bigcap_{f \in I} T(\langle f \rangle)$$

Why is tropical mathematics interesting?

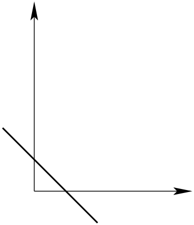
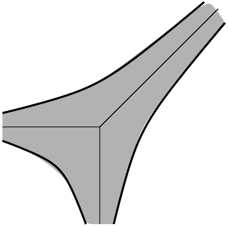
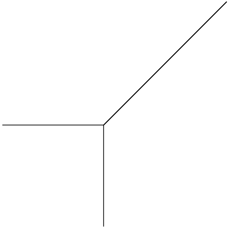
1. Combinatorial optimization.

   Example: Maximal matching

   $$= \text{tropdet} \begin{pmatrix} -\infty & 3 & 1 & -\infty \\ 4 & -\infty & -\infty & -\infty \\ -\infty & 2 & -\infty & 4 \\ -\infty & -\infty & 2 & 5 \end{pmatrix}$$

2. It shows the logarithmic limit behavior of a variety in $\mathbb{C}^n$:

   $V(x+y-1) \subseteq \mathbb{C}^2$ $\log(\text{abs}(V(x+y-1)))$ $T(x+y-1)$

3. Similarly for coordinatewise valuation $(\mathbb{C}\{\{t\}\}^*)^n \to \mathbb{Q}^n$.

4. Computing $T(I)$ is the first step of the polyhedral homotopy method for solving the polynomial system of I.

A polyhedral structure on the tropical variety

- From the definition:

$$T(I) = \{\omega \in \mathbb{R}^n | \text{in}_\omega(I) \text{ is monomial-free}\}$$

  it is clear that $T(I)$ is a union of Gröbner cones.
- We may think of the tropical variety as a polyhedral complex inheriting its structure from the Gröbner fan.
- The tropical variety is a subcomplex of the Gröbner fan.

A naive algorithm for computing the tropical variety

Algorithm

Input *Generators for homogeneous* $I \subseteq k[x_1, \ldots, x_n]$.

Output *The set of Gröbner cones contained in* $T(I)$.

- *Compute the maximal cones of the Gröbner fan*
- *Compute all the faces*
- *For each face C:*
  - *Compute* $\omega \in \mathrm{relint}(C)$
  - *Compute* $J := \mathrm{in}_\omega(I)$
  - *Check if* $\langle 1 \rangle = (J : x_1 \cdots x_n^\infty)$
  - *If not, output C*

Gröbner fan VS tropical variety

Let I be the ideal generated by the 3x3 minors of a 4x4 matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}$$

in the polynomial ring of 16 variables.

- The Gröbner fan consists of 163032 full-dimensional cones.
- T(I) is a 12-dimensional sub-complex with 936 maximal cones.
- We don't want to compute the Gröbner fan.

Four types of tropical (pre-)varieties handled by Gfan

Gfan can compute the following types of (pre-)varieties:

1. Tropical hypersurfaces
2. Intersections of hypersurfaces (not necessarily a variety)
3. Tropical curves.
4. Tropical varieties of prime ideals.

- A satisfactory output will contain a list of at least all maximal cones in the fan.

In the following we will assume that I is homogeneous in some positive grading.

Case 1: Tropical hypersurfaces

$T(f)$ is a subfan of the normal fan of the Newton polytope of f.

Algorithm

Input *A polynomial* $f \in k[x_1, \ldots, x_n]$

Output *A representation of* $T(f)$ *as a collection of cones*

- *Compute vertices of the Newton polytope of f*
- *Compute the edges*
- *Compute the normal cones of the edges*

Keywords: convex hull, linear programming

Case 2: Intersections of tropical hypersurfaces

Algorithm

Input *Polynomials* $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$

Output *A representation of* $\bigcap_i T(f_i)$ *as a collection of cones*

- *Compute representations of* $T(f_1), \ldots, T(f_r)$
- *Compute common refinement* $\bigwedge_i T(f_i)$ *of these polyhedral complexes by applying*

$$A \wedge B := \{a \cap b : a \in A, b \in B\}$$

in the most stupid way.

Keywords: Minkowski sums, mixed faces

Case 3: What is a tropical curve?

Definition

Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal.
The Gröbner cone of I equal to the equivalence class $C_0(I) = \{\omega \in \mathbb{R}^n : \mathrm{in}_\omega(I) = I\}$ is called the *homogeneity space*.

Lemma

- *The intersection of all Gröbner cones is $C_0(I)$.*
- *The Gröbner cones are invariant under translation by vectors in $C_0(I)$.*
- *The gradings for which I is homogeneous are exactly given by the vectors in $C_0(I)$.*

Definition

An ideal I is said to define a *tropical curve* $T(I)$ if

$$\text{Krull}\dim(k[x_1, \dots, x_n]/I) = 1 + \dim_{\mathbb{R}}(C_0(I)).$$

Case 3: Tropical curves - tropical bases

Definition

Let C be Gröbner cone not contained in $T(I)$. A polynomial $f \in I$ is a *witness* for C if $T(f) \cap \text{relint}(C) = \emptyset$.

Equivalently:
For any $\omega \in \text{rel int}(C)$ the initial form $\text{in}_\omega(f)$ is a monomial.

Definition

A finite generating set F of I is a *tropical basis* if

$$T(I) = \bigcap_{f \in F} T(f).$$

Theorem

Every Gröbner cone not contained in $T(I)$ has a witness.

Corollary

Every ideal $I \subseteq k[x_1, \ldots, x_n]$ has a tropical basis.

Case 3: Tropical curves

Algorithm

Input *Generators* $G = \{f_1, \ldots, f_s\}$ *for* $I \subseteq k[x_1, \ldots, x_n]$

Output *A tropical basis* G *for* I

- *Compute* $\bigwedge_{f \in G} T(f)$.
- *For every cone* $C \in \bigwedge_{f \in G} T(f)$:
  - *Let* $\omega \in C$ *be a generic relative interior point.*
  - *If* $\mathrm{in}_\omega(I)$ *contains a monomial then add a witness to* G *and restart the algorithm.*

This only works for curves.
To get a representation of $T(I)$ we compute the refinement $\bigwedge_{f \in G} T(f)$.

Case 4: Tropical varieties of prime ideals

A polyhedral complex is *pure* of dimension d if all maximal polyhedra have dimension d.
Let $d = \text{Krull dim}(k[x_1, \ldots, x_n]/I)$.

Theorem (Bieri-Groves)

The tropical variety $T(I)$ is pure of dimension d.

Definition

A *ridge path* is a list of maximal cones

$$C_0, \ldots, C_m$$

with any two consecutive cones sharing a facet.

Theorem (Speyer)

Any two d-dimensional cones C_0 and C_m of $T(I)$ are connected by a ridge path in $T(I)$.

Case 4: Tropical varieties of prime ideals

We wish to traverse the d-dimensional faces of $T(I)$.

- Consider a d-dimensional cone C.
- Let F be a facet with $\omega \in \text{relint}(F)$.
- Going to $\text{in}_\omega(I)$ see what happens locally:

$$\text{in}_{\omega+\varepsilon v}(I) = \text{in}_v(\text{in}_\omega(I))$$

- We wish to compute $T(\text{in}_\omega(I))$.
  - The dimension of $\text{in}_\omega(I)$ is d.
  - The homogeneity space has dimension $d - 1$.
  - Thus $\text{in}_\omega(I)$ defines a tropical curve!
- We may lift the result to $T(I)$ by applying Gröbner basis techniques.
- We now have a neighbour cone of C.

Case 4: Heuristics for finding a starting cone

How do we find just one d-dimensional cone in the tropical variety of a d-dimensional ideal I?

- If the d equals the $\dim(C_0(I))$ just use $C_0(I)$.
- Otherwise:
  - Pick random maximal Gröbner cones and extreme rays ω of the cones until $\mathrm{in}_\omega(I)$ is monomial-free.
  - Iterate recursively to compute a d-dimensional cone of $T(\mathrm{in}_\omega(I))$.
  - Combine the result to a d-dimensional cone of $T(I)$ using

$$\mathrm{in}_{\omega+\varepsilon v}(I) = \mathrm{in}_v(\mathrm{in}_\omega(I))$$

    and Gröbner basis techniques.

List of Gfan programs

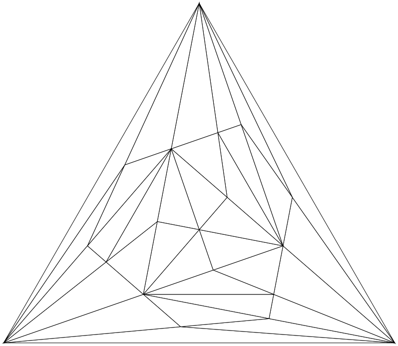
```
gfan
gfan_buchberger
gfan_doesidealcontain
gfan_facets
gfan_fvector
gfan_groebnercone
gfan_homogeneityspace
gfan_homogenize
gfan_initialforms
gfan_interactive
gfan_ismarkedgroebnerbasis
gfan_leadingterms
gfan_markpolynomialset
gfan_polynomialsetunion
gfan_render
gfan_renderstaircase
gfan_stats
gfan_substitute
gfan_tolatex
gfan_tropicalbasis
gfan_tropicalintersection
gfan_tropicalstartingcone
gfan_tropicaltraverse
gfan_weightvector
```

Combining Gfan programs on the UNIX shell

Commands `gfan|gfan_render > output.fig`

Input `{aab-c,bbc-a,cca-b}`

Output The following picture stored as an X-fig file



Do you see the symmetry?

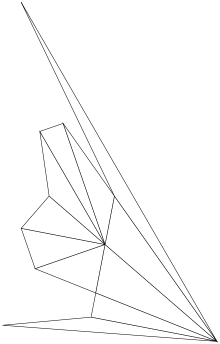
Exploiting symmetry

We wish to exploit the $A_3 \subseteq S_3$ symmetry:

Commands `gfan --symmetry | gfan_render > output.fig`

Input
```
{aab-c,bbc-a,cca-b}
{(1,2,0)}
```

Output The following picture stored as an X-fig file



We enumerate one Gröbner basis from each symmetry class.

Representing Gröbner cones

Practical ways for Gfan to represent Gröbner cones during computations.

- While n-dimensional Gröbner cones are naturally represented by reduced Gröbner bases ...
- ...a lower dimensional cone is represented by a pair of reduced Gröbner bases
  - one for the ideal and one for the initial ideal.

Finding a single d-dimensional cone in the tropical variety of a prime ideal

Command `gfan_tropicalstartingcone`

Input
```
{ bf-ah-ce, bg-ai-de, cg-aj-df,
ci-bj-dh, fi-ej-gh }
```

Output
```
{f*i-e*j,
d*h-c*i,
d*f+a*j,
d*e+a*i,
c*e+a*h}
{f*i-g*h-e*j,
d*h-c*i+b*j,
d*f-c*g+a*j,
d*e-b*g+a*i,
c*e-b*f+a*h}
```

Traversing the tropical variety of a prime ideal

Command `gfan_tropicaltraverse`

Input

```
{f*i-e*j, d*h-c*i, d*f+a*j,
d*e+a*i, c*e+a*h}
{f*i-g*h-e*j, d*h-c*i+b*j,
d*f-c*g+a*j, d*e-b*g+a*i,
c*e-b*f+a*h}
```

Output

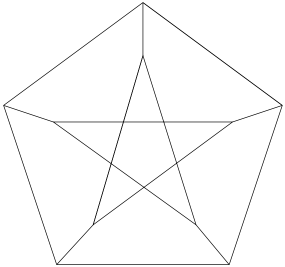
```
Rays:{
0:  (-1,0,0,0,0,0,0,0,0,0),
...
9:  (0,0,0,0,0,-1,0,0,0,0)}
Printing dimension 2 cones:
{
{0,4},
{4,5},
{1,4},
...
{3,8}}
...
```

A tropical example

- Consider the 10 2x2 minors of a 2x5 matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \end{pmatrix} \qquad \begin{array}{l} a = x_{11}x_{22} - x_{12}x_{21} \\ b = \dots \end{array}$$

- Notice $bf - ah - ce = 0$, $bg - ai - de = 0$ ...
- These relations generate the Grassmann-Plücker ideal I.
- The tropical variety of I is pure of dimension 7.
- We can draw it as the Petersen graph.



Applications

You can use Gfan

- to investigate combinatorial structure.
- to search for an initial ideal with a special property.
- to compute a universal Gröbner basis.

Computational example 1

Let I be the ideal generated by the 3x3 minors of a 4x4 matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}$$

in the polynomial ring of 16 variables.
The Gröbner fan consists of 163032 full dimensional cones.

- Without symmetry: 14 hours
- With symmetry: 7 minutes (289 orbits)

7 dimensional homogeneity space.

Computational example 2

Let I be the ideal generated by the 3x3 minors of a 4x4 matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}$$

in the polynomial ring of 16 variables.

- The tropical variety is 12 dimensional with a 7 dimensional homogeneity space.
- The F-vector is $(1, 50, 360, 1128, 1680, 936)$.
- Traversing the maximal cones with symmetry takes 2 minutes.

Computational example 3

Consider the 20 3x3 minors of a 3x6 matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix}$$

The relations on these generate the Grassmann-Plücker ideal I.

- The tropical variety of I is pure of dimension 10 and has a 6 dimensional homogeneity space.
- The 1035 maximal cones were computed in 3-4 hours (102 orbits).

Algorithms

- Collart, Kalkbrener, Mall:
  "Converting bases with the Gröbner walk" (1997)
- Fukuda, Jensen, Thomas:
  "Computing Gröbner fans" (2005)
- Bogart, Jensen, Speyer, Sturmfels, Thomas:
  "Computing tropical varieties" (2005)

Related software

- TiGERS (Huber, Thomas) and CaTS - for toric ideals
- TOPCOM (Rambau) - the Gröbner fan refines the secondary fan
- SAGE (Stein et al.) - Gfan is available in SAGE.

Would be interesting to use a more efficient Buchberger implementation or a faster code for computing mixed faces.