# OPTIMIZATION OF POLYNOMIALS ON THE UNIT SPHERE

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Typeset b

## The problem

Let

$$\mathbb{S}^{n-1} = \left\{ x \in \mathbb{R}^n, \quad x = (x_1, \ldots, x_n) : \quad \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

be the unit sphere and let

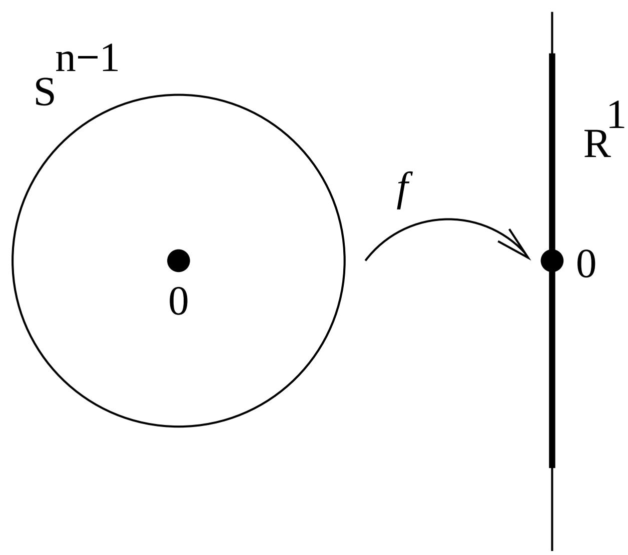
$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

be a homogeneous polynomial of degree $d$.

**Problem:** Compute exactly or approximately

$$\|f\|_\infty = \max_{x \in \mathbb{S}^{n-1}} |f(x)|.$$

Circumstances: $\deg f = d$ is fixed (small), $n$ varies (large).




We denote:

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

## Easy cases

Constants ($d = 0$):
$f(x) = c$, so $\|f\|_\infty = c$,

Linear forms ($d = 1$):
$f(x) = a_1x_1 + \ldots + a_nx_n$, so $\|f\|_\infty = \sqrt{a_1^2 + \ldots + a_n^2}$,

Quadratic forms ($d = 2$):
$f(x) = \sum_{ij} a_{ij}x_ix_j$, so $\|f\|_\infty$ is the largest in the absolute value eigenvalue of $A = (a_{ij})$.

## Hard cases

Higher-order forms ($d \geq 3$).

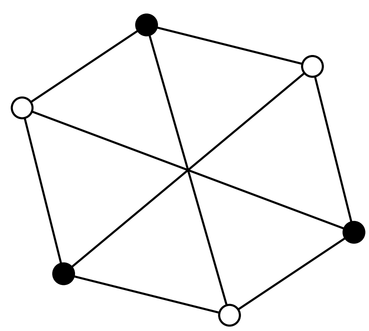
How much harder? NP-hard for sure, but could also be harder: can be hard to approximate.

Yu. Nesterov (in E. de Klerk formulation):

Let $G = (V, E)$ be a graph with the vertices $V = \{1, \dots, n\}$ and some edges $E$. Let $\alpha(G)$ be the *stability number* of $G$, that is, the largest cardinality of a subset of vertices, no two of which are connected by an edge.
Then

$$\sqrt{1 - \frac{1}{\alpha(G)}} = 3\sqrt{3} \max_{\|x\|^2 + \|y\|^2 = 1} \sum_{\{ij\} \notin E} y_{ij} x_i x_j.$$

Here $x_i$ correspond to the vertices and $y_{ij}$ to non-edges.



A graph with $\alpha(G) = 3$

The stability number is badly non-approximable (within a factor of $n^{1-\epsilon}$).

Hence, unless the computation complexity hierarchy collapses, there is no way to approximate the maximum of a cubic form within relative error $\epsilon$ in time polynomial in $n$ and $\epsilon^{-1}$.

Nevertheless, there could be an algorithm polynomial in $n$ for any *fixed* $\epsilon$.

## Why do we care?

Suppose we have a system of real homogeneous equations of degree $d$

$$f_i(x) = 0, \quad i \in I.$$

Pick a small $\epsilon > 0$ and consider

$$g = \frac{1}{|I|} \sum_{i \in I} \Big( \|x\|^{2d} - \epsilon f_i^2 \Big).$$

Thus $g$ is a form of degree $2d$ and

$$\|g\|_\infty = 1$$

if and only if the system has a solution $x \in \mathbb{S}^{n-1}$.

It would be nice to identify "clearly unsolvable" or "almost solvable" systems.

Let $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ be a homogeneous polynomial of degree $d$. Let

$$\|f\|_{2k} = \left( \int_{\mathbb{S}^{n-1}} f^{2k}(x)\ dx \right)^{1/2k}$$

and

$$\|f\|_\infty = \max_{x \in \mathbb{S}^{n-1}} |f(x)|.$$

Here $dx$ is the Haar probability measure and $k$ is a positive integer.

Then

$$\|f\|_{2k} \leq \|f\|_\infty \leq \binom{kd+n-1}{kd}^{1/2k} \|f\|_{2k}.$$

Observations:

For a fixed $k$ and $d$, the value of $\|f\|_{2k}$ is computable in time polynomial in $n$ (because it is easy to integrate monomials).

For a fixed $d$ and $\epsilon > 0$ one can find $k = k(d, \epsilon)$ so that the ratio of the upper and lower bounds is about $\epsilon n^{d/2}$.

For a fixed $d$, there is an $\alpha = \alpha(d) > 0$ such that

$$\|f\|_2 \leq \|f\|_\infty \leq \alpha n^{d/2} \|f\|_2.$$

## “Needle-like” polynomials

Let us consider polynomials $f$ for which

$$\|f\|_\infty \geq \beta n^{d/2} \|f\|_2 \quad \text{for some fixed} \quad \beta > 0,$$

such as

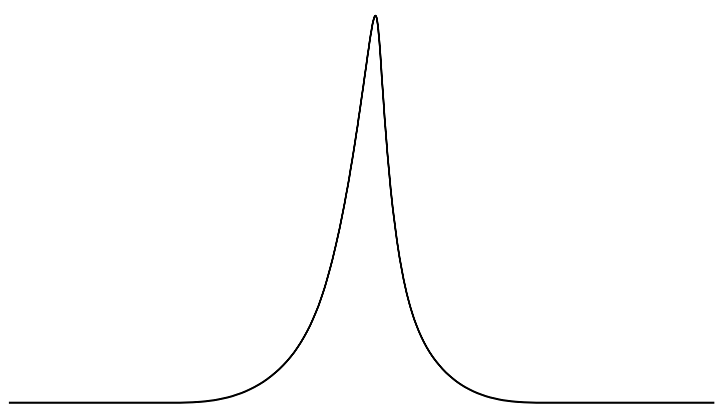
$$f(x) = x_1^d,$$

where we have

$$\|f\|_\infty = 1$$

and

$$\begin{aligned} \|f\|_2 &= \left( \frac{\Gamma(n/2)\Gamma(d+1/2)}{\Gamma(1/2)\Gamma(d+n/2)} \right)^{1/2} \\ &\approx d^{d/2} n^{-d/2} \quad \text{for} \quad n \gg d. \end{aligned}$$



The distribution of values: large values are rarely taken

## A toy example

Let $\langle x, y \rangle$ be the standard scalar product in $\mathbb{R}^n$.
Suppose that

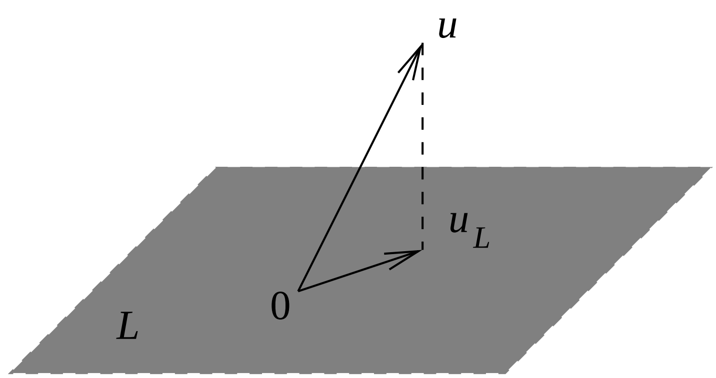
$$f(x) = \langle u, x \rangle^d \quad \text{for some} \quad u \in \mathbb{R}^n \setminus \{0\}.$$

Suppose further, that we know that $f$ has this structure but don't know $u$. How can we compute $\|f\|_\infty$, assuming that we can compute $f(x)$ for any given $x \in \mathbb{R}^n$?

Let us choose a $k \ll n$, and let us choose a random $k$-dimensional subspace $L \subset \mathbb{R}^n$ and consider the restriction $f_L$ of $f$ onto $L$. Hence

$$f_L(x) = \langle u_L, x \rangle^d \quad \text{for} \quad x \in L,$$

where $u_L$ is the orthogonal projection of $u$ onto $L$.




A vector and its projection

Chances are,

$$\|u_L\| \approx \sqrt{\frac{k}{n}} \|u\|$$

by the Johnson-Lindenstrauss Lemma.

Hence

$$\|f\|_\infty \approx \left(\frac{n}{k}\right)^{d/2} \|f_L\|_\infty.$$

We compute $\|f_L\|_\infty$ "by brute force".

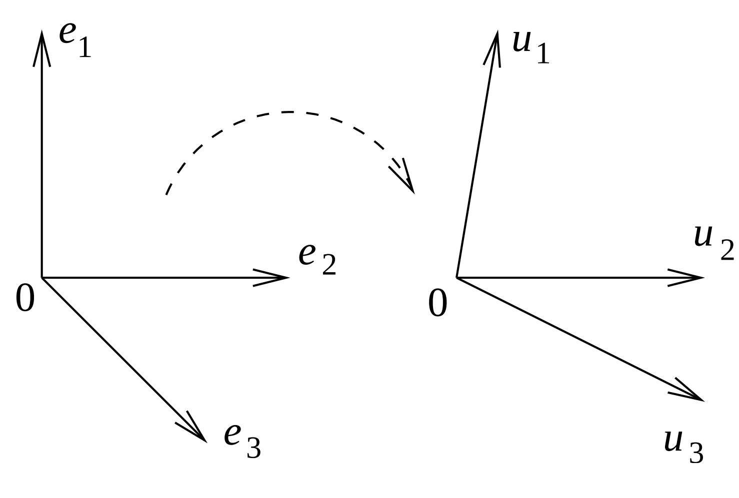
To achieve relative error $\epsilon > 0$, we take $k = O\left(\epsilon^{-2}\right)$.

## A generalization

We say that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is $(\delta, N)$-*focused* if

$$f(x) = \sum_I \alpha_I \prod_{i \in I} \langle u_i, x \rangle, \quad \text{where}$$

- non-zero vectors $u_1, \ldots, u_N$ are such that the cosine of the angle between each two is at least $\delta > 0$;
- the sum is taken over subsets $I \subset \{1, \ldots, N\}$ with $|I| = d$;
- we have $\alpha_I \geq 0$.




Focusing vectors

**Theorem.** *There is an absolute constant* $\gamma > 0$ *such that for any*

$$k \geq \gamma \epsilon^{-2} \delta^{-2} \ln(N+2)$$

*and a random* $k$*-dimensional subspace* $L \subset \mathbb{R}^n$ *the inequality*

$$(1-\epsilon)^{d/2} \|f_L\|_\infty \leq \left(\frac{k}{n}\right)^{d/2} \|f\|_\infty \leq (1-\epsilon)^{-d/2} \|f_L\|_\infty$$

*holds with probability at least* $2/3$.

### Remarks

For a fixed degree $d$, we get a polynomial time randomized approximation scheme to compute $\|f\|_\infty$ of $(\delta, N)$-focused polynomials $f$.

In fact, a bit stronger result holds: we can compute $L^p$ norms for positive even integer $p$.

For $g$ defined on a subspace $A$, let us define till the end of the page

$$\|g\|_p = \left(\int_{\mathbb{R}^n} f^p \; d\mu_A\right)^{1/p},$$

where $\mu$ is the standard Gaussian measure on $A$. Then

$$(1-\epsilon)^{d/2} \|f_L\|_p \leq \left(\frac{k}{n}\right)^{d/2} \|f\|_p \leq (1-\epsilon)^{-d/2} \|f_L\|_p.$$

$(\delta, N)$-focused polynomials are "needle-like":

$$\|f\|_\infty \geq \beta n^{d/2} \|f\|_2$$

for some $\beta = \beta(\delta, N) > 0$.

Various (maybe all?) "needle-like" polynomials are well-approximated by $(\delta, N)$-focused polynomials: sample $u_i$ at random from a biased distribution and take the expectation. For example, this way one can get complete symmetric polynomials.

## A question

Let us fix $d$ and $\beta > 0$ and let us consider the class of homogeneous polynomials $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of $\deg f = d$ and such that

$$(*) \qquad \|f\|_\infty \geq \beta n^{d/2} \|f\|_2.$$

Let us choose an $\epsilon > 0$. Is it true that there exists a

$$\gamma = \gamma(d, \beta, \epsilon)$$

such that if $L \subset \mathbb{R}^n$ is a random subspace of dimension

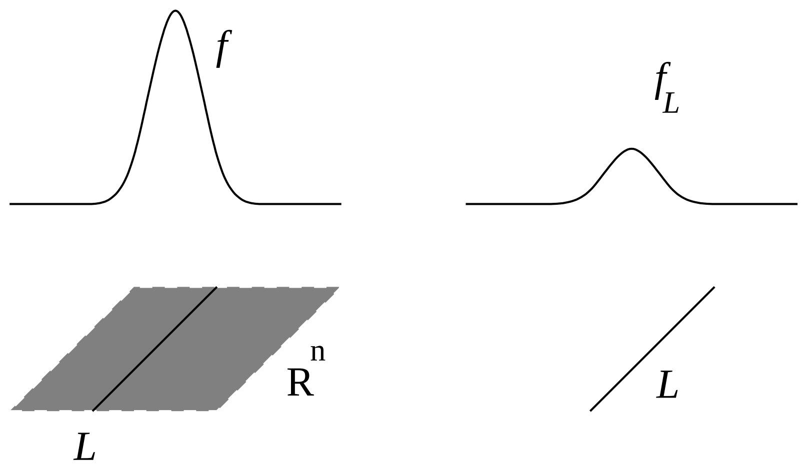
$$k \geq \gamma \ln n$$

then we have

$$(1-\epsilon)^{d/2} \|f_L\|_\infty \leq \left(\frac{k}{n}\right)^{d/2} \|f\|_\infty \leq (1-\epsilon)^{-d/2} \|f_L\|_\infty?$$

Polynomials satisfying $(*)$: for example, polynomials with the coefficients between 1 and 2.

More generally, suppose we know that $\|f\|_\infty$ is large enough compared to $\|f\|_2$. Can we figure out $\|f\|_\infty$ from $\|f_L\|_\infty$, where $L \subset \mathbb{R}^n$ is a random subspace of a small enough dimension?




If the distribution of $f$ on $\mathbb{S}^{n-1}$ is a peak,
will the distribution of $f_L$ on $\mathbb{S}^{n-1} \cap L$ be an accurately scaled peak?

Let

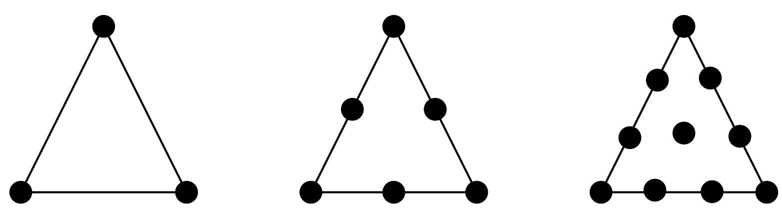
$$\Delta_{n-1} = \Big\{(x_1, \ldots, x_n) : \quad \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad x_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n\Big\}$$

be the standard simplex and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a homogeneous polynomial of degree $d$.

There is no symmetry anymore, so it makes sense to consider

$$\overline{f} = \max_{x \in \Delta_{n-1}} f(x) \quad \text{and} \quad \underline{f} = \min_{x \in \Delta_{n-1}} f(x).$$

Fix $d$ and $\epsilon > 0$. The values $\overline{f}$ and $\underline{f}$ can be computed in polynomial time within an error of $\epsilon\left(\overline{f} - \underline{f}\right)$, by a result of E. de Klerk, M. Laurent, and P. Parrilo.



The simplex has small (polynomial size) $\epsilon$-nets which save the day. The $k$th next consists of the points $x \in \Delta_{n-1}$ for which $kx \in \mathbb{Z}^n$.

In contrast, $\epsilon$-nets on the sphere have exponential size.

## Appendix: useful formulas

Integrating a monomial:

$$\int_{\mathbb{S}^{n-1}} x_1^{2m_1} \cdots x_n^{2m_n} \, dx = \frac{\Gamma(n/2) \prod_{i=1}^n \Gamma(m_1 + 1/2)}{\Gamma(1/2)\Gamma(m_1 + \ldots + m_n + n/2)}.$$

The Wick formula: let $u_1, \ldots, u_{2m} \in \mathbb{R}^n$ be vectors and let $C = (c_{ij})$ be their Gram matrix, $c_{ij} = \langle u_i, u_j \rangle$. Define

$$\operatorname{haf} C = \sum_{I=\Big\{\{i_1,j_1\},\ldots,\{i_m,j_m\}\Big\}} c_{i_1 j_1} \cdots c_{i_m, j_m},$$

where the sum is taken over all unordered partitions of $\{1, \ldots, 2m\}$ into unordered disjoint pairs $\{i_1, j_1\}, \ldots, \{i_m, j_m\}$. Then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{2m} \langle u_i, x \rangle \, d\mu_n(x) = \operatorname{haf} C,$$

where $\mu_n$ is the standard Gaussian measure with the density

$$(2\pi)^{-n/2} e^{-\|x\|^2/2}.$$