Mechanism and Robot Kinematics, Part I: Algebraic Foundations

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Overview

- Why kinematics is (mostly) algebraic geometry
  - Rigid bodies with algebraic surfaces in contact
    - A good approximation to the most common devices
- Mechanism & robot types
  - Families and sub-families
  - Classification via type graphs
- Types of kinematic problems
  - Analysis: how does this mechanism move?
    - Motion paths, workspace limits, singularities
  - Synthesis: what mechanism will move this way?
  - Exceptional ("overconstrained") mechanisms
Kinematics: Then & Now

Model of Watt Engine, 1784
(http://kmoddl.library.cornell.edu/tutorials/05/)

Stewart-Gough robot
(Fanuc F200)
Griffis-Duffy Platform

This is an algebraic curve of degree 40
Definition of a Mechanism

- A collection of links connected by joints.
  - Links are rigid bodies
    - In reality, not quite, but a good approximation
    - “Compliant mechanisms” are another story
  - Joints are mechanical contacts between surfaces of two links
    - Pin joint (hinge), Ball-and-socket, etc.
Rigid-Body Motion

- A rigid body has two defining properties:
  - Preservation of distance
  - Preservation of chirality (handedness)
Rigid-Body Motion (cont.)

- Suppose \( a, b, c, d \in \mathbb{R}^3 \) are given, non-coplanar.
- What is the set \( a', b', c', d' \in \mathbb{R}^3 \) such that homologous distances are preserved?

\[
(b'-a')^T(b'-a') = (b-a)^T(b-a) \\
(c'-b')^T(c'-b') = (c-b)^T(c-b)
\]

Etc.

\( \rightarrow 6 \) polynomials in 12 variables

\( \rightarrow \) Has 2 components, each 6 dimensional.

One component contains \( (a,b,c,d) \)

Other component: mirror image
Rigid Body Motion (Spatial)

- \( SE(3) = \mathbb{R}^3 \times SO(3) \) (Lie Group \( \mathbb{R}^3 \times SO(3) \))
  - Preserves distance & chirality
  - Let \( T \in SE(3) \) be given by \( (p, Q) \)
    - \( p \in \mathbb{R}^3, Q \in SO(3) \subset \mathbb{R}^{3 \times 3} \)
    - \( Q^T Q = I, \quad \det Q = 1 \)
    - \( T: \mathbb{R}^3 \to \mathbb{R}^3 \)
    - Operates on point \( x \in \mathbb{R}^3 \) as \( T(x) = p + Q \cdot x \)

- Facts
  - \( SE(3) \) is a 6-dimensional, algebraic subset of \( \mathbb{R}^{3 \times 4} \)
    - 3 translations, 3 rotations
  - \( T: \mathbb{R}^3 \to \mathbb{R}^3 \) is an algebraic map
Isomorphisms of SE(3)

- 4x4 Homogeneous matrix group
  - Products and inverses are in the same form
  - Transforms may be written as $T(x) = \begin{bmatrix} Q & p \\ 0 & 1 \end{bmatrix} x$
- SO(3) isomorphic to $P^3$
  - Quaternions
  - Unit quaternions double cover SO(3)
- Study’s soma (Study coordinates)
  - Study quadric in $P^7$ $[a \ b \ c \ d \ q \ r \ s \ t] \in P^7$
  - $aq + br + cs + dt = 0$
Subgroups of SE(3)

- Planar rigid-body motion
  - $SE(2) = \mathbb{R}^2 \times SO(2) \subset SE(3)$
  - $\text{dim } SE(2) = 3$

- Spherical rigid-body motion
  - $SO(3) = 0 \times SO(3) \subset SE(3)$
  - $\text{dim } SO(3) = 3$

- These are both of interest in kinematics too.
Definition: Link and Framed Link

- Let a **link** be a collection of features
  - Assume features are algebraic sets in \( \mathbb{R}^3 \).
  - Points, curves, surfaces
- Call an element of \( \text{SE}(3) \) a **frame**.
- Let **framed link** = link plus a frame.
Transformed features

- Let $A$ be a link
- Let $A = \{T,A\}$, $T \in SE(3)$, a framed link
- Feature point $x \in A$ transforms to $x' = T(x)$
- Feature given by $f(x) = 0$ transforms to $f(T^{-1}(x')) = 0$
\textbf{Joints}

- **Joint** = tangential contact between transformed features of two framed links.
  - Tangential contact =
    - Share at least one point, called “contact” point(s)
    - Surface tangent plane contains the tangent space of the opposing feature at contact point
## Contact types

<table>
<thead>
<tr>
<th></th>
<th>A ? B</th>
<th>B</th>
<th>point</th>
<th>curve</th>
<th>surface</th>
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<tr>
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<td>equality</td>
<td>inclusion</td>
<td>inclusion</td>
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<tr>
<td>curve</td>
<td>equality meet</td>
<td>inclusion</td>
<td>point contact</td>
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<tr>
<td>surface</td>
<td>equality curve contact</td>
<td>point contact</td>
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</table>

*Table showing contact types and their relationships.*
## Linear contact types

<table>
<thead>
<tr>
<th>A ⊗ B</th>
<th>B</th>
<th>point</th>
<th>line</th>
<th>plane</th>
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<td>plane</td>
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<td>equality</td>
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</table>
Example: Planar 7-bar Structure

Problem:
Assemble these 7 pieces, as labeled.
Example 2 Preparation

- Ball bar $\leftrightarrow$ Spherical surface
Example: Stewart-Gough platforms

- **Top link, B**
  - 6 feature points
    - $b_1, b_2, b_3, b_4, b_5, b_6$

- **Base link, A**
  - 6 feature spheres
    - Centers $a_1, a_2, a_3, a_4, a_5, a_6$
    - Radii $r_1, r_2, r_3, r_4, r_5, r_6$

- **Joints: point in surface**
  - Point $i$ of top link lies on sphere $i$ of base link, $i=1,\ldots,6$
Example (cont): Forming equations

- For $i=1,\ldots,6$
- Sphere equation:
  $$f_i(x) = (x-a_i)^T(x-a_i) - r_i^2 = 0$$
- Point inclusions:
  $$f_i(T(b_i)) = 0 \quad (*)$$
  Where $T$ is the transform for framed link $B$
- For given links, the $T$ that satisfy $(*)$ are the possible locations of $B$ w.r.t. $A$
Joints: Lower-order pairs

- What surface=surface joints allow relative motion?
  - Answer given by Reuleaux, 1875
  - Rigorous proof by Selig, 1989, IJRR
    - Lie groups

- Why engineers care:
  - Distributed stress = less wear

- And the answer is...
Joints: Lower-order pairs

- **Prismatic**: $f=1$, $c=5$
- **Rotational**: $f=2$, $c=4$
- **Helical (Screw)**: $f=1$, $c=5$ (Not Algebraic)
- **Cylindrical**: $f=2$, $c=4$

$f =$ freedom
$c =$ constraint in SE(3)

**Plane**: $f=3$, $c=3$
**Sphere**: $f=3$, $c=3$
5 of 6 lower-order pairs are linear

- **Prismatic (P)**: Line-in-plane
- **Cylindrical (C)**: Line
- **Sphere (S)**: Point
- **Rotational (R)**: Point-in-line
- **Plane (E)**: Plane
- **Equality between flags**
Linear link types

- Let link type index = integer triplet
  (# feature points, # feature lines, # f. planes)
- For each link type index, (a,b,c), there is an associated link type universal space
  \[ L_{abc} = \text{Aff}(0,3)^a \times \text{Aff}(1,3)^b \times \text{Aff}(2,3)^c \]
- A **link type** is an algebraic subset of a link type universal space, say
  \[ Y \subset L_{abc} \]
  - Inclusions, parallelism, perpendicularly, etc.
Mechanism Definition Revisited

- Limit our attention to linear features
- Mechanism family is given by
  - List of N framed links
    - Each link $i$ has a type $Y_i \subset L_{(abc)_i}$
      - $(a,b,c)_i$ is link type index
      - Each linear element is a feature
    - Framed link $i$ has a transform $T_i \in SE(3)$,
  - List of joints, each consisting of
    - A pair of features of two distinct links
    - A contact operation (equality, inclusion, meet)
  - Family is $M \subset SE(3)^N \times Y$,
    - where $Y$ is a given algebraic set $Y \subset Y_1 \times \cdots \times Y_N$
    - Equations for $M$ derive from the contact operations applied to transformed feature pairs.
Some Mechanism Families

- Native linear families
  - Those for which $Y = L_{(abc)}_1 \times \cdots \times L_{(abc)}_N$

- Lower-order pair families
  - All joints are one of P,R,C,E,S

- Planar mechanisms
  - Frames are in SE(2)
  - Joints are either:
    - P with lines parallel to reference plane
    - R with axes orthogonal to reference plane

- Spherical mechanisms
  - Frames are in SO(3)
  - Joints are all R with axes through the origin
Spherical four-bar

Courtesy of J. M. McCarthy, UC Irvine
Graphic by Hai-Jun Su
Mechanism Type Graphs

- For lower-order pair mechanisms
  - Joint type (P,R,C,E,S) implies feature type
- If there are no other conditions imposed (parallelism, orthogonality, etc.) then a colored graph defines the family
  - Links are nodes
  - Joints are colored (labeled) edges
  - Simple example: house door
- Extra conditions can be annotated separately
Examples

- Connecting rod
- Engine block
- Crankshaft
- Fuel and air
- Spark plug
- Compressed fuel and air
- Piston
- Rod
- Crankshaft
- Engine block
- Top plate
- Upper leg
- Lower leg
- Base plate
Type Enumeration

- For planar & spherical generically 1 DOF lower-pair mechanisms, we can enumerate all possible mechanism types up to $N=12$
  - Graph isomorphism is at issue

- For spatial, there are too many to bother

Two-bar

Four-bar

Watt six-bar

Stephenson six-bar

16 distinct eight-bars
Grounded Mechanism

- A mechanism, as defined above, floats freely in 3-space.
- A grounded mechanism is a mechanism with one link held stationary.
  - Say, $T_N = I$
  - Configuration space becomes $SE(3)^{N-1}$
- Different grounded mechanisms derived from the same mechanism are called inversions of the mechanism.

Watt I

Watt II
Input Joints

- Typically, some joints are actuated
- These are the input joints
- We may directly command the input angle (R joint) or input translation (P joint)
- The relative motion is parameterizable

\[
T_j = T_i A \begin{bmatrix}
    c_\theta & -s_\theta & 0 & 0 \\
    s_\theta & c_\theta & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} B
\]

\[
T_j = T_i A \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & \delta \\
    0 & 0 & 0 & 1
\end{bmatrix} B
\]

R joint  P joint

- This defines a map

\[
J : \text{SE}(3)^N \rightarrow S^1 \times \cdots \times S^1 \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 = J
\]
There is usually an output link or output joint(s) that directly interacts with the environment to achieve the purpose of the mechanism.

- Examples: robot hand, automobile wheel
- This gives an output function:

\[ K: \text{SE}(3)^N \rightarrow \mathbb{K} \]

- Output link, \( K: \text{SE}(3)^N \rightarrow \text{SE}(3) \)
- Rotational output joint, \( K: \text{SE}(3)^N \rightarrow S^1 \)
Big Picture

mechanism \( M \subseteq \text{SE}(3)^N \times Y \)

configuration space

configuration space

\( \text{SE}(3)^N \)

natural projections

\( \pi_1 \)

\( \pi_2 \)

mechanism type

input

output

J

K

J

K
Overview (revisited)

- Why kinematics is (mostly) algebraic geometry
  - Rigid bodies with algebraic surfaces in contact
    - A good approximation to the most common devices
- Mechanism & robot types
  - Families and sub-families
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- Types of kinematic problems
  - Analysis: how does this mechanism move?
    - Motion paths, workspace limits, singularities
  - Synthesis: what mechanism will move this way?
  - Exceptional ("overconstrained") mechanisms
Analysis

- How does a given mechanism move?
- Family $M \subset \text{SE}(3)^N \times Y$

$$
\begin{array}{c}
\pi_1 \\
\pi_2 \\
\end{array}
\begin{array}{c}
\text{SE}(3)^N \\
Y \\
\end{array}
$$

- $y_0 \in \pi_2(M) \subset Y$ is a mechanism in the family
- $\pi_1(\pi_2^{-1}(y_0))$ is its motion
- $\dim \pi_1(\pi_2^{-1}(y_0))$ is called its “degrees of freedom”
Analysis

- The motion $\pi_1(\pi_2^{-1}(y_0))$ may have several components.
  - Find the DOF of each component
  - Real dimension = complex dimension
- The DOF’s may be different for different points $y_0 \in \pi_2(M)$
  - For each irred. component of $\pi_2(M) \subset Y$
    find the generic DOF’s
Example: 7-bar Structure

Problem:
Assemble these 7 pieces, as labeled.
Result for Generic Links

18 rigid structures

• 8 real, 10 complex for this set of links.
• All isolated – can be found with traditional homotopy
Special Links (Roberts Cognates)

Dimension 1:
6\textsuperscript{th} degree four-bar motion

Dimension 0:
1 of 6 isolated (rigid) assemblies
Robot Workspace Analysis

- Find the workspace boundary (in the reals)
  - Real points of the singular set of a complex component of
    \[ K(\pi_1(\pi_2^{-1}(y_0))) \]
  - The set being analyzed here is the position of the hand
    - i.e., the image of a projection map from \( SE(3)^4 \) to \( \mathbb{R}^3 \)

A. Malek, U. Iowa
Forward and inverse kinematics

- Suppose
  \[ \dim J(C) = \dim K(C) = \dim C, \]
  \[ C = \pi_1(\pi_2^{-1}(y_0)), \quad y_0 \in \pi_2(M) \]

- Forward kinematics
  - Find \( K(J^{-1}(x)) \)

- Inverse kinematics
  - Find \( J(K^{-1}(x)) \)

- Find singularities of these maps
Synthesis

- Find a mechanism to approximate a given motion
  - Finite position synthesis = interpolate a finite set of given outputs
    \[ Z = \left\{ y \in \pi_2(M) \mid K(\pi_1(\pi_2^{-1}(y))) \supset \{ o_1, \ldots, o_m \} \right\} \]
    where \( o_1, \ldots, o_m \in \mathbf{K} \) are given
  - Input-output synthesis
    \[ Z = \left\{ y \in \pi_2(M) \mid [J(\pi_1(\pi_2^{-1}(y))), K(\pi_1(\pi_2^{-1}(y)))] \supset \{ [n_1, o_1], \ldots, [n_m, o_m] \} \right\} \]
    where \( [n_1, o_1], \ldots, [n_m, o_m] \in \mathbf{J} \times \mathbf{K} \) are given
Another view: Fiber Product

- Let \( P = \pi_2(M) \) be the parameter space.
- The fiber product \( \Pi_p^M = M \times_p \cdots \times_p M \) \( j \) times

is the space of multiple instances of the same mechanism in \( j \) different configurations.

- Let \( \pi_{1i} \) be the projection onto the \( i \)th configuration.
- Output synthesis sets

\[
K(\pi_{1i}(\Pi_p^M)) = o_i, \quad i = 1, \ldots, m
\]
Synthesis Example

- 9-Point Path Generation for Four-bars
  - Problem statement
    - Alt, 1923
  - Bootstrap partial solution
    - Roth, 1962
  - Complete solution
    - Wampler, Morgan & Sommese, 1992
      - m-hom. Continuation
        - 143,360 paths (2-way symmetry)
      - 4326 finite, isolated solutions
        - 1442 Robert cognate triples
Spherical four-bar

Courtesy of J. M. McCarthy, UC Irvine
Graphic by Hai-Jun Su
Exceptional Mechanisms

- A.K.A. “overconstrained mechanisms”
- For each component of $M$, there is a generic fiber dimension, $M$’s DOF
  
  $d^* = \dim_x \pi_2^{-1}(\pi_2(x)))$ for generic $x \in M$

- $M$ may have algebraic subsets where the fiber dimension is larger than $d^*$
  - These are exceptional mechanisms
  - Finding such mechanisms is challenging
Exceptional 7-bar (Roberts Cognates)

Dimension 1:

6\text{th} degree four-bar motion
Bennett four-bar

Courtesy of J. Michael McCarthy, UC Irvine
Exceptional Stewart-Gough Platform
Summary

- Rigid-body mechanisms with algebraic joint features are algebraic
  - Linear features include all lower-order pairs
    - P,R,C,E,S
  - Linear mechanism spaces are a rich source of interesting algebraic sets
- Kinematic problems fall into three main classes
  - Analysis
  - Synthesis
  - Discovery of Exceptional Mechanisms
- Many challenges await!