Computations in classical invariant theory of binary forms

Cheri Shakiban (Joint work with Peter Olver)

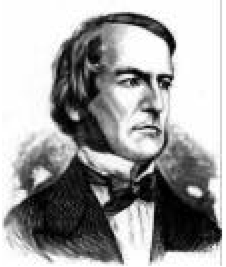
1. Historical Background
2. Invariants and Covariants of binary forms
3. Symbolic method of Aronhold
4. Bracket Polynomials
5. Digraphs and Molecules
6. Syzygies and the Algebra of Digraphs
7. Transvectants
8. Finding the Hilbert basis

1. Historical background

Classical invariant theory has died and been resurrected.

19th- Century

George Boole (1815-1864)

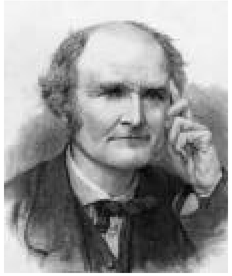


He was one of the first to notice that invariants were important.

Interesting note: Boole's daughter Alicia Boole Stott (1860-1949) was a mathematician.

Arthur Cayley (1821-1895)

Cayley was aware of invariants up to degree 6 but thought binary forms of degree 7 or more didn't have a basis for their Invariants.



Interesting note: Cayley won the first Smith prize at Cambridge.

Quiz: How many mathematical papers did he publish?

Paul Albert Gordan(1837 – 1912)

In 1868, Gordan gave a constructive proof that the covariants and invariants of a binary form of any degree has a finite basis. ,

Quiz: Who was Gordan's only student?

James Sylvester (1814-1897)

Sylvester and Cayley through their collaborations came to be known as *invariant twins*.

Quiz: Who was Sylvester's most famous student?
What famous journal did he start?

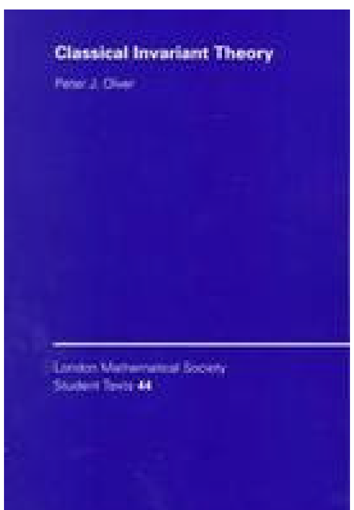
Sylvester produced tables for the bases of the invariants and covariants:

Degree	2	3	4	5	6	7	8	9	10	12
# Invariants	1	1	2	4	5	26(30)	9	89	104	109
# Covariants	2	4	5	23	26	124(130)	69	415	475	949

Degree 8: The invariants/covariants were checked by Shioda in 1967

Degree 7: Constructed by Dixmier in 1992.
New proof: Leonid Bedratyuk Feb 2006

http://front.math.ucdavis.edu/math.AG/0602373
http://front.math.ucdavis.edu/math.AG/0611122
Courtesy of Alicia Dickenstein



Classical Invariant Theory by Peter J. Olver

Death-blow: Hilbert's celebrated Basis Theorem.

Any finite system of homogeneous polynomials admits a finite basis for its invariants, as well as for its covariants (1888).

David Hilbert (1862-1943)

The first proof was existential.

Gordan’s comment: Das ist Theologie und nicht Mathematick

Hilbert published a second constructive proof.

Hilbert was unjustly saddled with the reputation of killing off constructive invariant theory. It was really a slow death.

As pointed out by Bernd Sturmfels:
Hilbert’s second proof combined with the theory of Gröbner bases can be used to construct an algorithm producing the Hilbert bases of a general system of forms.

Graphical Methods:

William Clifford (1845-1879)

Clifford: began developing a graphical method for the description of the invariants and covariants of binary forms.

Sylverster: unveiled his “ *algebro-chemical theory*”, whose aim was to apply the methods of classical invariant theory to the rapidly developing science of molecular chemistry.

Recent years: interest in classical invariant theory is on the rise.

- mathematical subject in it s own right

Important applications:

Dynamical systems
Solution of nonconvex variational problems
Elasticity
Molecular physics
Modular forms
Computer vision
Others

- **Revival of the computational approach**: partly due to current availability of symbolic manipulation computer programs.

2. Invariants and Covariants

An *invariant:* is something that is left unchanged by group of transformations.

invariant theory: study of quantities which are associated with polynomial equations and are left invariant under transformations of the variables.

form : homogeneous polynomial.

binary form :

$$Q(\mathbf{x}) = Q(x,y) = \sum_{i=0}^{n} a_i \binom{n}{i} x^{n-i} y^{i}$$

$\mathbf{x}$ = (x, y), a_i are either real or complex

n : *degree* of the form

Under the general linear changes of variables

$$(x, y) \to (a\,\tilde{x} + b\,\tilde{y}\,,\, c\,\tilde{x} + d\,\tilde{y})$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular $\in \; GL(2)$

The polynomial $Q(\mathbf{x}) \to \tilde{Q}(\tilde{\mathbf{x}})$ is given by

$$\tilde{\mathrm{Q}}(\tilde{x}, \tilde{y}) = \mathrm{Q}(a\,\tilde{x} + b\,\tilde{y}\,,\, c\,\tilde{x} + d\,\tilde{y})$$

$$a_i \to \tilde{a}_i$$

Definitions: An ***invariant*** **of *weight* g** of a binary form Q(x,y), of degree n, is a function

$$I(\tilde{\mathbf{a}}) = (\det A)^g I(\mathbf{a}), \qquad A \in GL(2)$$

A ***covariant*** **of *weight* g** is a function J(**a**,**x**)

$$J(\tilde{\mathbf{a}}, \tilde{\mathbf{x}}) = (\det A)^g J(\mathbf{a}, \mathbf{x}), \qquad A \in GL(2)$$

Example: Quadratic poly : $Q(\mathbf{x}) = a_0 x^2 + 2a_1 xy + a_2 y^2$

$$\text{discriminant} \quad \Delta = a_1^2 - a_0 a_2$$

$$\tilde{Q}(\tilde{x}) = \tilde{a}_0\tilde{x}^2 + 2\tilde{a}_1\tilde{x}\tilde{y} + \tilde{a}_2\tilde{y}^2$$

transformed coefficients

$$\tilde{a}_0 = a_0a^2 + 2a_1ac + a_2c^2,$$

$$\tilde{a}_1 = a_0ab + a_1(ad + bc) + a_2cd,$$

$$\tilde{a}_2 = a_0c^2 + 2a_1bd + a_2d^2$$

Thus the new discriminant is

$$\tilde{\Delta} = \tilde{a}_1^2 - \tilde{a}_0\tilde{a}_2 = (ad - bc)^2 \cdot (a_1^2 - a_0a_2) = (\det A)^2 \cdot \Delta$$

hence Δ is an invariant of weight 2.

Example: binary quartic

$$Q(\mathbf{x}) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$$

There are two fundamental invariants

A quadratic one

$$i = 2a_0a_4 - 8a_1a_3 + 6a_2^2, \qquad weight\ 4$$

A cubic one

$$j = 6 \det \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix}, \qquad weight\ 6$$

The most important covariant of a quartic, or, indeed, of any binary form Q is the Hessian

$$H(\mathbf{x}) = \frac{2}{n^2(n-1)^2} \left(Q_{xx} Q_{yy} - Q_{xy}{}^2 \right)$$

degree $2n - 4$, covariant of weight 2.

Besides Q itself , there is only one other independent covariant of the quartic: Jacobian of Q and H.

$$T = \frac{1}{16} \left(Q_x H_y - Q_y H_x \right)$$

Classical result : Any other polynomial invariant or covariant of a binary quartic can be written in terms of the covariants Q, H, i, j, and T.

3. The Symbolic Method (constructive technique, introduced by Aronhold)

Siegfried Aronhold (1819 - 1884)

The first German to work in this topic.

The motivating idea: We can pretend that
a binary form Q(x,y) is just the nth power of a linear form

$$(a\ \mathbf{x})^n = (a_1 x + a_2 y)^n = \sum_{i=0}^{n} \binom{n}{i} a_1^{\,n-i} a_2^{\,i} x^{n-i} y^i$$

$$a_i = a_1^{\,n-i} a_2^{\,i}, \qquad a = (a_1, a_2)$$ is a “*symbolic letter*”

This approach would not distinguish between different monomials. For example:

$$a_0 a_2 \rightarrow a_1^2 a_2^2 \leftarrow a_1^2$$

To resolve this ambiguity , we use a *different* symbolic letter for each occurrence of coefficients a_i:

$$a_0 a_2 \longrightarrow \textit{symbolic form} \quad a_1^2 b_2^2$$

$$a_1^2 \longrightarrow \textit{symbolic form} \quad a_1 a_2 b_1 b_2$$

The symbolic polynomial

$$a_1^2 b_1 b_2 g_2^2 xy^2 \longrightarrow a_0 a_1 a_2 xy^2$$

(The x's and y's are not affected by the symbolic method.)

Another ambiguity: a_0a_2 and a_2a_0, have different symbolic forms:

$a_1^2 b_2^2$ and $a_2^2 b_1^2$

All the different symbolic forms can be obtained one from the other merely by *permuting the symbolic letters*.

$$a_1^2 b_1 b_2 g_2^2 xy^2, a_2^2 b_1 b_2 g_1^2 xy^2, a_1 a_2 b_2^2 g_1^2 xy^2, a_1^2 b_2^2 g_1 g_2 xy^2$$

$a_0 a_1 a_2 xy^2$ $a_2 a_1 a_0 xy^2$ $a_1 a_2 a_0 xy^2$ $a_0 a_2 a_1 xy^2$

so on.

There is a unique *symmetric* symbolic form for any given polynomial, obtained by *symmetrizing* any given representative over all the symbolic letters occurring in it.

$$\Delta = a_0 a_2 - a_1^2$$ has symmetric symbolic form

$$\frac{1}{2}\{(a_1^2 b_2^2 - a_1 a_2 b_1 b_2) + (a_2^2 b_1^2 - a_1 a_2 b_1 b_2)\} =$$

$$\frac{1}{2}\{(a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2\} =$$

$$\frac{1}{2}(a_1 b_2 - a_2 b_1)^2$$

Theorem: Each polynomial $J(a,\mathbf{x})$ has a unique symmetric symbolic form $P(a,b,...,w,\mathbf{x})$

The number of different symbolic letters in a symbolic polynomial represents the degree of the polynomial in the coefficients a_i of the form.

Example: The invariant of the binary quartic,

$$i = 2a_0a_4 - 8a_1a_3 + 6a_2^2 \rightarrow$$
$$a_1^4b_2^4 - 4a_1^3a_2b_1b_2^3 + 6a_1^2a_2^2b_1^2b_2^2 - 4a_1a_2^3b_1^3b_2 + a_2^4b_1^4$$
$$= (a_1b_2 - a_2b_1)^4.$$

Again, we see a similar factorization as with the discriminant of the binary quadratic.

Example of a covariant: The Hessian of the quartic

$$\mathrm{H}(\mathbf{x}) = \frac{1}{72}\left(Q_{xx}Q_{yy} - Q_{xy}{}^{2} \right)$$

Symbolic covariant

$$(a_1 b_2 - a_2 b_1)^2 (a_1 x + a_2 y)^2 (b_1 x + b_2 y)^2$$

4. Bracket Polynomials.

Definition:

a) A *bracket factor of the first kind* is a linear monomial

$$(a\ \mathbf{x}) = a_1 x + a_2 y$$

b) A *bracket factor of the second kind* is

$$[a\ b] = \det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

where a, b are distinct symbolic letters.

The First Fundamental Theorem of Invariant Theory states that every *covariant* of a binary form can be written in symbolic form as a *bracket polynomial*.

a) The *degree* of the covariant in the coefficients a_i is equal to the number of distinct symbolic letters occurring in the bracket polynomial representative.

b) The *weight* of the covariant is equal to the number of bracket factors of the second kind in any monomial of P.

c) The degree of the covariant in the variables **x** is equal to the number of bracket factors of the first kind in any monomial of P.

Example: In the case of a quartic form,

The invariants

$$i \rightarrow (a_1 b_2 - a_2 b_1)^4 = [a\ b]^4$$

Similarly:

$$j \rightarrow [a\ b]^2 [a\ g]^2 [b\ g]^2$$

The Hessian $H \rightarrow [a\ b]^2 (a\ \mathbf{x})^2 (b\ \mathbf{x})^2$

The other covariant $T \rightarrow [a\ b]^2 [b\ g]\ (a\ \mathbf{x})^2 (b\ \mathbf{x})(g\ \mathbf{x})^3$

The symbolic form of a given covariant does *not* have a unique bracket polynomial representative, owing to the presence of certain *syzygies* among the bracket factors themselves.

There are three of these fundamental syzygies:

1. $[\alpha\, \beta] = -\, [\beta\, \alpha]$.
2. $[\alpha\, \beta]\, (\gamma\, \mathbf{x}) = [\alpha\, \gamma]\, (\beta\, \mathbf{x}) + [\gamma\, \beta]\, (\alpha\, \mathbf{x})$.
3. $[\alpha\, \beta]\, [\gamma\, \delta] = [\alpha\, \gamma]\, [\beta\, \delta] + [\alpha\, \delta]\, [\gamma\, \beta]$.

Here $\alpha, \beta, \gamma, \delta$ are distinct symbolic letters.

Remark on bracket polynomials:

If we know the degree of a covariant, and just the bracket factors of the second kind in any homogeneous bracket polynomial representative, we can reconstruct the bracket factors of the first kind.

Example: If we have a symbolic monomial of degree 3 in the coefficients a_i of the form whose bracket factors of the second kind are

$$[\alpha\ \beta]\ [\beta\ \gamma]^2$$

The full bracket monomial must be

$$[\alpha\,\beta]\,[\beta\,\gamma]^2\,(\alpha\,x)^{n-1}\,(\beta\,x)^{n-3}\,(\gamma\,x)^{n-2}, \quad n = \text{degree}$$

since α occurs once, β three times, and γ twice in the second factors.

If the degree of the covariant is 4

$[\alpha\,\beta]\,[\beta\,\gamma]^2\,(\alpha\,x)^{n-1}\,(\beta\,x)^{n-3}\,(\gamma\,x)^{n-2}\,(\delta\,x)^{n}$.

We can concentrate on the bracket factors of the second kind and drop $(\alpha\,x)^{n-1}\,(\beta\,x)^{n-3}\,(\gamma\,x)^{n-2}\,(\delta\,x)^{n}$….

We will call them just *brackets* for short.

5. Digraphs and Molecules

Graphical method: Consider a binary form of degree n, and let P be a bracket polynomial representing the symbolic form of some covariant. To each *monomial* in P we will associate a *"molecule"*, or, more mathematically, a *digraph*.

Algebro-chemical theory as proposed by Sylvester:

Let M be any unit bracket monomial (with coefficient 1).

To each distinct symbolic letter in M we associate an *atom*.

For a binary form of degree n, the atoms will all have "valence" n.

Example: Consider the Hessian of a binary form of degree n.

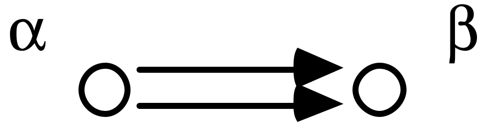
It has the symbolic form

$$[\alpha\ \beta]^2\ (\alpha\ x)^{n-2}\ (\beta\ x)^{n-2}.$$

Molecule will consist of two atoms.

Since the bracket factor $[\alpha\ \beta]$ occurs twice, there will be two directed bonds from atom α to atom β.

Thus the directed molecule representing the Hessian is



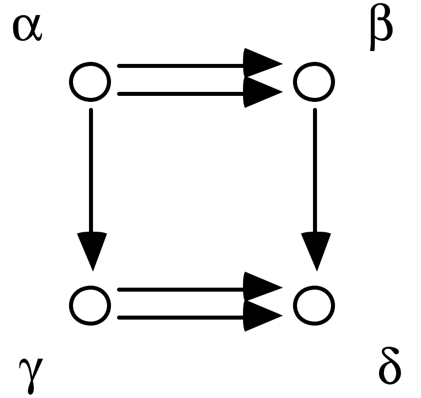
The discriminant of the binary cubic,

$$\Delta = 2a_0^2 a_3^2 - 6a_1^2 a_2^2 - 12 a_0 a_1 a_2 a_3 + 8 a_0 a_2^3 + 8 a_1^3 a_3$$

has symbolic bracket expression

$$[\alpha\ \beta]^2\ [\alpha\ \gamma]\ [\beta\ \delta]\ [\gamma\ \delta]^2,$$

It is represented by the neutral four-atom molecule

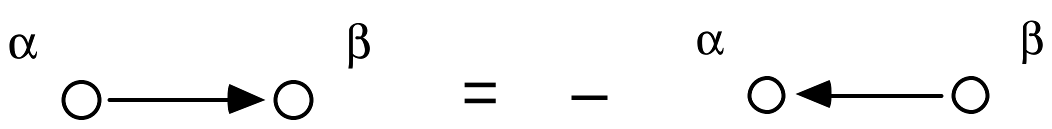


* The bonds in our molecule will correspond to all the bracket factors of the second kind occurring in M.

* If [α β] is a bracket in M, then we have a bond between the atom labeled α and the atom labeled β.

* If a bracket occurs to the k^{th} power - $[\alpha\ \beta]^k$ - in M, then there will be k bonds between atom α and atom β.

The key: To make use of *directed* (or *polarized*) *bonds,* which will enable us to distinguish between the bracket factors [α β] and [β α].



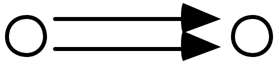
The molecular representation does not depend on how we label the constituent atoms.

Δ can represented by any of the equivalent forms

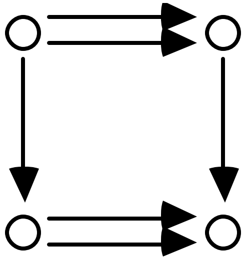
$[\alpha\ \delta]^2\ [\alpha\ \beta]\ [\delta\ \gamma]\ [\gamma\ \beta]^2$, $\qquad [\beta\ \gamma]^2\ [\beta\ \delta]\ [\gamma\ \alpha]\ [\alpha\ \delta]^2$,

etc.

We can drop the labels for the individual atoms, and concentrate on the pure "chemistry" of our molecule.



is the molecular representation of the Hessian.



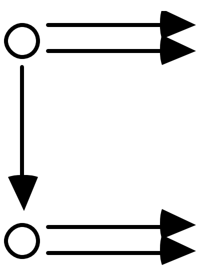
is the molecular representation of the discriminant of the binary cubic.

Representation of covariants

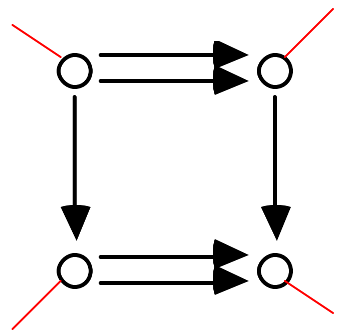
Ions: If there are one or more atoms with unused free bonding sites, the valence is strictly positive and we say we have an ion.

Neutral molecules : If an atom has exactly n bonds, and the entire molecule has valence 0. Neutral molecules correspond to invariants, while ions correspond to more general covariants.

saturated digraph



n=3 invariant



n=4 covariant

$[\alpha\ \beta]^2\ [\alpha\ \gamma]\ [\beta\ \delta]\ [\gamma\ \delta]^2$

irreducible

$[\alpha\ \beta]^2\ [\alpha\ \gamma]\ [\beta\ \delta]\ [\gamma\ \delta]^2\ (\alpha\ \mathbf{x})\ (\beta\ \mathbf{x})\ (\gamma\ \mathbf{x})\ (\delta\ \mathbf{x})$

reducible

Note: we must have exactly n of each symbols.

Linear combination of molecules:

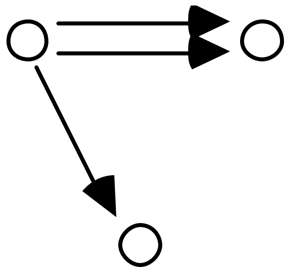
Example: In the case of a binary cubic, the bracket monomial

$$M_1 = [\alpha\ \beta]^2\ [\alpha\ \gamma]\ (\beta\ \mathbf{x})\ (\gamma\ \mathbf{x})^2$$

n=3

has molecular representation

D_1 =

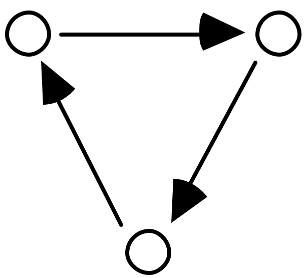


while

$$M_2 = [\alpha\ \beta]\ [\beta\ \gamma]\ [\gamma\ \alpha]\ (\alpha\ \mathbf{x})\ (\beta\ \mathbf{x})\ (\gamma\ \mathbf{x})$$

has molecular representation

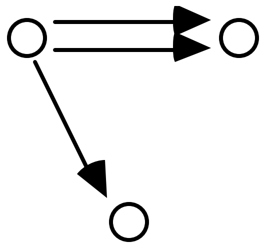
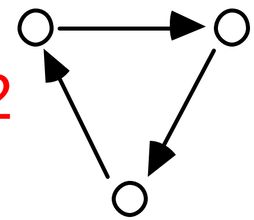
D_2 =



Therefore, the bracket polynomial

$$P = 2\,[\alpha\,\beta]^2\,[\alpha\,\gamma](\beta\,\mathbf{x})(\gamma\,\mathbf{x})^2 - 1/2\,[\alpha\,\beta][\beta\,\gamma]\,[\gamma\,\alpha]\,(\alpha\,\mathbf{x})\,(\beta\,\mathbf{x})(\gamma\,\mathbf{x})$$

has molecular representation

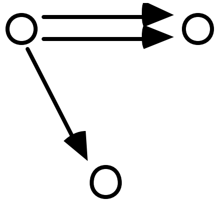
$2D_1 - 1/2\ D_2 = 2$  $-\ 1/2$ 

Chemical analogy: These linear combinations of molecules might be interpreted as "mixtures" of molecular substances, although the admission of negative coefficients stretches this analogy rather thin.

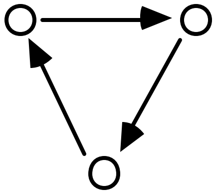
Mathematically: What we are doing is replacing each unit bracket monomial by a digraph.

Recall that a ***graph*** is: a collection of *vertices* and line segments connecting the *vertices*.

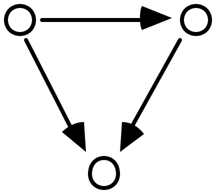
A ***digraph*** is: a graph in which the line segments are arrows.



D_1



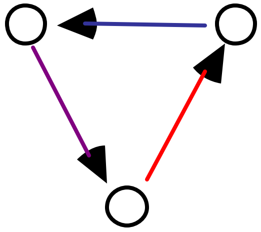
D_2



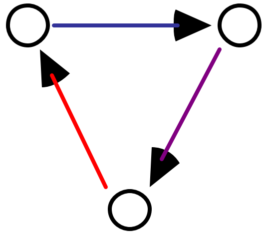
D_3

represent distinct digraphs.

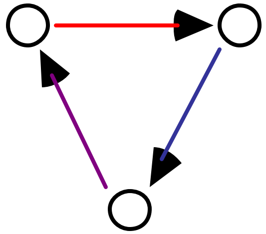
Note that the digraph



, the mirror image of D_2,



is really the same as D_2:



Any bracket monomial will have a ***unique*** digraph representation.

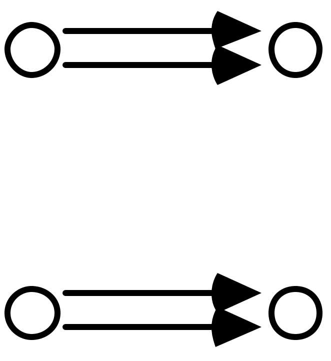
In a digraph, the vertices correspond to the atoms in the molecular representation, and the darts correspond to the directed bonds.

Theorem . Let Q be a binary form of degree n.

Then there is a one-to-one correspondence between bracket polynomials representing covariants of Q and elements of the space D_n of linear combinations of n-digraphs.

A digraph is *reducible* if it is the disjoint union of two subdigraphs.

Example: The reducible digraph on four vertices



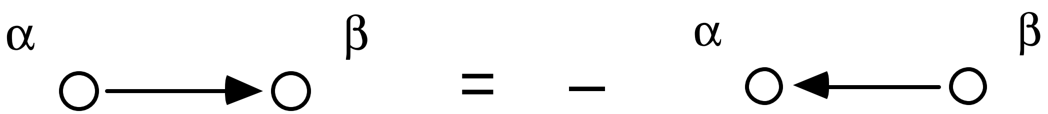
represents the square of the Hessian H of a form: H^2.

6. Syzygies and the Algebra of Digraphs

There are three basic rules in the algebra of digraphs:

Rule #1. From the first syzygy

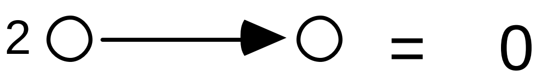
$$[\alpha\ \beta] = -\ [\beta\ \alpha]\ ,$$



Applications of rule #1:

a) Dropping the inessential symbolic labels for the vertices,

implies



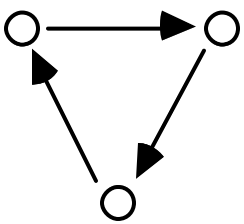
Hence represents the trivial covariant 0.

b) Consider the bracket monomial

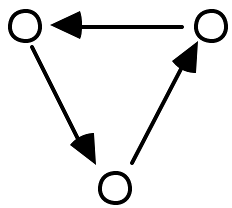
$$[\alpha\ \beta]\ [\beta\ \gamma]\ [\gamma\ \alpha]\ (\alpha\ x)^{n-2}\ (\beta\ x)^{n-2}\ (\gamma\ x)^{n-2}.$$

However, this monomial is a symbolic form of the trivial (zero) covariant. (Verify algebraically!).

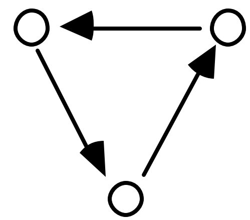
Graphically: If we reverse the direction of all three darts in the digraph, we see that

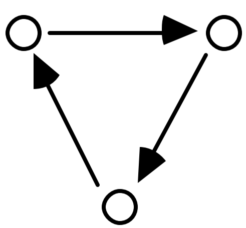


$= (-1)^3$



$= -$



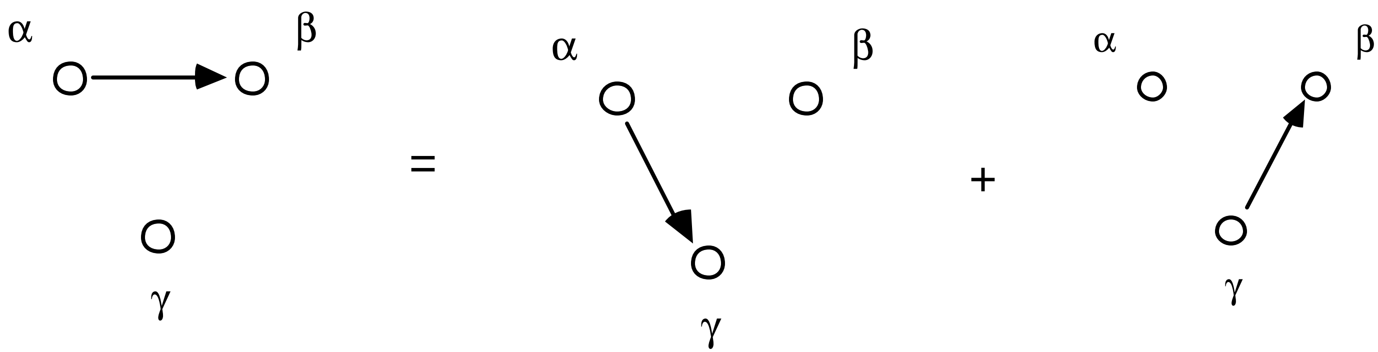
2  $= 0$

is equivalent to the trivial digraph.

Rule #2. The syzygy

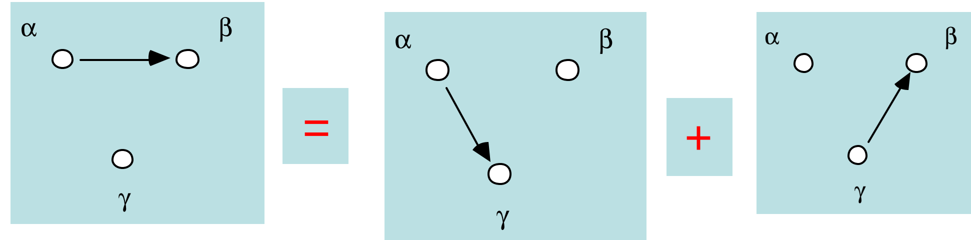
$$[\alpha\,\beta]\,(\gamma\ \mathrm{x}) = [\alpha\,\gamma]\,(\beta\ \mathrm{x}) + [\gamma\,\beta]\,(\alpha\ \mathrm{x}),$$

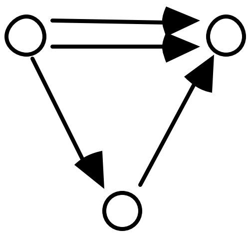
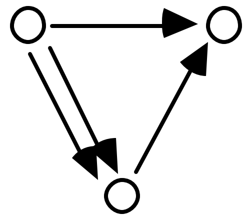
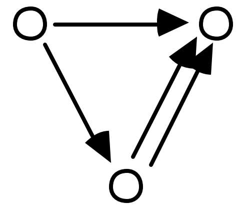
translates into the digraph rule



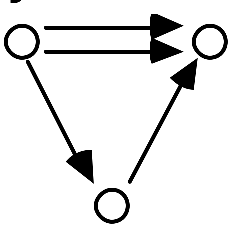
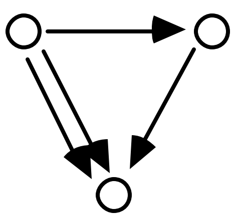
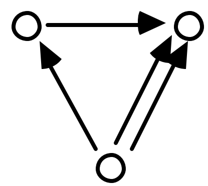
Applications of rule #2

a) Consider the digraph

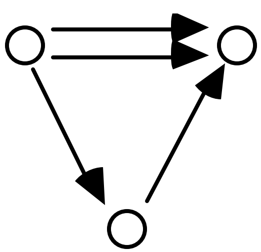


 =  + 

By Rule #1, we find

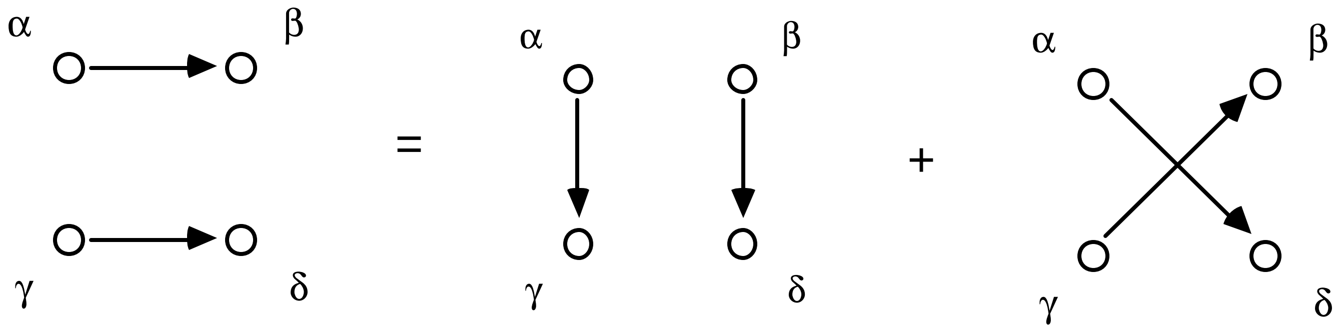
 = –  – 

But all three of these digraphs are exactly the same, hence

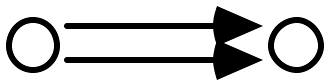
3  = 0

Rule #3. The remaining *syzygy* has the form

$$[\alpha\ \beta]\ [\gamma\ \delta] = [\alpha\ \gamma]\ [\beta\ \delta] + [\gamma\ \beta]\ [\alpha\ \delta].$$



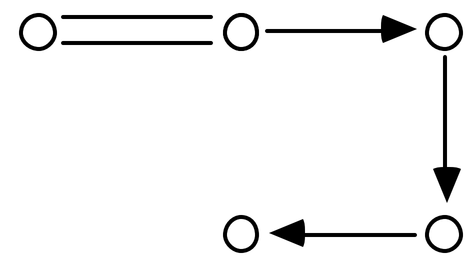
We can denote double bonds which point in the same direction by plain line segments, so



Is equivalent to

Applications of rule #3:

we can show that the digraph

D =  or 

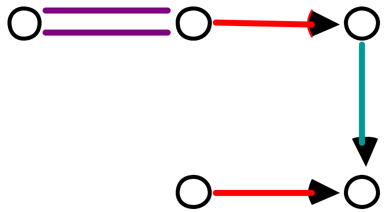
corresponding to the bracket monomial

$[\alpha\ \beta]^2\ [\beta\ \gamma]\ [\gamma\ \delta]\ [\delta\ \varepsilon]\ (\alpha\ x)^{n-2}\ (\beta\ x)^{n-3}\ (\gamma\ x)^{n-2}\ (\delta\ x)^{n-2}\ (\varepsilon\ x)^{n-1}$

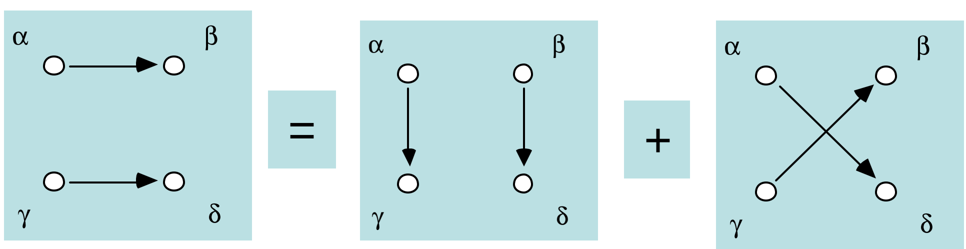
is equivalent to a reducible digraph. so this bracket monomial corresponds to a covariant which is the product of two simpler covariants.

Apply Rule #1 to the bottom dart,

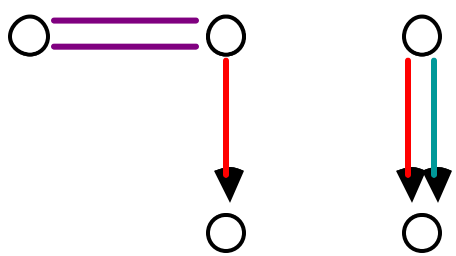
D = –



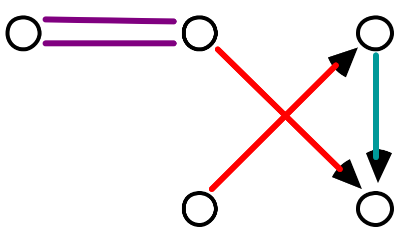
Apply Rule #3 to the top and bottom darts,



= –



–

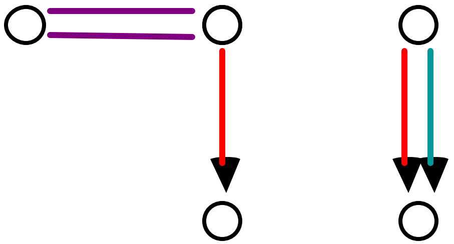


On the right hand side, the first digraph is reducible.

Untangling the second digraph, and reversing the directions of two darts, we get

D = – R – D,

R =



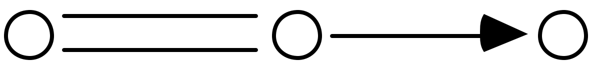
where R is reducible, hence D = – 1/2 R is also reducible.

7. Transvectants.

Given a molecular ion representing a covariant of a binary form, we can obtain new, more complicated molecules by "reacting" with other ions, in particular with free atoms. The invariant theoretic name for this reaction is *transvection*, and it provides a ready mechanism for constructing new covariants from old ones.

Example: Consider the digraph

T =

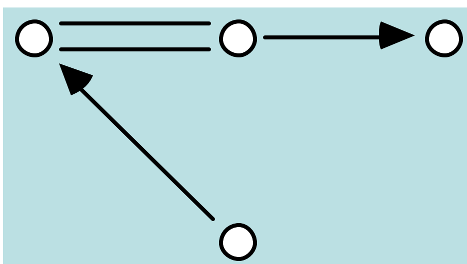


which represents the covariant T of the binary quartic.

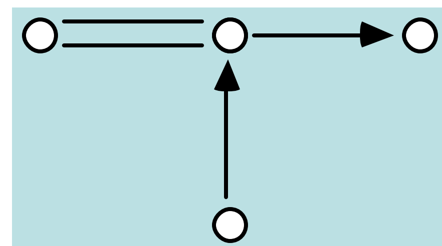
$$[a\ b]^2[b\ g]\ (a\ \mathbf{x})^2(b\ \mathbf{x})(g\ \mathbf{x})^3$$

The first transvectant $(Q,T)^{(1)}$. This will be a linear combination of all possible digraphs which can be obtained by joining a single atom or vertex, representing the quartic Q itself, to the digraph for T with a single dart. There are three possible such digraphs:

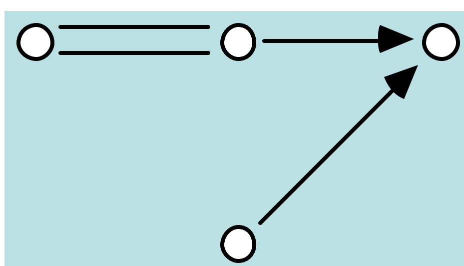
D_1 =



D_2 =



D_3 =



2 free bonds in D_1

1 free bond in D_2

3 free bonds in D_3

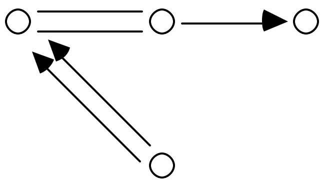
Therefore

$$(Q,T)^{(1)} = 2D_1 + D_2 + 3D_3.$$

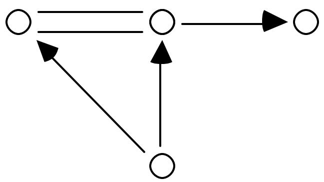
The second transvectant $(Q,T)^{(2)}$: It will be a linear combination of all possible digraphs which can be obtained by joining a single atom by two darts to the digraph for T.

There are five possible such digraphs:

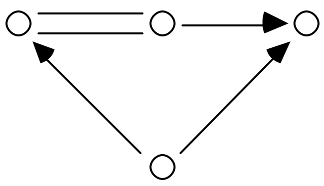
D_1 =



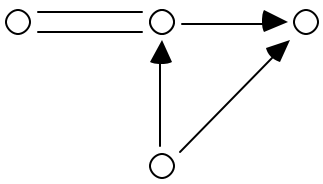
D_2 =



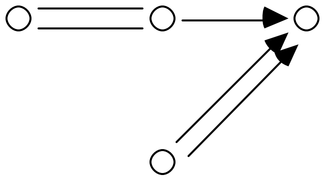
D_3 =



D_4 =



D_5 =



$$(Q,T)^{(2)} = 2D_1 + 4D_2 + 12D_3 + 6D_4 + 3D_5.$$

Finding the Hilbert basis

Gordan's method for constructing a Hilbert basis for the covariants of binary forms:

1. Start with the form Q itself.

2. Use successive transvectants with Q to recursively construct covariants of the next higher degree (using certain rules).

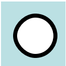
3. Use the syzygies to eliminate redundant covariants.

4. Stop when you don't get anything new.

Gordan proved that this algorithm terminates (1868).

Note: You only need to consider one digraph in each transvectant.

Quadratic case: For a quadratic, we are working in the space of 2-digraphs, so we can attach at most 2 darts to any given vertex.

We begin with the form Q itself. 

There are only two possible transvectants:

 and 

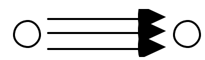
The first is trivial by Rule #1, and the second is the Hessian which is an invariant. we cannot get anything further by transvecting again. We are done.

That the only covariants of a binary quadratic are the form itself and its discriminant.

Cubic case : Turning to the binary cubic, we begin with Q

I_1 

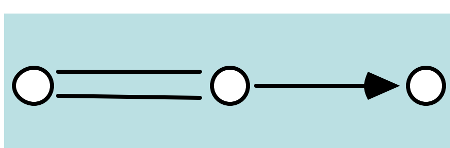
from which we can form three transvectants:



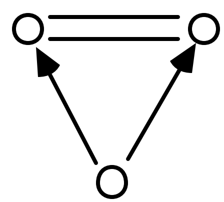
Trivial by Rule #1 — Trivial by Rule #1

I_2 H 

So we can form the two further transvectants:

I_3 

$$T = \frac{1}{16}\left(Q_x\, H_y - Q_y\, H_x\right)$$

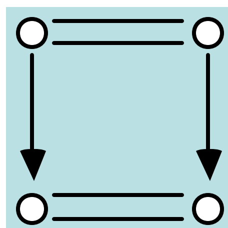


Trivial by Rule #2

Now T has valence three, so we can form three further transvectants.

zero by Rule #1

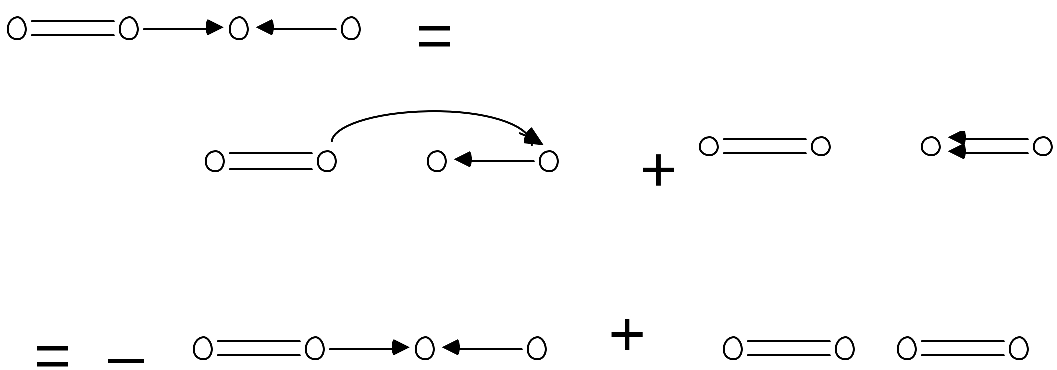
I_4



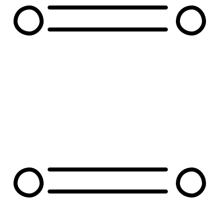
discriminant Δ

Only an invariant is left

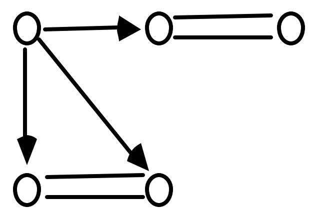
The first is equivalent to $-1/2\ H^2$ by Rule #2:



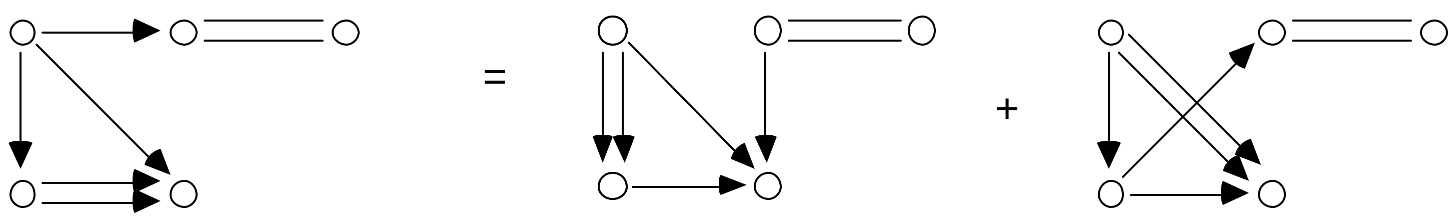
How about the reducible one



corresponding to H^2. Note that each component has valence two, so we can possibly form a non-reducible transvectant $(H^2)^{(3)}$:



Reducible, using Rule #3



$D = -D - D$ Therefore $D=0$

There are no more possible irreducible transvectants.

Basis: Q, the covariants T , H, and invariant Δ .

Example. The same method produces the Hilbert basis of covariants for the binary quartic.

I_1

Q

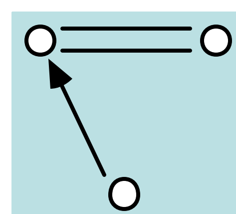
I_2

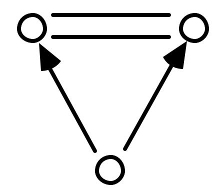
and

Hessian H , invariant i of the quartic.

I_3

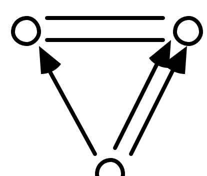
T

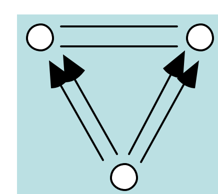




Trivial

Trivial

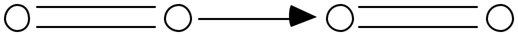


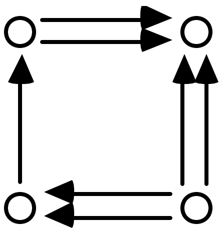


j

As j is an invariant, we can only get nontrivial transvectants from T.

There are four possibilities:





All four are either trivial, or equivalent to reducible digraphs.

A basis for the covariants of the binary quartic Q, H, T, i, j

Refereces

.

[1] Clifford, W., Extract of a letter to Mr. Sylvester from Prof. Clifford of University College, London, *Amer. J. Marh.* **1** (1878), 126-128.

[2] Grace, J.H. and Young, A., *The Algebra of Invariants*, Cambridge Univ. Press, Cambridge, 1903.

[3] Gurevich, G.B., *Foundations of the Theory of Algebraic Invariants*, P. Noordhoff Ltd., Groningen, Holland, 1964.

[4] Kempe, A.B., On the application of Clifford's graphs to ordinary binary quantics, *Proc. London Math. Soc.* **17** (1885), 107-121.

[5] Kung, J.P.S. and Rota, G.-C., "The invariant theory of binary forms", *Bull. Amer. Math. Soc.* **10** (1984), 27-85.

[6] Olver, P.J., Classical invariant theory *London Math. Soc. Cambridge Press, (1997).*

[7] Olver, P.J., Shakiban, Graph theory and Classical invariant theory *Advances in Math, Vol 75, No. 2 (1989), 212-244.*

[8] Sturmfels,B., Algorithms in Invariant Theory, Springer-Verlag, 1993.

[9] Sylvester, J.J., On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics, with three appendices, *Amer. J. Math.* **1** (1878), 64-125.

[10] Wybourne, B.G., *Classical Groups for Physicists*, John Wiley, New York, 1974.