Lecture 3: Discrete Compressed Sensing

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Traditional Paradigm for Signal Processing

- Model signals as bandlimited functions $x(t)$
- Support of $\hat{x}$ contained in $[-\Omega\pi, \Omega\pi]$
- Shannon-Nyquist
- Uniform time samples with spacing $h \leq \frac{1}{\Omega}$ allows for exact reconstruction
- A/D Converters: sample and quantize
- Problem: If Ω is too large we cant build circuitry to sample faithfully at the desired rate

Compressed Sensing

- Compressed Sensing seeks a way out of this dilemma
- Built on two new ingredients
- New model classes for signals
- Signals are sparse in some representation system: basis (frame)
- New meaning of samples
- Sample is a linear functional applied to the signal

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- Two issues: (i) Enough information in y; (ii) How to extract this information: Decoder

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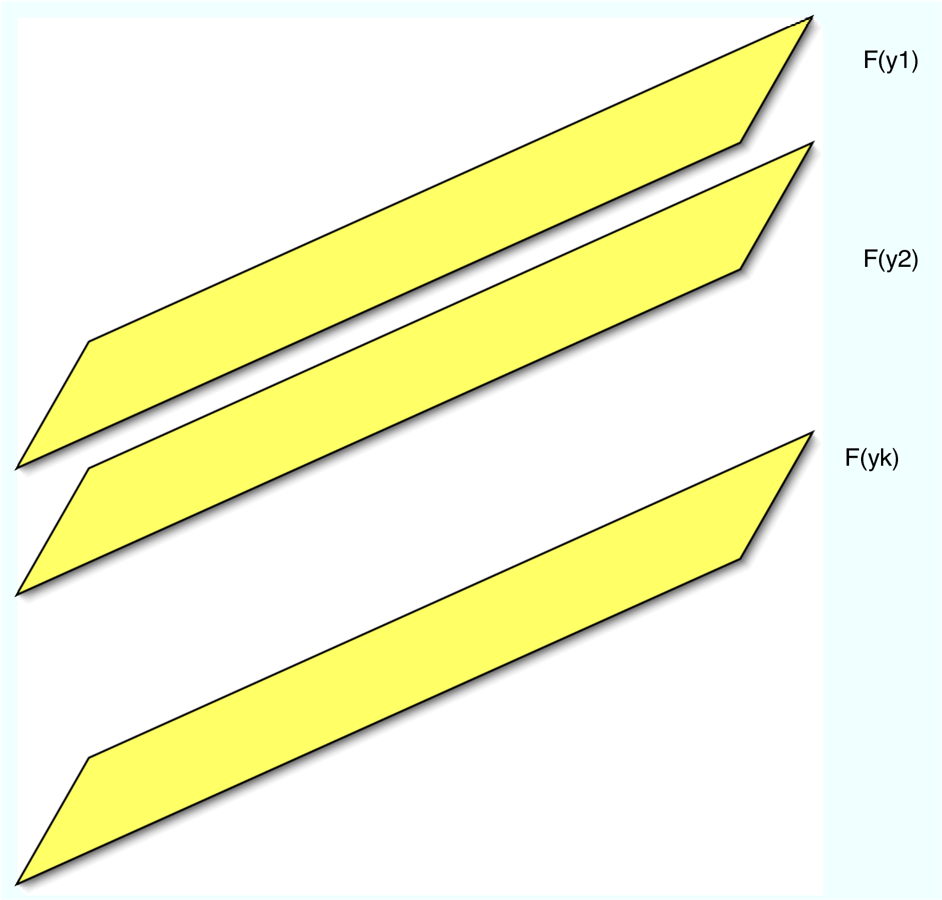

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- Pessimism: all $x \in \mathcal{F}(y)$ are approximated by the same $\bar{x}$

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- For the most part we shall assume the basis is known to us

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- $\Sigma_k := \{x : \ \#\mathrm{supp(x)} \leq \mathrm{k}\}$
- Note that Σ_k is a union of k dimensional subspaces: $\Sigma_k = \cup_{\#(T)=k} X_T$ where $X_T = \{x : \mathrm{supp(x)} \subset \mathrm{T}\}$

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- Note that this is equivalent to $\mathcal{F}(y)$ contains at most one vector from Σ_k for all $y \in I\!R^n$

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Theorem: If Φ is any $n \times N$ matrix, then the following are equivalent:

(i) $\mathcal{F}(y)$ contains at most one element of Σ_k for all $y \in I\!R^n$
(ii) $\Sigma_{2k} \cap \mathcal{N}(\Phi) = \{0\}$,
(iii) For any set T with $\#T = 2k$, the matrix Φ_T has rank $2k$.
(iv) For any set T with $\#T = 2k$, the $2k \times 2k$ matrix $\Phi_T^* \Phi_T$ is invertible: all its eigenvalues are positive.

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- (ii) $\to$ (i): Assume (ii) and let $y \in \mathbb{R}^n$. Suppose that there are two vectors $x_1, x_2 \in \mathbb{R}^N$ with $\Phi(x_1) = \Phi(x_2) = y$. Then $\eta := x_1 - x_2 \in \mathcal{N}$ and by (ii) $\eta = 0$. Thus $x_1 = x_2$

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- Vandermonde matrix. Choose $x_1 < x_2 < \cdots < x_N$

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- However there is a stability problem caused by poor conditioning of $\Phi_T^* \Phi_T$

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- Closely related signal classes are balls in ℓ_p^N

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- Here $\mathcal{A}_n = \{(\Phi, \Delta) : \ \Phi \text{ is n} \times \text{N}\}$

Gelfand widths

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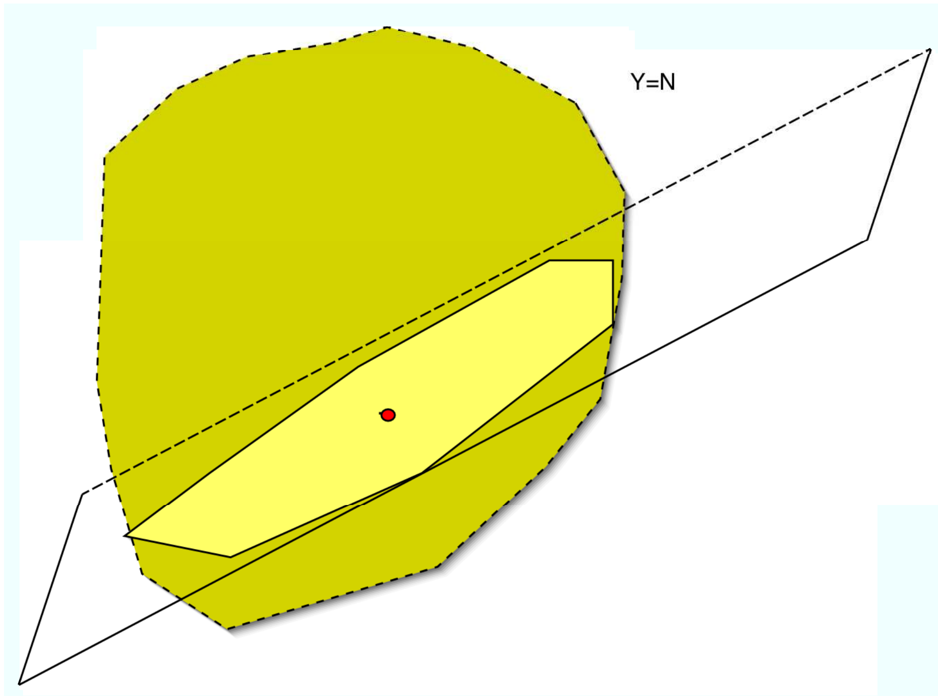
$$d^n(K)_X := \inf_{\mathrm{codim}(Y)=n} \sup_{x \in K \cap Y} \|x\|_X$$

The set $K \cap Y$



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- $Y \leftrightarrow \mathcal{N}$

Proof of Theorem

- Lower inequality: Suppose (Φ, Δ) gives bound $E_n(K)$. Choose $Y := \mathcal{N}$. If $x \in K \cap Y$ then

$$\|x\|_X \leq \max(\|x - \Delta(0)\|_X, \| - x - \Delta(0)\|_X) \leq E_n(K)_X$$

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- Upper inequality: Let Y be an optimal space for Gelfand width. Take an orthonormal basis for $Y^\perp$ and use these as the rows of Φ. Choose a decoder Δ so that $\Delta(y)$ is in K whenever $K \cap \mathcal{F}(y) \neq 0$. Now if $x \in K$ then $x - \Delta(\Phi(x))$ is in $K - K \subset CK$. It is also in $\mathcal{N}$. Hence $C^{-1}(x - \Delta(\Phi(x))$ is in $K \cap \mathcal{N}$. Hence

$$C^{-1}\|x - \Delta(\Phi(x))\| \leq d^n(K)_C$$

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$$C_1\Psi(n, N, p, q) \le d^n(K)_{\ell_p} \le C_2\Psi(n, N, p.q),$$

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Optimality on ℓ_p^N classes

- The asymptotic behavior of $E_n(K)_X$ is known for all $K = U(\ell_p^N)$ in all $X = \ell_q^N$
- Solved in 1970's and 1980's in Approximation Theory and Finite Dimensional Geometry
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- $\Psi(n, N, p, q) := [\min(1, N^{1-1/p} n^{-1/2})]^{\frac{1/p-1/q}{1/p-1/2}}, \quad 1 \leq n \leq N, \ 1 < p < q \leq 2$

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- These theorems were thought to be very deep at the time. We shall give rather simple proofs of them using compressed sensing.

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- In other words we get the same performance as if we took the N measurements (of the coordinates of x and then retained the largest k of them
- This is the advantage of compressed sensing: the number of measurements we need to make is only slightly larger than k

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- We will be interested in $X = \ell_q^N$
- Problem: for a given X and size $n \times N$ find the largest values of k for which we have instance-optimality and the encoder-decoder pairs (Φ, Δ) which admit these values of k

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- Elements in the null space should have no structure - look like noise

Main Result

Theorem (Cohen-Dahmen-DeVore) Given an $n \times N$ matrix Φ, a norm $\| \cdot \|_X$ and a value of k, then to have instance optimality in X with a constant C_0 a necessary and sufficient is that Φ has the null space of order $2k$ with a constant C_1 where $C_1 = C_0/2$ in the sufficient part and $C_1 = C_0$ in the necessary part.

Proof of Sufficiency

- define a decoder Δ for Φ

$$\Delta(y) := \operatorname*{Argmin}_{z \in \mathcal{F}(y)} \sigma_k(z)_X$$

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- The last inequality uses the fact that $\Delta(\Phi(x))$ minimizes $\sigma_k(z)$ over $\mathcal{F}(y)$