Wavelets Compression

Ronald DeVore

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- Images have an inherent complexity and should be classified according to this complexity
- This classification will determine how well we can do in such tasks as compression and denoising

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- $D_n(E_n(f))$ is our compressed image

The Issues

1. The metric: We shall employ typical compression convention and use least squares metric, i.e. $L_2([0,1]^2)$. This discrete form of this metric is equivalent to PSNR

2. The classes

3. Determine Entropy of Classes

4. Build near optimal Encoders/Decoders

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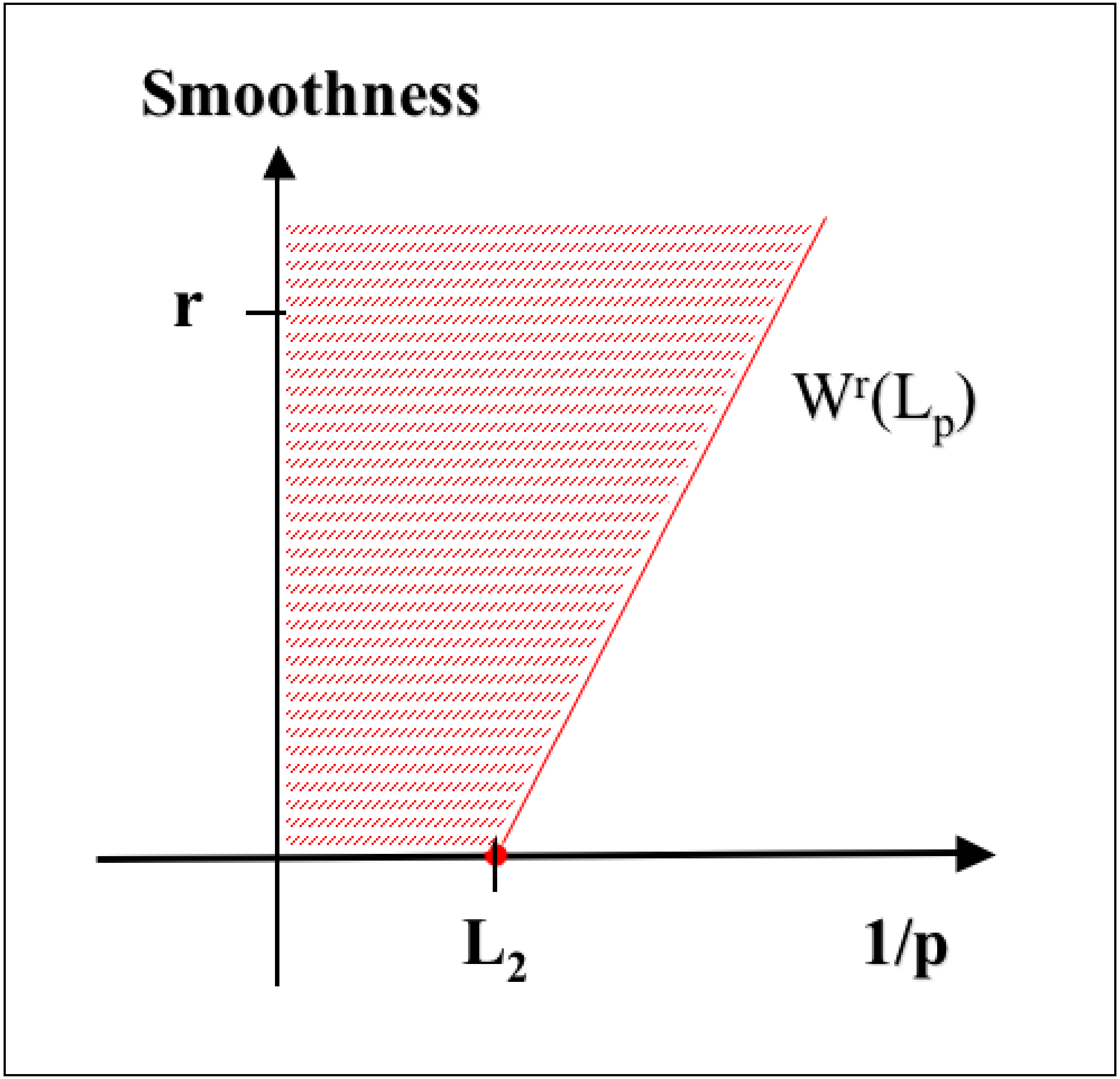
- BV - functions of Bounded Variation: $s = 1\ p = 1$

Sobolev embedding $d = 2$



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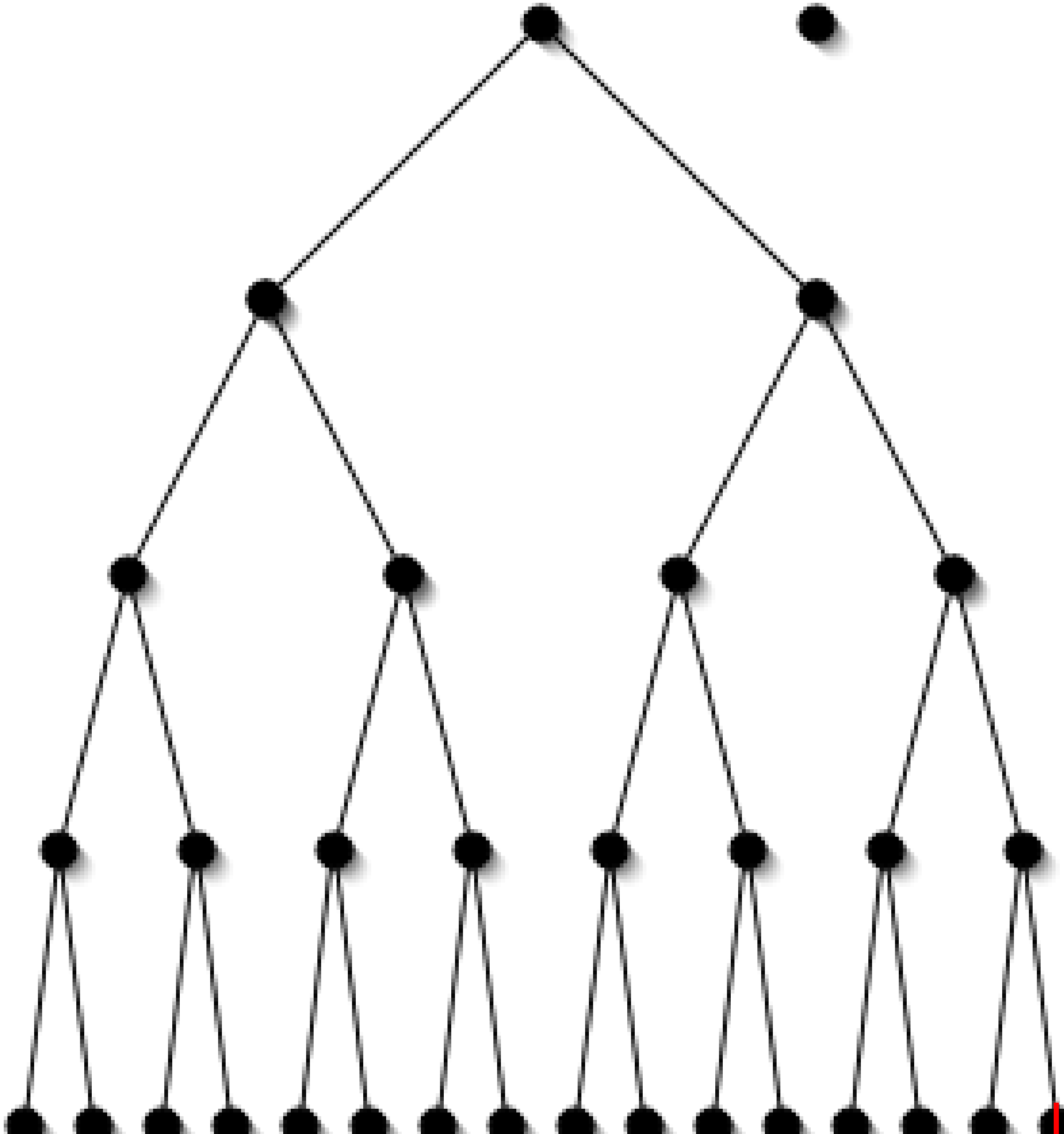
- Indexed on dyadic intervals: binary tree

Wavelet Tree



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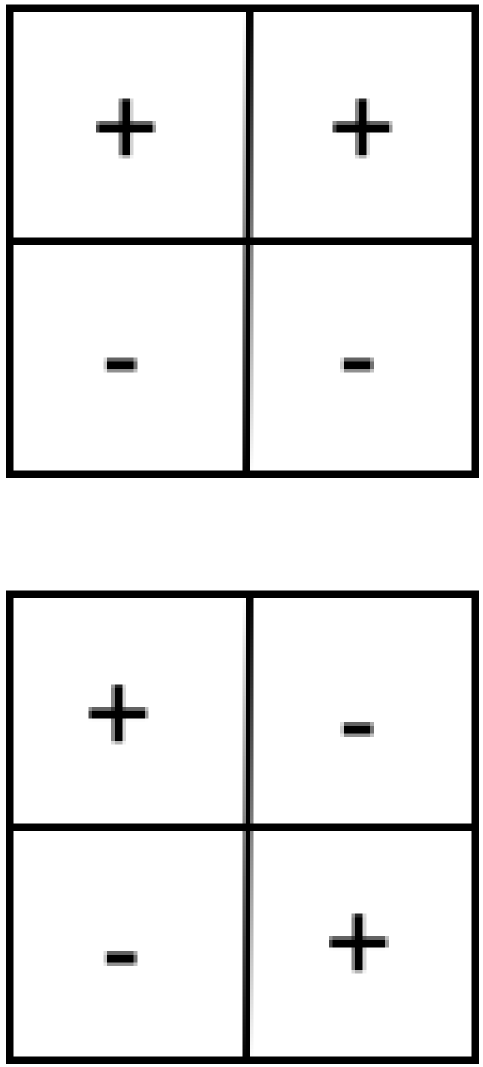
- On a finite domain there are four wavelets at the starting level and three on each other level

Example: Haar Functions



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- The same sequence is in ℓ_1 if and only if $s > 1$

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- This is best approximation using n terms
- Threshold at η: retains terms with coefficients larger than η

$$S_\eta(f) = \sum_{I \in \Lambda_\eta(f)} c_I(f)\psi_I$$

$$\Lambda_\eta(f) := \{I : |c_I(f)| > \eta\}$$

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- The same result holds if f is of bounded variation

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- Practically: for images scales are limited by pixel resolution
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- Using certain special wavelets and the fact that pixels are quantized the number of bits for complete resolution is finite

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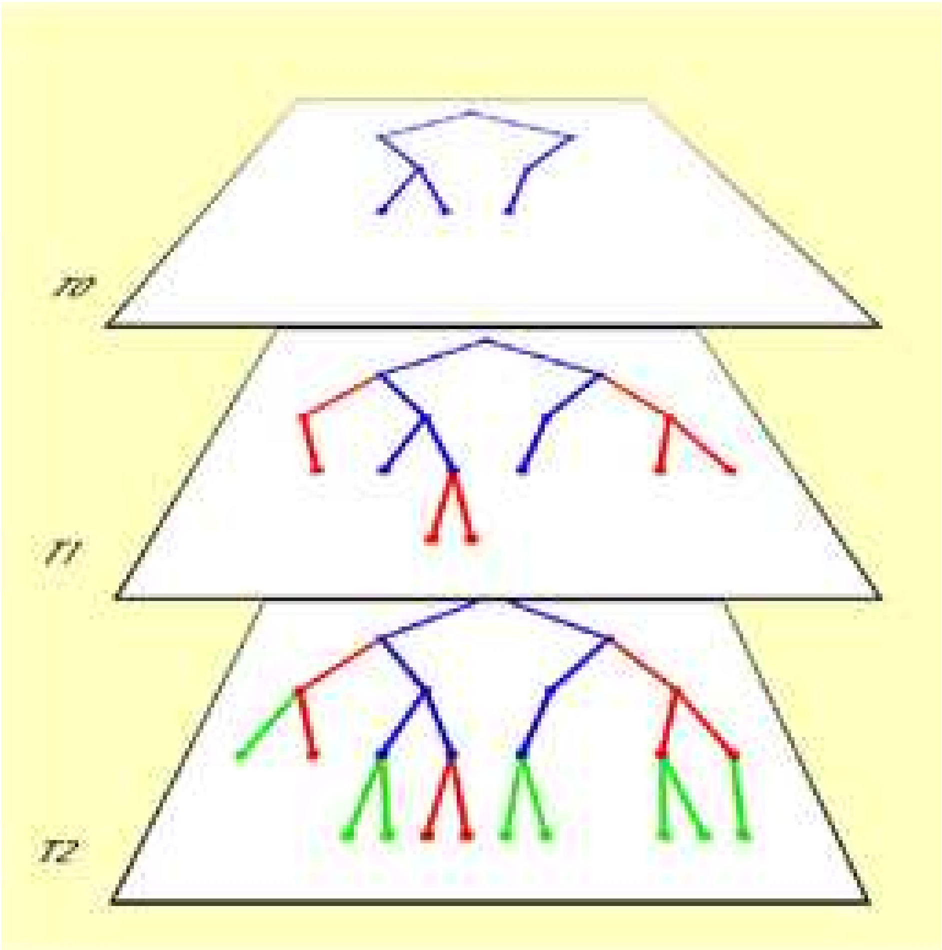
Progressive Trees



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- $c = \pm(b_1(c)2^{-1} + b_2(c)2^{-2} + \dots)$
- Note, if $I \in T_{k+1}(f) \setminus T_k(f)$ then $b_1(f) = b_2(f) = \cdots = b_k(f) = 0$

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- Note that after bits at level k have been received each wavelet coefficient of f is known to accuracy 2^{-k}

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- Besov smoothness order s is an exact predictor of tree encoder performance $O(n^{-s/2})$

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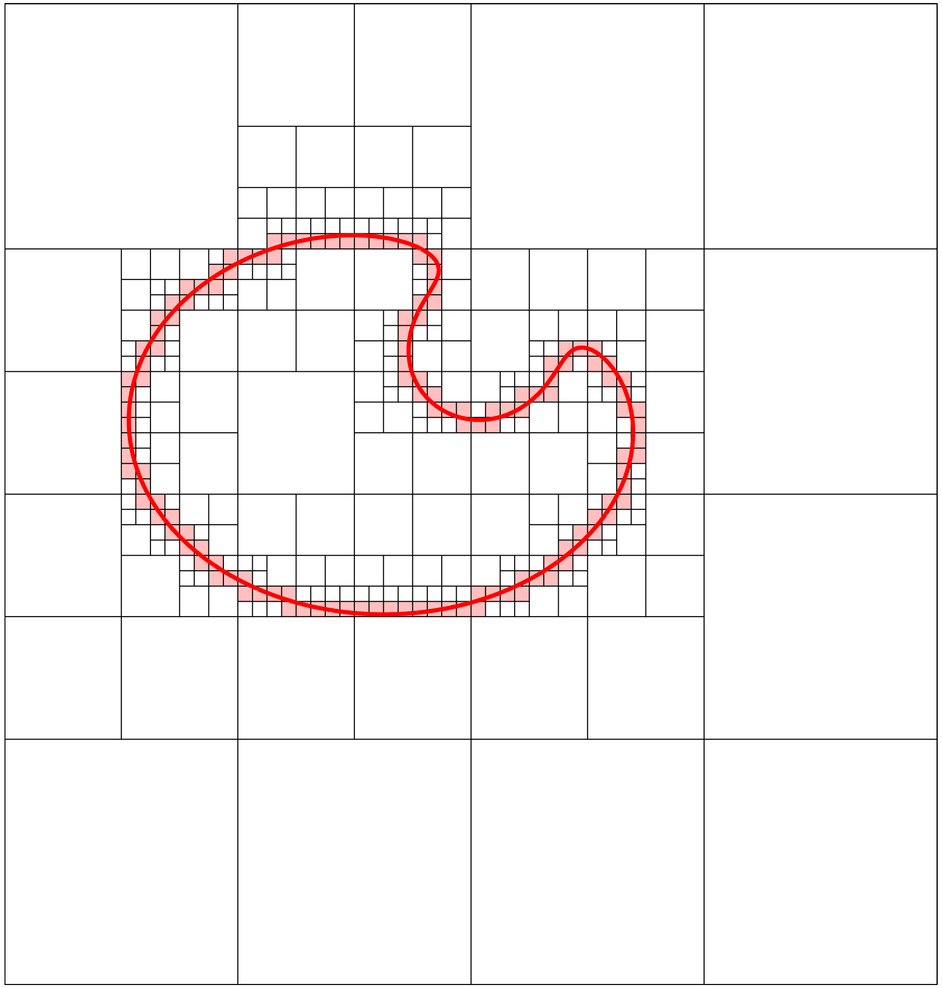
- Toy Problem: Horizon function

Horizon approximation



Possible remedies

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- Frames and redundant systems

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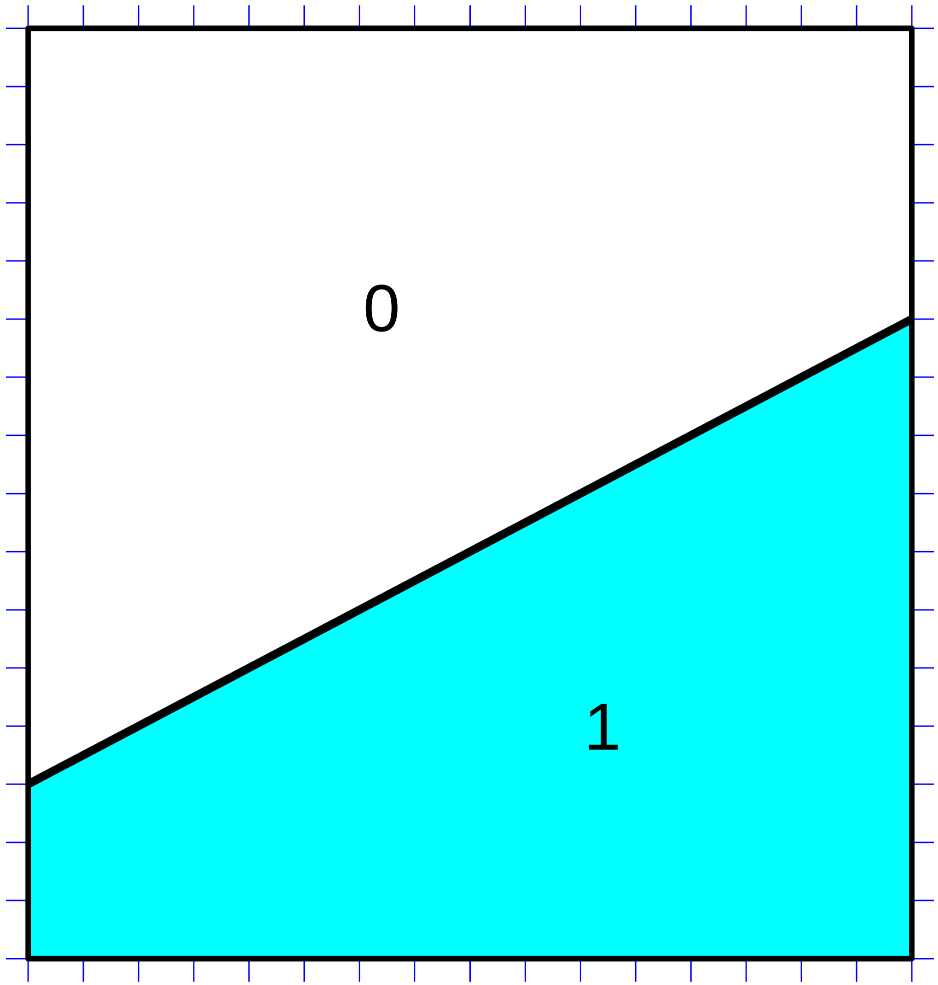
- A wedgelet is a piecewise constant function taking the values $0, 1$
- Demarcation is given by a line connecting grid points on boundary of I with spacing 2^{-2j}

Picture of a wedgelet



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- how to do this in a practical encoder

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- Idea is to decompose domain into regions done by either wavelets or wedgelets
- Ornate wavelet tree with label on how wavelet coefficients are to be computed
- interior nodes compute wavelet coefficients in standard way

BRW Continued

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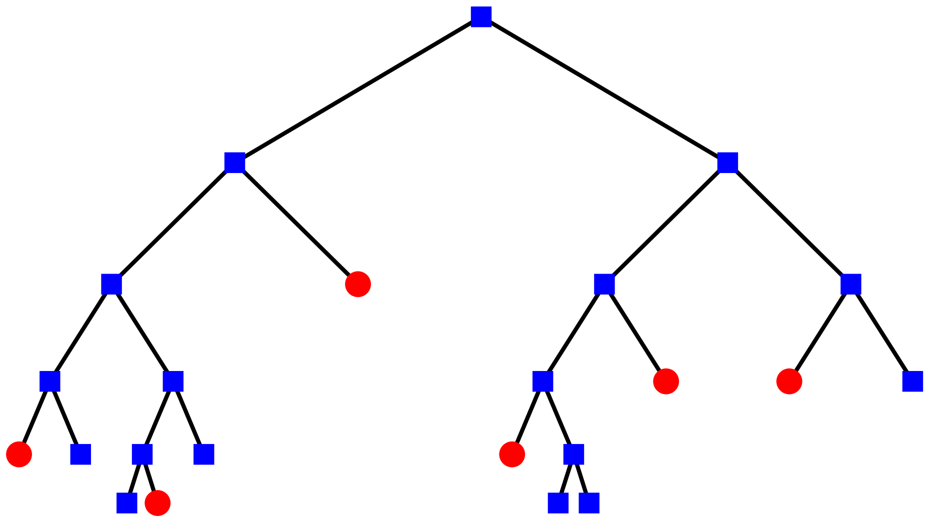
BRW Continued

- If leave (final node) of tree is ornated with a wavelet then wavelet coefficients for all nodes below leave are given value zero
- If leave is ornated with wedgelet, all coefficients below this node are computed as wavelet coefficients of that wedgelet (wedgeprint)

Wedgelet-wavelet tree

red = Wedgelets   blue = Wavelets



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- These classes should model the anisotropies found in real world images
- Horizon models are too simplistic
- Once these new classes are clearly defined then encoders can be designed to perform at near optimal compression rates for these classes