Object Recognition

Goal: recognize when two visual objects are equivalent

\[ g : \mathcal{O} \longleftrightarrow \tilde{\mathcal{O}} \]

Symmetry

Goal: find all self-equivalences of a visual object

\[ g : \mathcal{O} \longleftrightarrow \mathcal{O} \]
Equivalence, Symmetry & Groups

*Basic fact:*

Equivalence and symmetry transformations

\[ g : \mathcal{O} \longleftrightarrow \tilde{\mathcal{O}} \]

belong to a group:

\[ g \in G \]
## Computer Vision Groups

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$\sigma 3$
**Projective**  
*Preserves cross-ratios*

\[(x, y) \mapsto \left( \frac{ax + by + c}{gx + hy + j}, \frac{dx + ey + f}{gx + hy + j} \right)\]

\[
\det A = \det \begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & j
\end{pmatrix} = 1
\]

**Camera Rotations**

Projective orthogonal transformations:

\[
A = \begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & j
\end{pmatrix} \in \text{SO}(3)
\]

**Video Groups**

\[(x, y, t) \mapsto (\tilde{x}, \tilde{y}, \tilde{t})\]

*e.g. Galilean boosts (motion tracking)*

\[(x, y, t) \mapsto (x + at, y + bt, t)\]
Complications

- Occlusion
- Ducks \approx rabbits — Åström
- Outlines of 3D objects
- Bending, warping, etc.
  — pseudo-groups
- Thatcher illusion
Mathematical Setting

Ambient space:

\[ M = \mathbb{R}^n, \quad n = 2, 3, \ldots \quad \text{(manifold)} \]

Object:

\[ N \subset M \quad \text{submanifold} \]

Equivalences:

\[ G \quad \text{finite-dimensional Lie group} \]

acting on \( M \)

Basic equivalence problem:

\[ S \cong \overline{S} \iff \overline{S} = g \cdot S \quad \text{for} \quad g \in G \]

Symmetry (isotropy) subgroup:

\[ G_S = \{ g \in G \mid g \cdot S = S \} \subset G \]
Equivalence & Signature

Cartan’s main idea:

The equivalence and symmetry properties of submanifolds are entirely prescribed by their differential invariants.
Examples of Differential Invariants

Euclidean plane curves: \( C \subset \mathbb{R}^2 \) \((y = u(x))\)
\[
\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \quad \text{— Euclidean curvature}
\]
\(\kappa_s, \kappa_{ss}, \ldots \quad \text{— derivatives w.r.t. arc length}\)
\[
ds = \sqrt{1 + u_x^2} \; dx
\]

Euclidean space curves: \( C \subset \mathbb{R}^3 \)
\[
\kappa, \kappa_s, \kappa_{ss}, \ldots \quad \text{— curvature}
\]
\[
\tau, \tau_s, \tau_{ss}, \ldots \quad \text{— torsion}
\]

Equi-affine plane curves: \( C \subset \mathbb{R}^2 \)
\[
\kappa = \frac{5 u_{xx} u_{xxxx} - 3 u_{xxx}^2}{9 u_{xx}^{8/3}} \quad \text{— equi-affine curvature}
\]
\(\kappa_s, \kappa_{ss}, \ldots \quad \text{— derivatives w.r.t. equi-affine arc length}\)
\[
ds = \sqrt[3]{u_{xx}} \; dx
\]
Projective plane curves: \( C \subset \mathbb{RP}^2 \)
\[ \kappa = F(u^{(7)}) \] — projective curvature
\[ \kappa_s, \kappa_{ss}, \ldots \] — derivatives w.r.t. the projective arc length \( ds = P(u^{(5)}) \, dx \)

Euclidean surfaces: \( S \subset \mathbb{R}^3 \)
\[ K, H \] — Gauss and mean curvature
\[ K_1, K_2, H_1, H_2, K_{1,1}, \ldots \] — invariant derivatives w.r.t. the Frenet coframe \( \omega_1, \omega_2 \)

Equi-affine surfaces: \( S \subset \mathbb{R}^3 \)
\[ T \] — Pick invariant
\[ K_1, K_2, H_1, H_2, K_{1,1}, \ldots \] — invariant derivatives w.r.t. the equi-affine coframe \( \omega_1, \omega_2 \)
The Basis Theorem

Theorem. For “any” group $G$ acting on $p$-dimensional sub-manifolds $N \subset M$, there exists a finite generating set of differential invariants $I_1, \ldots, I_k$ and invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, so that every differential invariant

$$I = F(\ldots, \mathcal{D}_j I_\kappa, \ldots)$$

can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_j I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_i} I_\kappa$$

- Tresse
- Ovsiannikov, O \quad \text{dim} < \infty
- Kumpera, O–Pohjanpelto \quad \text{dim} = \infty
Complications

Curves (one-dimensional submanifolds) are well understood: $k = \dim M - 1$; no syzygies. (M. Green)

For higher dimensional submanifolds (surfaces):

- The number of generating differential invariants is difficult to predict in advance.
- The invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ do not commute.
- The differentiated invariants may be subject to certain functional relations or syzygies

$$S(\ldots, \mathcal{D}_J I_\kappa, \ldots) \equiv 0.$$ 

**Ex:** the Codazzi equation relating derivatives of the Gauss and mean curvatures of a Euclidean surface.
The method of moving frames (Cartan), especially as extended and generalized by O–Fels–Kogan–Pohjanpelto–... provides a completely constructive calculus for finding the differential invariants, invariant differential forms and differential operators, commutators, recurrence formulae, syzygies, signatures, invariant variational problems, etc. ⭐⭐
Equivalence and Invariants

- Equivalent submanifolds $N \approx \tilde{N}$ have the same invariants: $I = \tilde{I}$.

However, unless an invariant is constant

\[ \kappa = 2 \quad \iff \quad \tilde{\kappa} = 2 \]

\[ \implies \text{Constant curvature submanifolds} \]

it carries little information in isolation, since equivalence maps can drastically alter its dependence on the submanifold coordinates.

\[ \text{e.g.} \quad \kappa = x^3 \quad \text{versus} \quad \tilde{\kappa} = \sinh x \]

However, a syzygy

\[ I_k(x) = \Phi(I_1(x), \ldots, I_{k-1}(x)) \]

among multiple invariants is intrinsic

\[ \text{e.g.} \quad \tau = \kappa^3 - 1 \quad \iff \quad \tilde{\tau} = \tilde{\kappa}^3 - 1 \]
Equivalence & Syzygies

**Theorem.** (Cartan)
Two submanifolds are (locally) equivalent if and only if they have the same syzygies among *all* their differential invariants.

- Universal syzygies — Codazzi
- Distinguishing syzygies.

**Proof**: 

Cartan’s technique of the graph:
Construct the graph of the equivalence map as the solution to a (Frobenius) integrable differential system, which can be integrated by solving ordinary differential equations.
Finiteness of Syzygies

★ ★ Higher order syzygies are consequences of a finite number of the lowest order syzygies.

Example. If
\[ \kappa_s = H(\kappa) \]
then
\[ \kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa) \kappa_s = H'(\kappa) H(\kappa) \]

Thus, for Euclidean (or equi-affine or projective or ...) plane curves we need only know a single syzygy between \( \kappa \) and \( \kappa_s \) in order to establish equivalence!
The Signature Map

The generating syzygies are encoded by the signature map

$$\Sigma : N \rightarrow S$$

parametrized by the fundamental differential invariants:

$$\Sigma(x) = (I_1(x), \ldots, I_m(x)) \quad \text{for} \quad x \in N.$$ 

We call

$$S = \text{Im} \; \Sigma$$

the signature subset (or submanifold) of $N$. 
The Signature Theorem

**Theorem.** Two submanifolds are equivalent

\[ \bar{N} = g \cdot N \]

if and only if their signatures are identical

\[ S = \bar{S} \]
Differential Invariant Signatures

Plane Curves:

The signature curve \( S \subset \mathbb{R}^2 \) of a plane curve \( C \subset \mathbb{R}^2 \) is parametrized by the first two differential invariants \( \kappa \) and \( \kappa_s \):

\[
S = \left\{ \left( \kappa, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2
\]

Theorem. Two curves \( C \) and \( \overline{C} \) are equivalent

\[
\overline{C} = g \cdot C
\]

if and only if their signature curves are identical

\[
\overline{S} = S
\]
More Differential Invariant Signatures

Space Curves:
The signature curve of a space curve $C \subset \mathbb{R}^3$ is parametrized by

$$S = \left\{ \left( \kappa, \frac{d\kappa}{ds}, \tau \right) \right\} \subset \mathbb{R}^3$$

$\implies$ DNA recognition (Shakiban)

Euclidean Surfaces:
The signature surface of a (generic) surface $N \subset \mathbb{R}^3$ under the Euclidean group is parametrized by

$$S = \left\{ \left( K, H, K_1, K_2 \right) \right\} \subset \mathbb{R}^4$$

$\implies$ umbilic points
Advantages of the Signature

- Completely local
- Applies to curves, surfaces and higher dimensional submanifolds
- Symmetries and approximate symmetries
- Occlusions and reconstruction
Symmetry Groups

Symmetry subgroup of a submanifold:

\[ G_N = \{ g \in G \mid g \cdot N = N \} \subset G \]

Theorem. The dimension of the symmetry group of a (regular) submanifold equals the codimension of its signature:

\[ \dim G_N = \dim N - \dim S \]

Corollary.

\[ 0 \leq \dim G_N \leq p = \dim N \]

\[ \implies \text{Only totally singular submanifolds can have larger symmetry groups!} \]
Maximally Symmetric Submanifolds

**Theorem.** The following are equivalent:
- The submanifold $N$ has a $p$-dimensional symmetry group
- The signature $S$ degenerates to a point:
  \[ \dim S = 0 \]
- The submanifold has all constant differential invariants
- \( N = H \cdot \{ z_0 \} \) is the orbit of a $p$-dimensional subgroup $H \subset G$

\[ \Rightarrow \] In Euclidean geometry, these are the circles, straight lines, spheres & planes.

\[ \Rightarrow \] In equi-affine plane geometry, these are the conic sections.
Discrete Symmetries

Definition. The index of a submanifold $N$ equals the number of points in $\mathcal{C}$ which map to a generic point of its signature $\mathcal{S}$:

$$\iota_N = \min \left\{ \# \Sigma^{-1}\{w\} \mid w \in \mathcal{S} \right\}$$

$\implies$ Self-intersections

Theorem. The number of symmetries of $N$ equals its index:

$$\# G_N = \iota_N$$

$\implies$ Approximate symmetries
Signature Metrics

- Hausdorff
- Monge–Kantorvich transport metric
- Electrostatic repulsion
- Latent semantic analysis (Shakiban)
- Histograms (Kemper–Boutin)
- Geodesic distance
- Diffusion metric
- Gromov–Hausdorff
Noise Reduction

The key objection to the differential invariant signature is its dependence on (high order) derivatives, and hence sensitivity to noise.

---

Noise Reduction Strategy #1: Smoothing

Apply (group-invariant) smoothing to the object.

Curvature flows:

\[ C_t = -\kappa \mathbf{n} \]
\[ u_t = -\frac{u_{xx}}{1 + u_x^2} \]

\[ \implies \text{Hamilton–Gage–Grayson} \]
Noise Reduction Strategy #2: Use lower order invariants to construct a signature.

Joint Invariants

A joint invariant is an invariant of the \( k \)-fold Cartesian product action of \( G \) on \( M \times \cdots \times M \):

\[
I(g \cdot z_1, \ldots, g \cdot z_k) = I(z_1, \ldots, z_k)
\]

A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points \( z_1, \ldots, z_k \in N \) on the submanifold:

\[
I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) = I(z_1^{(n)}, \ldots, z_k^{(n)})
\]
Joint Euclidean Invariants

**Theorem.** Every joint Euclidean invariant is a function of the interpoint distances

\[ d(z_i, z_j) = \| z_i - z_j \| \]

---

Joint Equi–Affine Invariants

**Theorem.** Every joint planar equi–affine invariant is a function of the triangular areas

\[ [i \ j \ k] = \frac{1}{2} (z_i - z_j) \wedge (z_i - z_k) \]
Joint Projective Invariants

**Theorem.** Every joint projective invariant is a function of the planar cross-ratios

\[ C(z_i, z_j, z_k, z_l, z_m) = \frac{AB}{CD} \]
Euclidean Joint Differential Invariants

— Planar Curves

• One–point

\[ \kappa = \frac{\dot{z} \wedge \ddot{z}}{\| \dot{z} \|^3} \]

\( \Rightarrow \) curvature

• Two–point

\( \Rightarrow \) distances \[ \| z_1 - z_0 \| \]

\( \Rightarrow \) tangent angles \[ \phi_k = \psi(z_1 - z_0, \dot{z}_k) \]
Equi–Affine Joint Differential Invariants  
— Planar Curves

• One–point

⇒ affine curvature

\[
\kappa = \frac{(z_t \wedge z_{tttt}) + 4(z_{tt} \wedge z_{ttt})}{3(z_t \wedge z_{tt})^{5/3}} - \frac{5(z_t \wedge z_{ttt})^2}{9(z_t \wedge z_{tt})^{8/3}}
\]

\[
= z_s \wedge z_{ss}
\]

• Two–point

⇒ tangent triangle area ratio

\[
\frac{\dot{z}_0 \wedge \ddot{z}_0}{[(z_1 - z_0) \wedge \dot{z}_0]^3} = \frac{\dot{\mathbf{0}} \dddot{\mathbf{0}}}{\begin{bmatrix} \dot{0} & \dddot{0} \\ 0 & 1 & \dot{0} \end{bmatrix}^3}
\]

• Three–point

⇒ triangle area

\[
\frac{1}{2} (z_1 - z_0) \wedge (z_2 - z_0) = \frac{1}{2} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}
\]
Projective Joint Differential Invariants
— Planar Curves

- One–point
  ⇒ projective curvature
  \[ \kappa = \ldots \]

- Two–point
  ⇒ tangent triangle area ratio
  \[
  \begin{vmatrix}
  0 & 1 & \dot{0} \\
  \end{vmatrix}^3 \begin{vmatrix}
  \dot{1} & \dot{1} \\
  \end{vmatrix}
  \begin{vmatrix}
  0 & 1 & \dot{1} \\
  \end{vmatrix}^3 \begin{vmatrix}
  \dot{0} & \dot{0} \\
  \end{vmatrix}
  \]

- Three–point
  ⇒ tangent triangle ratio
  \[
  \begin{vmatrix}
  0 & 2 & \dot{0} \\
  \end{vmatrix} \begin{vmatrix}
  0 & 1 & \dot{1} \\
  \end{vmatrix} \begin{vmatrix}
  1 & 2 & \dot{2} \\
  \end{vmatrix}
  \begin{vmatrix}
  0 & 1 & \dot{0} \\
  \end{vmatrix} \begin{vmatrix}
  1 & 2 & \dot{1} \\
  \end{vmatrix} \begin{vmatrix}
  0 & 2 & \dot{2} \\
  \end{vmatrix} \cdot
  \]

- Four–point
  ⇒ area cross–ratio
  \[
  \begin{vmatrix}
  0 & 1 & 2 \\
  \end{vmatrix} \begin{vmatrix}
  0 & 3 & 4 \\
  \end{vmatrix}
  \begin{vmatrix}
  0 & 1 & 3 \\
  \end{vmatrix} \begin{vmatrix}
  0 & 2 & 4 \\
  \end{vmatrix}
  \]

[σ 31]
Joint Euclidean Signature

For the Euclidean group $G = \text{SE}(2)$ acting on curves $C \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) we need at least four points

$$z_0, z_1, z_2, z_3 \in C$$

Joint invariants:

$$
\begin{align*}
    a &= \|z_1 - z_0\| \\
    b &= \|z_2 - z_0\| \\
    c &= \|z_3 - z_0\| \\
    d &= \|z_2 - z_1\| \\
    e &= \|z_3 - z_1\| \\
    f &= \|z_3 - z_2\|
\end{align*}
$$

$\implies$ six functions of four variables

Joint Signature: $\Sigma : C^4 \longrightarrow S \subset \mathbb{R}^6$

$\dim S = 4$ $\implies$ two syzygies

$$
\begin{align*}
\Phi_1(a, b, c, d, e, f) &= 0 \\
\Phi_2(a, b, c, d, e, f) &= 0
\end{align*}
$$

Universal Cayley–Menger syzygy:

$$
\det \begin{vmatrix}
2a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\
2b^2 & b^2 + c^2 - f^2 & 2c^2 \\
2c^2 & a^2 + c^2 - e^2 & b^2 + c^2 - f^2
\end{vmatrix} = 0
$$

$\iff C \subset \mathbb{R}^2$

The joint invariant signature encodes the distance matrix!
Four-Point Euclidean Joint Signature
Joint Equi–Affine Signature

Requires 7 triangular areas:

\[ [0 \ 1 \ 2], [0 \ 1 \ 3], [0 \ 1 \ 4], [0 \ 1 \ 5], [0 \ 2 \ 3], [0 \ 2 \ 4], [0 \ 2 \ 5] \]
Joint Invariant Signatures

- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies (Boutin)
- It includes the differential invariant signature and semi-differential invariant signatures as its “coalescent boundaries”.
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.
Histograms

**Theorem.** (Boutin–Kemper)

All point configurations

\[(z_1, \ldots, z_n) \in M^{\times n} \setminus V\]

lying outside a certain algebraic subvariety \(V\) are uniquely determined by their Euclidean distance histograms.
Invariant Numerical Approximations

$G$ — Lie group acting on $M$

Basic Idea:

Every invariant finite difference approximation to a differential invariant must expressible in terms of the joint invariants of the transformation group.

Differential Invariant

$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$

Joint Invariant

$J(g \cdot z_0, \ldots, g \cdot z_k) = J(z_0, \ldots, z_k)$

Semi-differential invariant = Joint differential invariant

★★ Approximate differential invariants by joint invariants
Euclidean Invariants

Joint Euclidean invariant:

\[ d(z, w) = \| z - w \| \]

Euclidean curvature:

\[ \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \]

Euclidean arc length:

\[ ds = \sqrt{1 + u_x^2} \, dx \]

Higher order differential invariants:

\[ \kappa_s = \frac{d\kappa}{ds}, \quad \kappa_{ss} = \frac{d^2\kappa}{ds^2}, \quad \ldots \]

Euclidean–invariant differential equation:

\[ F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0 \]
Numerical approximation to curvature

Heron’s formula

\[ \tilde{\kappa}(A, B, C) = 4 \frac{\Delta}{abc} = 4 \frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc} \]

\[ s = \frac{a + b + c}{2} \quad \text{— semi-perimeter} \]
Higher order invariants

\[ \kappa_s = \frac{d\kappa}{ds} \]

Invariant finite difference approximation:

\[ \tilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = \frac{\tilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \tilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{d(P_i, P_{i-1})} \]

Unbiased centered difference:

\[ \tilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_j) = \frac{\tilde{\kappa}(P_i, P_{i+1}, P_{i+2}) - \tilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{d(P_{i+1}, P_{i-1})} \]

Better approximation (M. Boutin):

\[ \tilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = 3 \frac{\tilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \tilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{d_{i-2} + 2d_{i-1} + 2d_i + d_{i+1}} \]

\[ d_j = d(P_j, P_{j+1}) \]
Affine Joint Invariants

\[ \mathbf{x} \to A\mathbf{x} + b \quad \text{det} \, A = 1 \]

Area is the fundamental joint affine invariant

\[ [i \, j \, k] = (P_i - P_j) \wedge (P_i - P_k) \]

\[
\begin{vmatrix}
  x_i & y_i & 1 \\
  x_j & y_j & 1 \\
  x_k & y_k & 1 \\
\end{vmatrix}
\]

= Area of parallelogram

= 2 \times \text{Area of triangle } \Delta(P_i, P_j, P_k)

Syzygies:

\[ [i \, j \, l] + [j \, k \, l] = [i \, j \, k] + [i \, k \, l] \]

\[ [i \, j \, k] [i \, l \, m] - [i \, j \, l] [i \, k \, m] + [i \, j \, m] [i \, k \, l] = 0 \]
Affine Differential Invariants

Affine curvature

\[ \kappa = \frac{3u_{xx}u_{xxxx} - 5u^2_{xxx}}{9(u_{xx})^{8/3}} \]

Affine arc length

\[ ds = \sqrt[3]{u_{xx}} \, dx \]

Higher order affine invariants:

\[ \kappa_s = \frac{d\kappa}{ds}, \quad \kappa_{ss} = \frac{d^2\kappa}{ds^2}, \ldots \]
Conic Sections

\[ Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \]

Affine curvature:

\[ \kappa = \frac{S}{T^{2/3}} \]

\[ S = AC - B^2 = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix} \]

\[ T = \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} \]

Ellipse:

\[ \kappa = \left( \frac{\pi}{A} \right)^{2/3} \]

\[ A = \pi \frac{T}{S^{3/2}} = \text{Area} \]

Affine arc length of ellipse:

\[
\int_P^Q ds = \frac{T^{1/3}}{S^{1/2}} \arcsin \sqrt{\frac{-CT}{S^2}} \left( x + \frac{CD - BE}{S} \right) \bigg|_P^Q \\
= 2ST^{-2/3} A(P, Q)
\]
\( A(P, Q) : \)

Triangular approximation:

\( \Delta(O, P, Q) : \)

Total affine arc length:

\[
L = 2 \sqrt[3]{A} = -2\pi \frac{\sqrt[3]{T}}{\sqrt{S}}
\]
Conic through five points $P_0, \ldots, P_4$:

$$[013][024][x_{12}][x_{34}] = [012][034][x_{13}][x_{24}]$$

$x = (x, y)$

Affine curvature and arc length:

$$\kappa = \frac{S}{T^{2/3}}$$

$$ds = \text{Area } \Delta(O, P_1, P_3) = \frac{1}{2}[O, P_1, P_3] = \frac{N}{2S}$$

$$4T = \prod_{0 \leq i < j < k \leq 4} [ijk]$$

$$4S = [013]^2[024]^2([124] - [123])^2 + [012]^2[034]^2([134] + [123])^2 - 2[012][034][013][024]([123][234] + [124][134])$$

$$4N = -[123][134] \{[023]^2[014]^2([124] - [123]) + [012]^2[034]^2([134] + [123]) + [012][023][014][034]([134] - [123])\}$$

$$\sigma \equiv 45$$