The geometry of the space of planar shapes - geodesics and curvature

Peter W. Michor

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joint work with David Mumford

$\text{Diff}(S^1)$ a regular Lie group,

$\text{Emb} = \text{Emb}(S^1, \mathbb{R}^2)$,

$\text{Imm} = \text{Imm}(S^1, \mathbb{R}^2)$,

$B_e = B_e(S^1, \mathbb{R}^2) = \text{Emb}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$, the manifold of 1-dimensional connected submanifolds of $\mathbb{R}^2$,

$B_i = B_i(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$, an infinite dimensional 'orbifold'

Notation. We work mostly with arclength ds, arclength derivative D_s and the unit tangent vector v to the curve:

$$ds = |c_\theta| d\theta$$
$$D_s = \partial_\theta / |c_\theta|$$
$$v = c_\theta / |c_\theta|$$

Attention: Given a family of curves $c(\theta, t)$, then ∂_θ and ∂_t commute but D_s and ∂_t don't. Rotation through 90 degrees (complex multiplication by $\sqrt{-1}$) will be denoted by:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The unit normal vector to the image curve is thus

$$n = Jv.$$

Curvature and length on $\mathrm{Imm}(S^1,\mathbb{R}^2)$

$$\kappa : \mathrm{Imm}(S^1,\mathbb{R}^2) \to C^\infty(S^1,\mathbb{R}),$$

$$\kappa(c) = \frac{\det(c_\theta, c_{\theta\theta})}{|c_\theta|^3} = \langle n, D_s v\rangle$$

$$d\kappa(c)(h) = \langle D_s^2(h), n\rangle - 2\kappa\langle D_s(h), v\rangle$$

The length function

$$\ell : \mathrm{Imm}(S^1,\mathbb{R}^2) \to \mathbb{R}, \qquad \ell(c) = \int_{S^1} |c_\theta|\, d\theta$$

$$d\ell_c(h) = \int_{S^1} \frac{\langle h_\theta, c_\theta\rangle}{|c_\theta|} d\theta = \int_{S^1} \langle D_s(h), v\rangle ds$$

$$= -\int_{S^1} \langle h, D_s(v)\rangle ds = -\int_{S^1} \kappa(c)\langle h, n\rangle ds$$

The degree of immersions. The degree of an immersion $c : S^1 \to \mathbb{R}^2$ is the winding number around 0 of the tangent $c' : S^1 \to \mathbb{R}^2$. $\operatorname{Imm}(S^1, \mathbb{R}^2)$ decomposes into the disjoint union of the open submanifolds $\operatorname{Imm}^k(S^1, \mathbb{R}^2)$ for $k \in \mathbb{Z}$ according to the degree k.

Theorem. *The manifold* $\operatorname{Imm}^k(S^1, \mathbb{R}^2)$ *of immersed curves of degree* k *contains* S^1 *as a strong smooth deformation retract.*
For $k \neq 0$ *the manifold*
$B_i^k(S^1, \mathbb{R}^2) := \operatorname{Imm}^k(S^1, \mathbb{R}^2) / \operatorname{Diff}^+(S^1)$
is contractible.
For $k = 0$ *we have (surprise, surprise) (K. M.)*

$$\pi_1(B^0(S^1, \mathbb{R}^2)) = \mathbb{Z},$$
$$\pi_2(B^0(S^1, \mathbb{R}^2)) = \mathbb{Z},$$
$$\pi_k(B^0(S^1, \mathbb{R}^2)) = 0 \qquad \text{for } k > 2.$$

The tangent bundle is
$T\,\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathrm{Imm}(S^1,\mathbb{R}^2)\times C^\infty(S^1,\mathbb{R}^2)$, the cotangent bundle is
$T^*\,\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathrm{Imm}(S^1,\mathbb{R}^2) \times \mathcal{D}(S^1)^2$ where the second factor consists of periodic distributions.

We consider smooth Riemannian metrics on $\text{Imm}(S^1, \mathbb{R}^2)$, i.e., smooth mappings

$$G : \text{Imm}(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2) \to \mathbb{R}$$
$$(c, h, k) \mapsto G_c(h, k), \qquad \text{bilinear in } h, k$$
$$G_c(h, h) > 0 \qquad \text{for } h \neq 0.$$

Each such metric is *weak* in the sense that G_c, viewed as bounded linear mapping

$$\begin{aligned} G_c : T_c \text{Imm}(S^1, \mathbb{R}^2) = C^\infty(S^1, \mathbb{R}^2) \to \\ \to T_c^* \text{Imm}(S^1, \mathbb{R}^2) = \mathcal{D}(S^1)^2 \\ G : T\,\text{Imm}(S^1, \mathbb{R}^2) \to T^*\,\text{Imm}(S^1, \mathbb{R}^2) \\ G(c, h) = (c, G_c(h, \quad)) \end{aligned}$$

is injective, but can never be surjective.

We shall need also its tangent mapping which is given by

$$TG : T(T\,\mathrm{Imm}(S^1, \mathbb{R}^2)) \to T(T^*\,\mathrm{Imm}(S^1, \mathbb{R}^2))$$

$$TG(c, h; k, \ell) =$$

$$= (c, G_c(h, \quad); k, dG(c)(k)(h, \quad) + G_c(\ell, \quad))$$

In the sequel we shall further assume that that *the weak Riemannian metric G itself admits G-gradients with respect to the variable c in the following sense:*

$$\boxed{dG_c(m)(h,k) = G_c(m, H_c(h,k)) = G_c(K_c(m,h),k)}$$

$H, K : \operatorname{Imm} \times C^\infty \times C^\infty \to C^\infty$

$(c,h,k) \mapsto H_c(h,k), K_c(h,k)$

smooth and bilinear in h,k.

We will check and compute these gradients for several concrete metrics below.

The fundamental symplectic form on $T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ pulled back from the canonical symplectic form on the contangent bundle via the mapping $G : T\,\mathrm{Imm}(S^1,\mathbb{R}^2) \to T^*\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ is then:

$$\begin{aligned}
&\omega_{(c,h)}((k_1,\ell_1),(k_2,\ell_2)) = \\
&= -dG_c(k_1)(h,k_2) - G_c(\ell_1,k_2) \\
&\quad + dG_c(k_2)(h,k_1) + G_c(\ell_2,k_1) \\
&= G_c(k_2, H_c(h,k_1) - K_c(k_1,h)) \\
&\quad + G_c(\ell_2,k_1) - G_c(\ell_1,k_2)
\end{aligned}$$

The Hamiltonian vector field $\operatorname{grad}^\omega(f)$ associated to a function f on $\operatorname{Imm}(S^1,\mathbb{R}^2) \times C^\infty(S^1,\mathbb{R}^2))$ admitting smooth G-gradients in both factors:

$$\boxed{\begin{aligned}&\operatorname{grad}^\omega_1(f)(c,h) = \operatorname{grad}^G_2(f)(c,h)\\ &\operatorname{grad}^\omega_2(f)(c,h) = -\operatorname{grad}^G_1(f)(c,h)\\ &+ H_c\Big(h, \operatorname{grad}^G_2(f)(c,h)\Big) - K_c(\operatorname{grad}^G_2(f)(c,h),h)\end{aligned}}$$

For a smooth function f on $T\operatorname{Imm}(S^1,\mathbb{R}^2)$ the ω-gradient exists if and only if both G-gradients exist.

The geodesic equation. The Hamiltonian vector field of the Riemann energy function

$$E(c,h) = \frac{1}{2} G_c(h,h), \qquad E : T\,\mathrm{Imm}(S^1, \mathbb{R}^2) \to \mathbb{R}$$

is the geodesic vector field:

$$\begin{aligned} \mathrm{grad}_1^{\omega}(E)(c,h) &= h \\ \mathrm{grad}_2^{\omega}(E)(c,h) &= \tfrac{1}{2} H_c(h,h) - K_c(h,h) \end{aligned}$$

and the geodesic equation becomes:

$$\begin{cases} c_t &= h \\ h_t &= \tfrac{1}{2} H_c(h,h) - K_c(h,h) \end{cases}$$

$$\boxed{c_{tt} = \tfrac{1}{2} H_c(c_t,c_t) - K_c(c_t,c_t)}$$

The momentum mapping for a G-isometric group action. Consider a (possibly infinite dimensional regular) Lie group with Lie algebra $\mathfrak{g}$ with a right action $g \mapsto r^g$ by isometries on $\text{Imm}(S^1, \mathbb{R}^2)$. Fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(\text{Imm}(S^1, \mathbb{R}^2))$, a bounded Lie algebra homomorphism, given by

$$\zeta_X(c) = \partial_t|_0 r^{\exp(tX)}(c).$$

momentum map $j : \mathfrak{g} \to C^\infty_G(T\,\text{Imm}(S^1, \mathbb{R}^2), \mathbb{R})$:

$$\boxed{j_X(c, h) = G_c(\zeta_X(c), h).}$$

$$\mathcal{J} : T\,\text{Imm}(S^1, \mathbb{R}^2) \to \mathfrak{g}', \quad \langle \mathcal{J}(c, h), X \rangle = j_X(c, h).$$

It fits into the following commmutative diagram and is a homomorphism of Lie algebras:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0 & \xrightarrow{i} & C^\infty_G & \xrightarrow{\text{grad}^\omega} & \mathfrak{X}_\omega & \longrightarrow & H^1 & \longrightarrow & 0 \\ & & & & & \nwarrow^{j} & \uparrow{\scriptstyle \zeta^{T\,\text{Imm}}} & & & & \\ & & & & & & \mathfrak{g} & & & & \end{array}$$

$\mathcal{J}$ is equivariant for the group action. Along any geodesic $t \mapsto c(t, \quad)$ this momentum mapping is constant, thus for any $X \in \mathfrak{g}$

$$\boxed{\langle \mathcal{J}(c, c_t), X \rangle = j_X(c, c_t) = G_c(\zeta_X(c), c_t)}$$

is constant in t.

We can apply this construction to the following group actions on $\text{Imm}(S^1, \mathbb{R}^2)$.

• The smooth right action of the group $\text{Diff}(S^1)$ on $\text{Imm}(S^1, \mathbb{R}^2)$, given by composition from the right: $c \mapsto c \circ \varphi$ for $\varphi \in \text{Diff}(S^1)$. For $X \in \mathfrak{X}(S^1)$ the fundamental vector field is then given by

$$\zeta_X^{\text{Diff}}(c) = \zeta_X(c) = \partial_t|_0(c \circ \text{Fl}_t^X) = c_\theta.X.$$

The *reparametrization momentum*, for any vector field X on S^1 is thus:

$$j_X(c, h) = G_c(c_\theta.X, h).$$

Assuming the metric is reparametrization invariant, it follows that on any geodesic $c(\theta, t)$, the expression $G_c(c_\theta.X, c_t)$ is constant for all X.

• The left action of the Euclidean motion group $M(2) = \mathbb{R}^2 \rtimes SO(2)$ on $\text{Imm}(S^1, \mathbb{R}^2)$ given by $c \mapsto e^{aJ}c + B$ for $(B, e^{aJ}) \in \mathbb{R}^2 \times SO(2)$. The fundamental vector field mapping is

$$\zeta_{(B,a)}(c) = aJc + B$$

The *linear momentum* is thus $G_c(B, h), B \in \mathbb{R}^2$ and if the metric is translation invariant, $G_c(B, c_t)$ will be constant along geodesics. The *angular momentum* is similarly $G_c(Jc, h)$ and if the metric is rotation invariant, then $G_c(Jc, c_t)$ will be constant along geodesics.

• The action of the scaling group of $\mathbb{R}$ given by $c \mapsto e^r c$, with fundamental vector field $\zeta_a(c) = a.c$. If the metric is scale invariant, then the *scaling momentum* $G_c(c, c_t)$ will also be invariant along geodesics.

If the Riemannian metric G on Imm is invariant under the action of Diff(S^1) it induces a metric on the quotient B_i as follows. For any $C_0, C_1 \in B_i$, consider all liftings $c_0, c_1 \in$ Imm such that $\pi(c_0) = C_0, \pi(c_1) = C_1$ and all smooth curves $t \mapsto (\theta \mapsto c(t,\theta))$ in Imm$(S^1, \mathbb{R}^2)$ with $c(0,\cdot) = c_0$ and $c(1,\cdot) = c_1$. Since the metric G is invariant under the action of Diff(S^1) the arc-length of the curve $t \mapsto \pi(c(t,\cdot))$ in $B_i(S^1, \mathbb{R}^2)$ is given by

$$
\begin{aligned}
&L_G^{\text{hor}}(c) := L_G(\pi(c(t,\cdot))) \\
&= \int_0^1 \sqrt{G_{\pi(c)}(T_c\pi.c_t, T_c\pi.c_t)}\, dt \\
&= \int_0^1 \sqrt{G_c(c_t^\perp, c_t^\perp)}\, dt \\
&\text{dist}_G^{B_i(S^1,\mathbb{R}^2)}(C_1, C_2) = \inf_c L_G^{\text{hor}}(c).
\end{aligned}
$$

The general almost local metric G^Φ.

$$G_c^\Phi(h,k) := \int_{S^1} \Phi(\ell_c, \kappa_c(\theta)) \langle h(\theta), k(\theta)\rangle ds.$$

The metric G^Φ is invariant under the reparametizatio group $\text{Diff}(S^1)$ and under the Euclidean motion group.

We compute the G^Φ-gradients of $c \mapsto G^\Phi_c(h,k)$:

$$dG^\Phi(c)(m)(h,k) = G^\Phi_c(K^\Phi_c(m,h),k)$$
$$= G^\Phi_c\Big(m, H^\Phi_c(h,k)\Big),$$
$$K^\Phi_c(m,h) = -\Big(\int_{S^1} \kappa_c \langle m,n\rangle ds\Big)\frac{\partial_1\Phi(\ell,\kappa)}{\Phi(\ell,\kappa)}h$$
$$+ \frac{\partial_2\Phi(\ell,\kappa)}{\Phi(\ell,\kappa)}\Big(\langle D_s^2(m),n\rangle - 2\kappa\langle D_s(m),v\rangle\Big)h$$
$$+ \langle D_s(m),v\rangle h$$
$$H^\Phi_c(h,k) = \frac{1}{\Phi(\ell,\kappa)}\bigg(- \Big(\kappa_c \int \partial_1\Phi(\ell,\kappa)\langle h,k\rangle ds\Big)n$$
$$+ D_s^2\Big(\partial_2\Phi(\ell,\kappa)\langle h,k\rangle n\Big)+$$
$$+ 2D_s\Big(\partial_2\Phi(\ell,\kappa)\kappa\langle h,k\rangle v\Big) - D_s\Big(\Phi(\ell,\kappa)\langle h,k\rangle v\Big)\bigg)$$

Conserved momenta for G^Φ along any geodesic $t \mapsto c(\quad, t)$:

$\Phi(\ell_c, \kappa_c)\langle v, c_t\rangle |c_\theta|^2 \in \mathfrak{X}(S^1)$	reparam. mom.
$\int_{S^1} \Phi(\ell_c, \kappa_c) c_t ds \in \mathbb{R}^2$	linear moment.
$\int_{S^1} \Phi(\ell_c, \kappa_c)\langle Jc, c_t\rangle ds \in \mathbb{R}$	angular moment.

Setting the reparametrization momentum to 0 and doing symplectic reduction amounts exactly to investigating the quotient space

$B_i(S^1, \mathbb{R}^2) = \operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$

and using horizontal geodesics for doing so; a horizontal geodesic is G^Φ-normal to the $\operatorname{Diff}(S^1)$-orbits. If it is normal at one time it is normal forever (since the reparametrization momentum is conserved).

Horizontality for G^Φ.

$T_c(c \circ \mathrm{Diff}(S^1)) = \{X.c_\theta : X \in C^\infty(S^1,\mathbb{R})\}$. Thus the bundle of horizontal vectors is

$$\begin{aligned}\mathcal{N}_c &= \{h \in C^\infty(S^1,\mathbb{R}^2) : \langle h, v\rangle = 0\}\\ &= \{a.n \in C^\infty(S^1,\mathbb{R}^2) : a \in C^\infty(S^1,\mathbb{R})\}\end{aligned}$$

A tangent vector $h \in T_c\mathrm{Imm}(S^1,\mathbb{R}^2) = C^\infty(S^1,\mathbb{R}^2)$ has an orthonormal decomposition

$$\begin{aligned}h &= h^\top + h^\perp \in T_c(c \circ \mathrm{Diff}^+(S^1)) \oplus \mathcal{N}_c\\ h^\top &= \langle h, v\rangle v \in T_c(c \circ \mathrm{Diff}^+(S^1)),\\ h^\perp &= \langle h, n\rangle n \in \mathcal{N}_c,\end{aligned}$$

into smooth tangential and normal components, independent of the choice of $\Phi(\ell,\kappa)$.

Lemma.

For any smooth path c *in* $\mathrm{Imm}(S^1, \mathbb{R}^2)$ *there exists a smooth path* φ *in* $\mathrm{Diff}(S^1)$ *with* $\varphi(0, \quad) = \mathrm{Id}_{S^1}$ *depending smoothly on* c *such that the path* e *given by* $e(t, \theta) = c(t, \varphi(t, \theta))$ *is horizontal:* $e_t \perp e_\theta$.

Consider a path $t \mapsto c(\cdot,t)$ in the manifold $\mathrm{Imm}(S^1,\mathbb{R}^2)$. It projects to a path $\pi\circ c$ in $B_i(S^1,\mathbb{R}^2)$ whose energy is called the *horizontal energy* of c:

$$\begin{aligned} E^{\text{hor}}_{G^\Phi}(c) &= \tfrac{1}{2}\int_a^b\int_{S^1}\Phi(\ell_c,\kappa_c)\langle c_t,n\rangle^2\,d\theta dt \\ &= \tfrac{1}{2}\int_{[a,b]\times S^1}\Phi(\ell_c,\kappa_c)\frac{|n_S^0|^2}{\sqrt{1-|n_S^0|^2}}\,d\mu_S \end{aligned}$$

Here the final expression is only in terms of the surface S and its fibration over the time axis, and is valid for any path c. This anisotropic area functional has to be minimized in order to prove that geodesics exists between arbitrary curves (of the same degree) in $B_i(S^1,\mathbb{R}^2)$.

The horizontal geodesic equation.

Let $c(\theta,t)$ be a horizontal geodesic for the metric G^Φ. Then $c_t(\theta,t) = a(\theta,t).n(\theta,t)$. Denote the integral of a function over the curve with respect to arclength by a bar. Then the geodesic equation for horizontal geodesics is:

$$\boxed{\begin{aligned} a_t = \frac{1}{2\Phi}\Big(&\left(-\kappa\Phi + \kappa^2 \partial_2\Phi\right) a^2 \\ &- D_s^2\left(\partial_2\Phi \cdot a^2\right) + 2\partial_2\Phi \cdot a D_s^2(a) \\ &- 2\partial_1\Phi \cdot \overline{(\kappa a)} \cdot a + \overline{(\partial_1\Phi \cdot a^2)} \cdot \kappa \Big) \end{aligned}}$$

Special case: the metric G^A.

If we choose $\Phi(\ell_c, \kappa_c) = 1 + A\kappa_c^2$ then we obtain the metric we have investigated before:

$$G_c^A(h,k) = \int_{S^1} (1 + A\kappa_c(\theta)^2)\langle h(\theta), k(\theta)\rangle ds.$$

The horizontal geodesic equation for the G^A-metric reduces to

$$\begin{aligned} a_t = \frac{1}{1 + A\kappa_c^2}\Big(&-\tfrac{1}{2}\kappa_c a^2 \\ &+ A\Big(a^2(-D_s^2(\kappa_c) + \tfrac{1}{2}\kappa_c^3) \\ &\quad - 4D_s(\kappa_c)aD_s(a) - 2\kappa_c D_s(a)^2\Big)\Big) \end{aligned}$$

Horizontal Geodesics for $A = 0$

$\langle c_t, c_\theta \rangle = 0$ and $c_t = an = aJ\frac{c_\theta}{|c_\theta|}$ for $a \in C^\infty(S^1, \mathbb{R})$.
We use functions a, $s = |c_\theta|$, and κ, only holonomic derivatives:

$$s_t = -a\kappa s, \quad a_t = \tfrac{1}{2}\kappa a^2,$$
$$\kappa_t = a\kappa^2 + \frac{1}{s}\left(\frac{a_\theta}{s}\right)_\theta = a\kappa^2 + \frac{a_{\theta\theta}}{s^2} - \frac{a_\theta s_\theta}{s^3}.$$

We may assume $s|_{t=0} \equiv 1$. Let $v(\theta) = a(0,\theta)$, the initial value for a. Then
$\frac{s_t}{s} = -a\kappa = -2\frac{a_t}{a}$, so $\log(sa^2)_t = 0$, thus
$s(t,\theta)a(t,\theta)^2 = s(0,\theta)a(0,\theta)^2 = v(\theta)^2$,
a conserved quantity along the geodesic. We substitute $s = \frac{v^2}{a^2}$ and $\kappa = 2\frac{a_t}{a^2}$ to get

$$a_{tt} - 4\frac{a_t^2}{a} - \frac{a^6 a_{\theta\theta}}{2v^4} + \frac{a^6 a_\theta v_\theta}{v^5} - \frac{a^5 a_\theta^2}{v^4} = 0,$$
$$a(0,\theta) = v(\theta),$$

a nonlinear hyperbolic second order equation. Note that wherever $v = 0$ then also $a = 0$ for all t. So substitute $a = vb$. The outcome is

$$\boxed{\begin{aligned} &(b^{-3})_{tt} = -\frac{v^2}{2}(b^3)_{\theta\theta} - 2vv_\theta(b^3)_\theta - \frac{3vv_{\theta\theta}}{2}b^3, \\ &b(0,\theta) = 1. \end{aligned}}$$

This is the codimension 1 version where Burgers' equation is the codimension 0 version.

Along a geodesic $t \mapsto c(t, \quad)$ we have the following conserved quantities:

$(1 + A\kappa_c^2)\langle v, c_t\rangle \lvert c_\theta\rvert^2 \in \mathfrak{X}(S^1)$	reparam. mom.
$\int_{S^1}(1 + A\kappa_c^2)c_t ds \in \mathbb{R}^2$	linear momentum
$\int_{S^1}(1 + A\kappa_c^2)\langle Jc, c_t\rangle ds \in \mathbb{R}$	angular momentum

Now the big surprise for $A = 0$, for the L^2-metric:

Theorem. *For* $c_0, c_1 \in \mathrm{Imm}(S^1, \mathbb{R}^2)$ *there exists always a variation through immersions* $t \mapsto c(t,\cdot)$ *with* $c(0,\cdot) = c_0$ *and* $\pi(c(1,\cdot)) = \pi(c_1)$ *for any given immersions* c_0 *and* c_1 *such that* $L^{hor}_{G^0}(c)$ *is arbitrarily small.*

Thus the distance $\mathrm{dist}^{B_i}_{H^0}$ on $B_i(S^1, \mathbb{R}^2)$ vanishes.

Lipschitz continuity of $\sqrt{\ell} : B_i \to \mathbb{R}_{\geq 0}$**.**
For C_0 *and* C_1 *in* $B_i = \mathrm{Imm}\,/\,\mathrm{Diff}(S^1)$ *we have for* $A > 0$*:*

$$\sqrt{\ell(C_1)} - \sqrt{\ell(C_0)} \leq \frac{1}{2\sqrt{A}}\,\mathrm{dist}^{B_i(S^1,\mathbb{R}^2)}_{G^A}(C_1, C_2).$$

Area swept out bound.
If c is any path from C_0 to C_1, then

$$\begin{pmatrix} \textit{area of the region} \\ \textit{swept out by the} \\ \textit{variation } c \end{pmatrix} \leq \max_t \sqrt{\ell(c(t,\cdot))} \cdot L^{hor}_{G^A}(c).$$

Maximum distance bound.
Consider $\epsilon < \min\{\sqrt{A\ell}/4, \ell^{3/4}/\sqrt{8}\}$ and let $\eta = 4(\ell^{3/4}A^{-1/4} + \ell^{1/4})\sqrt{\epsilon}$. Then for any path c starting at C_0 whose length $L^{hor}_{G^A}$ is ϵ, the final curve lies in the tubular neighborhood of C_0 of width η. More precisely, if we choose the path $c(t,\theta)$ to be horizontal, then
$\max_\theta |c(0,\theta) - c(1,\theta)| < \eta$.

Corollary.

For any $A > 0$*, the map from* $B_i(S^1, \mathbb{R}^2)$ *in the* G^A *metric to the space* $B_i^{cont}(S^1, \mathbb{R}^2)$ *in the Frechet metric is continuous, and, in fact, uniformly continuous on every subset where the length* ℓ *is bounded. In particular,* G^A *is a separating metric on* $B_i(S^1, \mathbb{R}^2)$*. Moreover, the completion* $\overline{B_i}(S^1, \mathbb{R}^2)$ *of* $B_i(S^1, \mathbb{R}^2)$ *in this metric can be identified with a subset of* $B_i^{lip}(S^1, \mathbb{R}^2)$*.*

Explicit equicontinuity bounds, under appropriate parameterization.

Corollary.

If a path $c(\theta, t), 0 \leq t \leq 1$ *satisfies:*

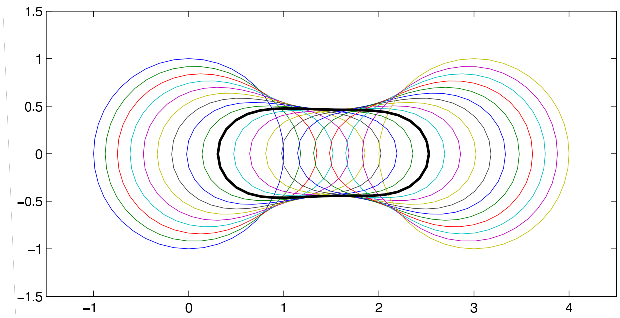
- $|c_\theta(\theta, t)| \equiv \ell(t)/2\pi$ *for all* θ, t,
- $\langle c_t, c_\theta \rangle(0, t) \equiv 0$ *in a base point* 0 *for all* t
- $\int_{C_t}(1 + A\kappa^2_{C_t})|\langle c_t, ic_\theta\rangle|^2 d\theta/|c_\theta| \equiv L^2$ *for all* t,

then

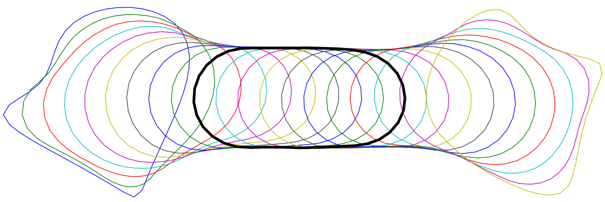
$$|c(\theta_1, t_1) - c(\theta_2, t_2)| \leq \frac{\ell_{\text{max}}}{2\pi}|\theta_1 - \theta_2| + \\ + 7(\ell_{\text{max}}^{3/4}/A^{1/4} + \ell_{\text{max}}^{1/4})\sqrt{L(t_1 - t_2)} \quad (1)$$

whenever $|t_1 - t_2| \leq \min(2\sqrt{A\ell_{\text{min}}}, \ell_{\text{min}}^{3/2})/(8L)$.

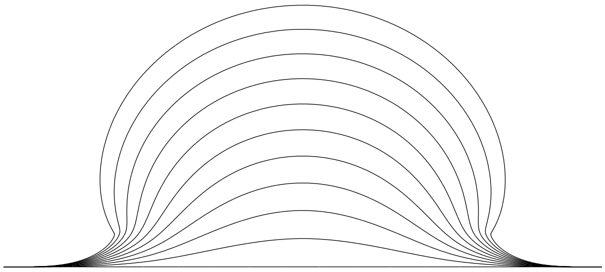
A numerical simulation of the geodesic connecting two circles. Minimize $E^{\text{hor}}_{G^1}(c)$ for variations c with initial and end curves unit circles at distance 3 produced the following image for the geodesic:



The geodesic joining 2 'random' shapes of size about 1 at distance 5 apart with $A = .25$ (using 20 time samples and a 48-gon approximation for all curves).

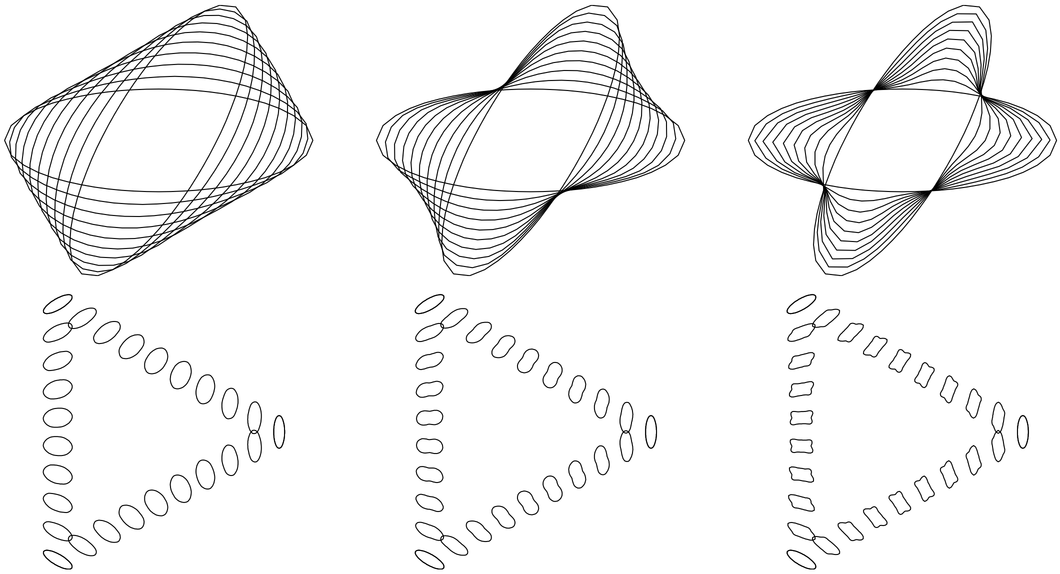


The forward integration of the geodesic equation when $A = 0$, starting from a straight line in the direction given by a smooth bump-like vector field. Note that two corner like singularities with curvature going to ∞ are about to form.



Top Row: Geodesics in 3 metrics joining the same two ellipses. Ellipses have eccentricity 3, same center and are rotated at 60° degree.

$A = 1$; $A = 0.1$; $A = 0.01$.



Bottom Row: Geodesic triangles in B_e formed by joining three ellipses at angles 0, 60 and 120 degrees, for the same three values of A. Here the intermediate shapes are just rotated versions of the geodesic in the top row but are laid out on a plane triangle for visualization purposes.

The sectional curvature on B_i

$$R_0(a,b,a,b) = G_0^A(R_0(a,b)a,b) =$$
$$= \int_{S^1} \left(\tfrac{1}{2}(A\kappa^2 - 1)(ab' - a'b)^2 + A(ab'' - a''b)^2 \right) d\theta$$
$$+ \int_{S^1} \frac{A\kappa^2 - A^2\kappa^4 + 2A^2\kappa\kappa'' - 4A^2\kappa'^2}{1 + A\kappa^2}(ab' - a'b)^2 d\theta$$
$$= \int_{S^1} \frac{-(A\kappa^2 - 1)^2 + 4A^2\kappa\kappa'' - 8A^2\kappa'^2}{2(1 + A\kappa^2)} W(a,b)^2 \, d\theta$$
$$+ \int_{S^1} A\, W(a,b)'^2 \, d\theta$$

where $W(a,b) = ab' - a'b$ is the Wronskian of a and b.

Special case: the conformal metrics

$\Phi(\ell(c), \kappa(c)) = \Phi(\ell(c))$, metric proposed by Menucci and Yezzi and, for Φ linear, independently by Shah:

$$G_c^{\Phi}(h,k) = \Phi(\ell_c) \int_{S^1} \langle h,k \rangle ds = \Phi(\ell_c) G_c^0(h,k).$$

All these metrics are conformally equivalent to the basic L^2-metric G^0.

As they show, the infimum of path lengths in this metric is positive so long as Φ satifies an inequality $\Phi(\ell) \geq C.\ell$ for some $C > 0$.

More precisely (Shah), if Area(c) is area swept over by the path c,

$$\begin{aligned}
\mathrm{dist}_{G^\ell}(C_0, C_1) &= \inf_c \mathrm{Area}(c) \\
\sqrt{Ae}.\inf_c \mathrm{Area}(c) &\le \mathrm{dist}_{G^{e^{A\ell}}}(C_0, C_1) \le \\
&\le \sqrt{Ae}.e^{A\ell_{\max}} \inf_c \mathrm{Area}(c)
\end{aligned}$$

The horizontal geodesic equation reduces to:

$$a_t = -\frac{\kappa}{2}a^2 + \frac{\partial_1\Phi}{\Phi}\cdot\left(\tfrac{1}{2}\left(\int a^2.ds\right)\kappa - \left(\int \kappa.a.ds\right)a\right)$$

If we change variables and write $b(s,t) = \Phi(\ell(t)).a(s,t)$, then this equation simplifies to:

$$\boxed{b_t = -\frac{\kappa}{2\Phi}\left(b^2 - \frac{\partial_1\Phi}{\Phi}\int b^2\,ds\right)}$$

Along a geodesic $t \mapsto c(t, \quad)$ we have the following conserved quantities:

$\Phi(\ell_c)\langle v, c_t\rangle |c'(\theta)|^2 \ \in \mathfrak{X}(S^1)$ reparam. moment.

$\Phi(\ell_c)\int_{S^1} c_t ds \ \in \mathbb{R}^2$ linear moment.

$\Phi(\ell_c)\int_{S^1} \langle Jc, c_t\rangle ds \in \mathbb{R}$ angular moment.

Curvature on B_i for the conformal metrics. Sectional curvature has been computed by J. Shah. Let g, h be orthonomal, then

Curv. in plane $\langle g.h \rangle$

$$= \frac{\Phi}{2} \cdot \overline{(g.D_s(h) - h.D_s(g))^2} + \frac{\partial_1 \Phi}{4\Phi} \cdot \left(\overline{g^2.\kappa^2} + \overline{h^2.\kappa^2} \right)$$
$$+ \frac{3\partial_1 \Phi^2 - 2\Phi.\partial_1^2 \Phi}{4\Phi^2} \cdot \left(\overline{(g.\kappa)}^2 + \overline{(h.\kappa)}^2 \right)$$
$$- \frac{\partial_1 \Phi}{2\Phi} \cdot \left(\overline{D_s(g)^2} + \overline{D_s(h)^2} + \frac{\partial_1 \Phi}{2\Phi^2} \cdot \overline{\kappa^2} \right)$$

Note that the first two lines are positive while the last line is negative. The first term is the curvature term for the H^0-metric. The key point about this formula is how many positive terms it has.

Special case: the smooth scale invariant metric G^{SI}

$\Phi(\ell, \kappa) = \ell^{-3} + A\frac{\kappa^2}{\ell}$ gives the metric:

$$G_c^{SI}(h,k) = \int_{S^1} \left(\frac{1}{\ell_c^3} + A\frac{\kappa_c^2}{\ell_c} \right) \langle h, k \rangle ds.$$

The beauty of this metric is that (a) it is scale invariant and (b) $\log(\ell)$ is Lipschitz, hence the infimum of path lengths is always positive.

Horizontal geodesics in this metric as special case of the equation for G^{Φ}:

$$a_t = \frac{1}{1+A(\ell\kappa)^2}\Bigg(\left(-1+A(\ell\kappa)^2\right)\frac{\kappa a^2}{2}$$
$$- 2A\ell^2\kappa D_s(a)^2 - 4A\ell^2 D_s(\kappa) a D_s(a)$$
$$+ \left(3+A(\ell\kappa)^2\right)\overline{(a\kappa)}\cdot a - \frac{3}{2}\overline{(a^2)}\cdot\kappa$$
$$- \frac{A\ell^2}{2}\overline{(\kappa a)^2}\cdot\kappa\Bigg)$$

where the "overline" stands now for the *average* of a function over the curve, i.e. $\int \cdots ds/\ell$.

Since this metric is scale invariant, there are now *four* conserved quantities, instead of three:

$$\Phi(\ell,\kappa)\langle v, c_t\rangle|c'(\theta)|^2 \in \mathfrak{X}(S^1)$$ reparam. mom.

$$\int_{S^1}\Phi(\ell,\kappa)c_t ds \in \mathbb{R}^2$$ linear moment.

$$\int_{S^1}\Phi(\ell,\kappa)\langle Jc, c_t\rangle ds \in \mathbb{R}$$ angular moment.

$$\int_{S^1}\Phi(\ell,\kappa)\langle c, c_t\rangle ds \in \mathbb{R}$$ scaling moment.

The Wasserstein metric and a related G^{Φ}-metric. The Wasserstein metric (also known as the Monge-Kantorovich metric) is a metric between probability measures on a common metric space. Let μ and ν be 2 probability measures on a metric space (X, d). Consider all measures ρ on $X \times X$ whose marginals under the 2 projections are μ and ν. Then:

$$d_{\text{wass}}(\mu, \nu) = \inf_{\rho} \iint_{X \times X} d(x, y) d\rho(x, y).$$

where inf is over all ρ with $\text{pr}_{1,*}(\rho) = \mu$ and $\text{pr}_{2,*}(\rho) = \nu$.

The Wasserstein norm is sandwiched between $G^{\ell^{-1}}$ and G^{Φ_W} for $\Phi_W = \ell^{-1}.(1 + \ell\kappa)^2/12$.

Immersion-Sobolev metrics on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ **and on** B_i

Note that $D_s = \frac{\partial_\theta}{|c_\theta|}$ is anti self-adjoint for the metric G^0, i.e., for all $h, k \in C^\infty(S^1.\mathbb{R}^2)$ we have

$$\int_{S^1} \Big\langle D_s(h), k \Big\rangle ds = \int_{S^1} \Big\langle h, -D_s(k) \Big\rangle ds$$

The metric:

$$\begin{aligned} G_c^{\mathrm{imm},n}(h,k) &= \int_{S^1} \left(\langle h, k \rangle + A.\langle D_s^n h, D_s^n k \rangle \right) .ds \\ &= \int_{S^1} \langle L_n(h), k \rangle ds \qquad \text{where} \end{aligned}$$

$L_n(h)$ or $L_{n,c}(h) = I + (-1)^n A.D_s^{2n}(h)$

For fixed constant speed c of length ℓ, $L_{n,c}$ is the constant coeff. ODE $f \mapsto f + (-1)^n A.f^{(n)}$ on the s-line modulo $\ell.\mathbb{Z}$. Green's function:

$$K_n(x) = \frac{1}{2n} \cdot \sum_{\lambda^{2n}=(-1)^n/A} \frac{\lambda}{1-e^{\lambda\ell}} e^{\lambda x}.$$

Thus the dual metric $\check{G}_c^{\text{imm},n} = (G_c^{\text{imm},n})^{-1}$ on the *smooth cotangent space* $C^\infty(S^1,\mathbb{R}^2) \cong G_c^0(T_c \text{Imm}(S^1,\mathbb{R}^2)) \subset \mathcal{D}(S^1)^2$ is the integral operator L^{-1}, convolution by K_n:

$$\check{G}_c^{\text{imm},n}(h,k) =$$
$$= \iint_{S^1\times S^1} K_n(s_1-s_2).\langle h(s_1), k(s_2)\rangle.ds_1.ds_2.$$

Geodesics in the $H^{\mathrm{imm},n}$-metric

$$
\begin{aligned}
(L_n(c_t))_t &= -\langle L_n(c_t), D_s(c_t)\rangle v \\
&\quad - \frac{|c_t|^2\kappa(c)}{2} n - \langle D_s(c_t), v\rangle L_n c_t \\
&\quad + \frac{A}{2}. \sum_{j=1}^{2n-1} (-1)^{n+j} \langle D_s^{2n-j} c_t, D_s^j c_t\rangle \kappa(c) n
\end{aligned}
$$

The conserved momenta of $G^{\mathrm{imm},n}$ along any geodesic $t \mapsto c(t, \quad)$:

$\langle c_\theta, L_{n,c}(c_t)\rangle \lvert c'(\theta)\rvert \ \in \mathfrak{X}(S^1)$	repar. moment.
$\int_{S^1} L_{n,c}(c_t)\, ds \ \in \mathbb{R}^2$	linear moment.
$\int_{S^1} \langle Jc, L_{n,c}(c_t)\rangle\, ds \in \mathbb{R}$	angular moment.

Horizontality for $G^{\text{imm},n}$ $h \in T_c \text{Imm}(S^1, \mathbb{R}^2)$ is $G_c^{\text{imm},n}$-orthogonal to the $\text{Diff}(S^1)$-orbit through c if and only if

$$0 = G_c^{\text{imm},n}(h, \zeta_X(c)) = \int_{S^1} X.\langle L_{n,c}(h), c_\theta \rangle \, ds$$

for all $X \in \mathfrak{X}(S^1)$. So the $G^{\text{imm},n}$-normal bundle is given by

$$\mathcal{N}_c^n = \{h \in C^\infty(S, \mathbb{R}^2) : \langle L_{n,c}(h), v \rangle = 0\}.$$

The G^n-orthonormal projection $T_c \text{Imm} \to \mathcal{N}_c^n$, denoted by $h \mapsto h^\perp = h^{\perp, G^n}$ and the complementary projection $h \mapsto h^\top \in T_c(c \circ \text{Diff}(S^1))$ are determined as follows:

$$h^\top = X(h).v \text{ where } \langle L_{n,c}(h), v \rangle = \langle L_{n,c}(X(h).v), v \rangle$$

Thus we are led to consider the linear differential operators associated to $L_{n.c}$

$$L_c^\top, L_c^\perp : C^\infty(S^1) \to C^\infty(S^1),$$

$$L_c^\top(f) = \langle L_{n,c}(f.v), v\rangle = \langle L_{n,c}(f.n), n\rangle,$$
$$L_c^\perp(f) = \langle L_{n,c}(f.v), n\rangle = -\langle L_{n,c}(f.n), v\rangle.$$

The operator $L_c^\top$ is of order $2n$ and also unbounded, self-adjoint and positive on $L^2(S^1, |c_\theta|\,d\theta)$. In particular, $L_c^\top$ is injective. $L_c^\perp$, on the other hand is of order $2n-1$ and a similar argument shows it is skew-adjoint. For example, if $n=1$, then one finds that:

$$L_c^\top = -A.D_s^2 + (1 + A.\kappa^2).I$$
$$L_c^\perp = -2A.\kappa.D_s - A.D_s(\kappa).I$$

The operator $L_c^\top : C^\infty(S^1) \to C^\infty(S^1)$ *is invertible.*

We want to go back and forth between the 'natural' horizontal space of vector fields $a.n$ and the $G^{\text{imm},n}$-horizontal vector fields $\{h \mid \langle Lh, v\rangle = 0\}$: We use $C_c : C^\infty(S^1, \mathbb{R}^2) \to C^\infty(S^1)$ given by

$$C_c(h) := (L_c^\top)^{-1} \circ L_c^\perp,$$

a pseudo-differential operator of order -1 so that

$$a.n + C(a).v \quad \text{is } H^{\text{imm},n}\text{-horizontal}$$

The restriction of the metric $G^{\text{imm},n}$ to horizontal vector fields $h_i = a_i.n + b_i.v$ can be computed like this:

$$\begin{aligned} G_c^{\text{imm},n}(h_1, h_2) &= \int_{S^1} \langle Lh_1, h_2 \rangle.ds \\ &= \int_{S^1} \left(L^\top + L^\perp \circ C \right) a_1.a_2.ds. \end{aligned}$$

Thus the metric restricted to horizontal vector fields is given by the pseudo differential operator $L^{\text{red}} = L^\top + L^\perp \circ (L^\top)^{-1} \circ L^\perp$.

The metric on the cotangent space to B_i, is simple. On the smooth cotangent space $C^\infty(S^1, \mathbb{R}^2) \cong G_c^0(T_c \text{Imm}(S^1, \mathbb{R}^2)) \subset \mathcal{D}(S^1)^2$ the dual metric is given by convolution with the elementary kernel K_n.

$$\begin{aligned} \check{G}_c^n(a_1, a_2) = \iint_{S^1 \times S^1} & K_n(s_1 - s_2). \\ & .\langle n_c(s_1), n_c(s_2) \rangle.a_1(s_1).a_2(s_2).ds_1 ds_2. \end{aligned}$$

Horizontal geodesics

For any smooth path c in $\mathrm{Imm}(S^1, \mathbb{R}^2)$ there exists a smooth path φ in $\mathrm{Diff}(S^1)$ with $\varphi(t, \quad) = \mathrm{Id}_{S^1}$ depending smoothly on c such that the path e given by $e(t,\theta) = c(t, \varphi(t,\theta))$ is horizontal: $\langle L_{n,c}(e_t), e_\theta\rangle = 0$.

We may specialize the general geodesic equation to horizontal paths and then take the v and n parts of the geodesic equation. For a horizontal path we may write $L_{n,c}(c_t) = \tilde{a}n$ for $\tilde{a}(t,\theta) = \langle L_{n,c}(c_t), n\rangle$. The v part of the equation turns out to vanish identically and then n part gives us

$$\boxed{\begin{aligned}\tilde{a}_t = -\frac{|c_t|^2\kappa(c)}{2} - \langle D_s c_t, v\rangle \tilde{a} + \\ + \frac{\kappa(c)}{2}\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_s^{2n-j}c_t, D_s^j c_t\rangle\end{aligned}}$$

A Lipschitz bound for arclength in $G^{\mathsf{imm},n}$

$$\boxed{|\sqrt{\ell(C_1)} - \sqrt{\ell(C_0)}| \le \frac{1}{2}\,\mathsf{dist}_{G^n}^{B_i}(C_1, C_0)}$$

Sobolev metrics on $\text{Diff}(\mathbb{R}^2)$ **and its quotients** $\text{Emb}(S^1, \mathbb{R}^2)$ **and** $B_e(S^1, \mathbb{R}^2)$

Right invariant metric on the Lie group $\text{Diff}(\mathbb{R}^2)$ induced by the inner product

$$H^n(X, Y) = \int_{\mathbb{R}^2} \langle LX, Y \rangle \, dx \qquad \text{where}$$

$$L = L_{A,n} = (1 - A\Delta)^n, \qquad \Delta = \partial^2_{x^1} + \partial^2_{x^2}.$$

with fundamental solution $L_{A,n}(F_{A,n}) = \delta_0$ given by

$$F_{A,n}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\langle x,\xi\rangle} \frac{1}{(1 + A|\xi|^2)^n} \, d\xi$$

$$= \frac{c}{A^{(n-1)/2}}.|x|^{n-1}.K_{n-1}(\frac{|x|}{\sqrt{A}}), \qquad *$$

for the classical modified Bessel functions K_r.

The geodesic equation on Diff$(\mathbb{R}^2)$ is V.Arno'ld's equation EPDiff:

$$t \mapsto \varphi(t, \quad) \in \text{Diff}(\mathbb{R}^2)$$
$$v(t) = (\partial_t \varphi) \circ \varphi^{-1} \in \mathfrak{X}(\mathbb{R}^2), \quad u(t) = L(v(t)),$$
$$\frac{\partial u_i}{\partial t} + \sum_j \left(v^j . \frac{\partial u_i}{\partial x^j} + u^j . \frac{\partial v^j}{\partial x^i} \right) + \text{div}\, v . u_i = 0.$$

The quotient $\mathrm{Emb}(S^1, \mathbb{R}^2)$**.**

$\mathrm{Diff}(\mathbb{R}^2) \to \mathrm{Emb}(S^1, \mathbb{R}^2)$

$\varphi \mapsto \varphi \circ i$, where $i : S^1 \subset \mathbb{R}^2$.

If $c = \varphi \circ i$, the fiber through φ is

$\varphi.\{\psi : \psi \circ i = i\} = \{\psi : \psi \circ c = c\}.\varphi$.

The tangent space to the fiber is (right translated by φ)

$\{X \in X(\mathbb{R}^2) : X \circ c = 0\}$.

The horizontal subspace is the translate by φ of

$\{Y : \int_{\mathbb{R}^2} \langle LY, X \rangle \, dx = 0, \text{ if } X \circ c = 0\}$.

If Y is C^∞ then $Y = 0$. So we need

$LY = c_*(p(\theta).ds)$ for $p \in C^\infty(S^1, \mathbb{R}^2)$, a distribution carried by c. Thus

$Y(x) = \int_{S^1} F(x - c(\theta)) p(\theta) \, ds$

$Y(x) = \int_{S^1} F(x - c(\theta)) p(\theta)\, ds$
Mapped to $T_c \text{Emb}$ we get

$$(Y \circ c)(\theta) = \int_{S^1} F(c(\theta) - c(\theta_1)).p(\theta_1).|c'(\theta_1)| d\theta_1$$
$$=: (F_c * p)(\theta) \qquad \text{where}$$
$$F_c(\theta_1, \theta_2) := F(c(\theta_1) - c(\theta_2))$$

is an elliptic pseudo differential operator kernel of order $-2n + 1$ which is real and positive, so the operator $p \mapsto F_c * p$ is self-adjoint and positive, so injective, and by index deformation it is bijective between the Sobolev spaces on S^1. The inverse operator $(F_c * \quad)^{-1}$ has kernel $L_c(\theta, \theta_1)$ which is a pseudo differential operator kernel of order $2n - 1$.

Write $h = Y \circ c \in T_c\,\mathrm{Emb}$ and express the horizontal lift $Y = Y_h$ in terms of h:
$h = Y_h \circ c = F * (c_*(p.ds)) = F_c * p$ so $p = L_c * h$

$$Y = Y_h = F * (c_*((L_c * h).ds))$$

$$Y_h(x) = \int_{S^1} F(x - c(\theta)).$$
$$. \int_{S^1} L_c(\theta, \theta_1) h(\theta_1) |c'(\theta_1)| d\theta_1 |c'(\theta)| d\theta$$

Finally the metric:

$$G_c^{\mathrm{diff},n}(h,k) = \int_{\mathbb{R}^2} \langle LY_h, Y_k \rangle dx$$
$$= \iint_{S^1 \times S^1} L_c(\theta, \theta_1) \langle h(\theta_1), k(\theta) \rangle ds_1\, ds.$$

We can now compute K and H and the geodesic equation. It becomes simpler if written for the 1-current $L_c * c_t = p.|c_\theta| =: q$:

$$\boxed{q_t(\theta_0) = -\int_{S^1} F_c'(\theta_0, \theta_1)\langle q(\theta_0), q(\theta_1)\rangle \, d\theta_1}$$

where $F_c'(\theta_1, \theta_2) = \operatorname{grad} F(c(\theta_1) - c(\theta_2))$.

Conserved momenta: Along a geodesic c,

$$G_c^{\mathrm{diff},n}(c_\theta.X, c_t) =$$
$$= \iint_{S^1\times S^1} L_c(\theta,\theta_1)\langle c_\theta(\theta_1)X(\theta_1), c_t(\theta)\rangle ds_1\, ds$$

is conserved for every vector field X on S^1; the conserved **reparametrization momentum** is $\langle c_\theta, L_c * c_t\rangle = \langle c_\theta, q\rangle$.
Also $\iint_{(S^1)^2} L_c(\theta,\theta_1)c_t(\theta)\rangle ds_1\, ds = \int_{S^1} q(\theta)\, ds$ is the conserved **linear momentum**.

$$\iint_{S^1\times S^1} L_c(\theta,\theta_1)\langle Jc(\theta_1), c_t(\theta)\rangle ds_1\, ds =$$
$$= \int_{S^1}\langle Jc(\theta), q(\theta)\rangle ds$$

is the conserved **angular momentum**.

Horizontal geodesics.

A field h along c is horizontal if $\langle L_c * h, c_\theta \rangle = 0$. For a horizontal path we have $\langle q, c_\theta \rangle = 0$, so let $q = \tilde{a}.n$. Then the horizontal geodesic equation is

$$\boxed{\begin{aligned}&\tilde{a}_t(\theta) = \langle q_t, n \rangle(\theta) = \\ &= -\int_{S^1} \langle F_c'(\theta,\theta_1), n(\theta)\rangle \tilde{a}(\theta)\tilde{a}(\theta_1)\langle n(\theta), n(\theta_1)\rangle \, d\theta_1\end{aligned}}$$

Note that also $n = Jc_\theta/|c_\theta|$ appears.