Multiterminal Network Tomography

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Networks

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• Let $G$ be a directed graph without loops and without multiple directed edges.
• Sources will be called incoming terminals, sinks will be called outgoing terminals, all other vertices will be called intermediate terminals.
• $G$ is called a multiterminal network.
An Example
Another Example
The Inverse Problem

• Assign a number to each arrow of $G$, the probability that a message goes through that arrow.
The Inverse Problem

- Assign a number to each arrow of $G$, the *probability* that a message goes through that arrow.
The Inverse Problem

• Assign a number to each arrow of $G$, the probability that a message goes through that arrow.

• Goal: to recover these numbers from measurements at the incoming and outgoing terminals.
Why is this tomography?
Why is this tomography?

Detector

X-ray source
Why is this tomography?
Why is this tomography?
Why is this tomography?
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An Example
An Example

• Consider the numbers $\delta_i$ which represent the probability that a message goes from source to sink in $i$ steps.
An Example

• Consider the numbers

\[ \delta_i = \begin{cases} 
(a c + b d) f^k e^k & i = 2k + 2 
\end{cases} \]
An Example

• Consider the numbers

$$\delta_i = \begin{cases} 
(ac + bd) f^k e^k & i = 2k + 2 \\
(af d + bec) f^k e^k & i = 2k + 3 
\end{cases}$$
An Example

- Consider the numbers $\delta_i \implies$ probability that a message goes from source to sink in $i$ steps.
An Example

• Consider the numbers
  \( \delta_i \Rightarrow \) probability that a message goes from source to sink in \( i \) steps

• These numbers determine
  \[ ac + bd, \quad afd + bec, \quad fe \]
Observations

• The numbers $\delta_2, \delta_3, \delta_4$ contain all the information.
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$$Q^{(i)} = \sum_k k^i \delta_k$$
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• The numbers $\delta_2, \delta_3, \delta_4$ contain all the information.

• We can determine three quantities, when we’d like six.

• More reasonable to measure:

$$Q^{(i)} = \sum_{k} k^i \delta_k$$

the moments of travel time.
Another Example
Another Example

Get everything from $Q^{(0)}$, $Q^{(1)}$
Another Example

Get everything from $Q^{(0)}, Q^{(1)}$ quickly!
Plan for the rest of the talk
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- Explain how to solve the “4pixelman” problem. (Grünbaum and Patch; Grünbaum and M.)
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• Construct a large class of networks for which this method works.
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• Construct a large class of networks for which this method works.

Setting up the equations
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\[ P_{HH} = \begin{pmatrix} 0 & f \\ e & 0 \end{pmatrix} \]
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\[ P_{HO} = \begin{pmatrix} c \\ d \end{pmatrix} \]
Setting up the equations

• Construct four matrices:

\[ P_{IO}, P_{IH}, P_{HH}, P_{HO} \]

• Want to recover these matrices from:

\[
Q^{(0)} = P_{IO} + \sum_{k=0}^{\infty} P_{IH} P_{HH}^k P_{HO} = P_{IO} + P_{IH} (I - P_{HH})^{-1} P_{HO}
\]

\[
Q^{(1)} = Q^{(0)} + P_{IH} (I - P_{HH})^{-2} P_{HO}
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- These are nonlinear equations!
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  \[
  Q^{(1)} = Q^{(0)} + P_{IH} (I - P_{HH})^{-2} P_{HO}
  \]

- These are nonlinear equations!(polynomial)
Change Variables!
Change Variables!

- Consider the transformation

\[ A := P_{HO}^{-1}, \quad X := P_{IO}A, \]
\[ W := AP_{HH}, \quad Y := XA^{-1}W - P_{IH} \]
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- Need the same number of terminals of all types!
Change Variables!

- Consider the transformation

\[ A := P_{HO}^{-1}, \quad X := P_{IO} A, \]
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- Need the same number of terminals of all types! (OK in the case of 4pixelman)
Change Variables!

• Consider the transformation

\[ A := P_{HO}^{-1}, \quad X := P_{IOA}, \]
\[ W := AP_{HH}, \quad Y := XA^{-1}W - P_{IH} \]

• Fiddle with the equations to get

\[ Q^{(0)}(A - W) = X - Y \]
\[ (R - Q^{(0)})A = RW - X \]

where

\[ R = Q^{(1)} - Q^{(0)} \]
Change Variables!

- Consider the transformation
  \[ A := P_{HO}^{-1}, \quad X := P_{IO} A, \]
  \[ W := AP_{HH}, \quad Y := XA^{-1}W - P_{IH} \]

- Fiddle with the equations to get
  \[ Q^{(0)}A - X = -Q^{(0)}W - X \]
  \[ (R - Q^{(0)})A = RW - X \]

- These are linear.
Change Variables!

• Solve the linear equations

\[ Q^{(0)}(A - W) = X - Y \]
\[ (R - Q^{(0)})A = RW - X \]
Change Variables!

- Solve the linear equations

\[ Q^{(0)} (A - W) = X - Y \]
\[ (R - Q^{(0)}) A = RW - X \]

- The block structure in the case of 4pixelman allows explicit solving.
Change Variables!

• Solve the linear equations

\[ Q^{(0)} (A - W) = X - Y \]

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• The block structure in the case of 4pixelman allows explicit solving.

• It also guarantees recovery of all data.
Change Variables!

- Solve the linear equations
  \[ Q^{(0)}(A - W) = X - Y \]
  \[ (R - Q^{(0)})A = RW - X \]
- The block structure in the case of 4pixelman allows explicit solving.
- It also guarantees recovery of all data up to a natural gauge.
Some formulas

\[(P_{IO})_{88} = \left( \frac{Q[4,5,8;2,3,8]}{Q[4,5;2,3]} \lambda + \frac{Q[4,5,8;1,2,3]}{Q[4,5;2,3]} \right) - \left( \frac{Q[4,5,8;1,6,7]}{Q[4,5;6,7]} + \frac{Q[4,5,8;6,7,8]}{Q[4,5;6,7]} \mu \right) \frac{\lambda - \mu}{\lambda - \mu} \]
Some formulas

\[(P_{IO})_{88} = \left( \frac{Q[4,5,8;2,3,8]}{Q[4,5;2,3]} \lambda + \frac{Q[4,5,8;1,2,3]}{Q[4,5;2,3]} \right) - \left( \frac{Q[4,5,8;1,6,7]}{Q[4,5;6,7]} + \frac{Q[4,5,8;6,7,8]}{Q[4,5;6,7]} \mu \right) \frac{\lambda - \mu}{\lambda - \mu} \]

\[\lambda = \frac{R_{11} Q[4,5;3,2] + (R_{12} + Q_{12}) Q[4,5;1,3] + (R_{13} + Q_{13}) Q[4,5;2,1]}{R_{18} Q[4,5;2,3] + (R_{12} + Q_{12}) Q[4,5;3,8] + (R_{13} + Q_{13}) Q[4,5;8,2]} \]
Some formulas

\[(P_{IO})_{88} = \frac{\left(\frac{Q[4,5,8;2,3,8]}{Q[4,5;2,3]} \lambda + \frac{Q[4,5,8;1,2,3]}{Q[4,5;2,3]}\right) - \left(\frac{Q[4,5,8;1,6,7]}{Q[4,5;6,7]} + \frac{Q[4,5,8;6,7,8]}{Q[4,5;6,7]} \mu\right)}{\lambda - \mu}\]

\[\lambda = \frac{R_{11}Q[4, 5; 3, 2] + (R_{12} + Q_{12})Q[4, 5; 1, 3] + (R_{13} + Q_{13})Q[4, 5; 2, 1]}{R_{18}Q[4, 5; 2, 3] + (R_{12} + Q_{12})Q[4, 5; 3, 8] + (R_{13} + Q_{13})Q[4, 5; 8, 2]}\]

\[\mu = \frac{R_{11}Q[4, 5; 7, 6] + (R_{16} + Q_{16})Q[4, 5; 1, 7] + (R_{17} + Q_{17})Q[4, 5; 6, 1]}{R_{18}Q[4, 5; 6, 7] + (R_{16} + Q_{16})Q[4, 5; 7, 8] + (R_{17} + Q_{17})Q[4, 5; 8, 6]}\]
A different way to draw
A different way to draw
A different way to draw
A different way to draw
A different way to draw
A different way to draw
Now add arrows as before…
And replace the square!
And replace the square!
And replace the square!
Recall - this graph:
Gives this network:
So many limitations…
So many limitations…

• We can deal with a limited class of networks.
So many limitations…

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• Realistically, many more intermediate terminals than outgoing or incoming.
So many limitations…

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- Messages should carry addresses.
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• Messages can interfere with each other.
So many limitations…

- We can deal with a limited class of networks.
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- Messages should carry addresses.
- Messages can interfere with each other.
- ( ……… )
This is just the beginning...
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Thanks!