Appraisal Analysis in Geophysical Inverse Problem: Tool for Image Interpretation and Survey Design

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Image Interpretation Problem

• What scale of structure we can believe?
• What features can be resolved?
• How much of this picture has come from prior information?

Sources
★ Receivers

Slowness (1/Velocity) Map
Survey Design Problem

- Can we improve the model by designing a new survey?
- Where should we place our sources/receivers with logistical constraints?
- How can we improve signal-to-noise ratio?
**Basic Goal of Interpretation & Survey Design**

**Interpretation:** We do the experiment and try to find the “right” support volume from the data.

**Survey Design:** We define the support volume and design the “right” experiment for enhanced resolution.
Physics

Model

Imaging/Inversion

Perfect Data

Ideal World
Outline of the Talk

- **Linearized Appraisal Analysis**: (Backus & Gilbert, 1970)
  - Point Spread Function (PSF)
  - Averaging Kernel

- **PSF for survey design**

- **Nonlinear Appraisal Analysis and image interpretation**
  - Nonlinear PSF
  - Region of Data Influence Analysis (Oldenburg & Li, 1999)
  - Funnel Function Analysis (Oldenburg, 1983)
  - Nonlinear Resolution with Inverse Scattering Approach (Snieder, 1991)

- **Conclusions and a Recipe**
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**Linear Problem**

Forward Problem: \[ d^{\text{obs}} = Gm + \epsilon \]

Inverse Problem:
\[
\min \phi = \left\| W_{d} (Gm - d^{\text{obs}}) \right\|^2 + \beta \left\| W_{m} (m - m_0) \right\|^2
\]

Solution:
\[
\hat{m} = (G^T W_d W_d G + \beta W_m W_m)^{-1} (G^T W_d W_d d^{\text{obs}} + \beta W_m W_m m_0)
\]

\[
\hat{m} = Rm + (I - R)m_0 + e
\]

Where:
\[
R = (G^T W_d W_d G + \beta W_m W_m)^{-1} (G^T W_d W_d G)
\]
\[
e = (G^T W_d W_d G + \beta W_m W_m)^{-1} (G^T W_d W_d \epsilon)
\]

R: Resolution Matrix
Appraisal Analysis
What can we tell about the model?

Model

Limitation from Resolution Operator

Bias from prior Information

Errors in Data

Model Estimate
Linear Resolution Matrix

\[ R = (G^T W_d^T W_d G + \beta W_m^T W_m)^{-1} (G^T W_d^T W_d G) \]

Rows of R are **Averaging Functions**

Columns of R are **Point Spread Functions**
Nonlinear Problem

Forward Problem: \( d^{obs} = F(m) + \varepsilon \)

Inverse Problem: \( \min \phi = \left\| W_d (F(m + \delta m) - d^{obs}) \right\|^2 + \beta \left\| W_m (m + \delta m - m_0) \right\|^2 \)

Solution:

\[
(J_n^T W_d^T W_d J_n + \beta W_m^T W_m) \delta \hat{m} = J_n^T W_d^T W_d (d^{obs} - F(m_n) - P(\delta \hat{m})) - \beta W_m^T W_m (m_n - m_0)
\]

Linearization Error

\[
d^{obs} = F(m_n + \delta m) + \varepsilon
\]

\[
d^{obs} = F(m_n) + J_n \delta m + Q(\delta m) + \varepsilon
\]

\[
\delta \hat{m} = R_n \delta m + A_n^{-1} J_n^T W_d^T W_d (Q(\delta m) - P(\delta \hat{m}) + \varepsilon) + \beta A_n^{-1} W_m^T W_m \delta m_0
\]

Linearized Resolution Matrix
Nonlinear Problem: Linearized Resolution

\[ R_n = (J_n^T(m_n)W_d^TW_dJ_n(m_n) + \beta W_m^TW_m)^{-1} J_n^T(m_n)W_d^TW_dJ_n(m_n) \]

Physics:
- Model
- Acquisition Parameters
- Discretization

Bayesian Formulation

\[ R_n = (J_n^T(m_n)C_d^{-1}J_n(m_n) + \beta C_m^{-1})^{-1} J_n^T(m_n)C_d^{-1}J_n(m_n) \]

Data Noise
Prior Information
Regularization
Linear Problem

\[ d_j = (g_j(x), m(x)) + \epsilon_j \]

**Kernel**

[kernel plots]

**Model**

[graph showing model output]

**Data**

[plot of data]

**kernels:** \[ g_j(x) = \text{Damped Sine & Cosine functions} \]

**Experiment:**
- cosine \((j = 0, 17)\)
- sine \((j = 1, 7)\)
Linear Problem

True Model

Inverted Model

Resolution Matrix
Point Spread Functions

Resolution decreases with depth
Point Spread Functions

- Kernels (# and type)
- Data accuracy
- Details of model objective function
- Reference model

Survey

Prior information

Routh et. al (2005); Routh et. al, (in prep)
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- Conclusions and a Recipe
PSF for comparing between two surveys

Model From survey-1
11 Kernels

Model From survey-2
24 Kernels

PSF
Survey Optimization using PSF

Why PSF?

- Physics
- Data noise
- Prior information
- Regularization
- Model discretization
Survey Optimization using PSF

“Given a region of interest where target is most likely to occur, can we design survey to provide enhanced resolution?”

Methodology

• We pose the inverse problem by **maximizing a resolution measure** i.e. point spread function (PSF).

• **Very Nonlinear** Constrained Optimization Problem:
  - Newton strategy based on **primal interior point**.
  - Global optimization using **simulated annealing**.
Survey Optimization using PSF

\[ \min \left\| W_k \left( p_k (\xi) - \Delta_k \right) \right\|^2 \quad \text{s.t.} \quad \xi_{\text{min}} \leq \xi \leq \xi_{\text{max}} \]

\( \xi \): Survey Parameters
Survey Optimization using PSF: Newton Method

Minimize: \( \psi (\xi) = \| W_k (p_k(\xi) - \Delta_k) \|^2 - \lambda \sum_{k=1}^{N} \log \left( 1 - \frac{\xi_k}{\xi_{k_{\max}}} \right) \)

\[ - \lambda \sum_{k=1}^{N} \log \left( \frac{\xi_k - \xi_{k_{\min}}}{\xi_{k_{\max}}} \right) \]

Solve:
\[
\begin{bmatrix}
W_k J \\
\sqrt{\lambda} X \\
\sqrt{\lambda} Y
\end{bmatrix}
\begin{bmatrix}
\delta \xi
\end{bmatrix} =
\begin{bmatrix}
W_k (\Delta_k - p_k(\xi)) \\
\sqrt{\lambda} e \\
- \sqrt{\lambda} e
\end{bmatrix}
\]

Update:
\( \xi^{(n+1)} = \xi^{(n)} + \alpha \delta \xi \)
Tomography Example

Inverted Model

Raypaths
Tomography Example: Raypaths

Before

After (Newton)

After (SA)
Tomography Example

PSF (Original Survey): Region I

PSF (Newton)

PSF (SA)
Tomography Example

PSF (Original Survey): Region II

PSF (Newton)     PSF (SA)
Tomography Example: BG Averaging Functions

Averaging Function before survey optimization

AVG (Newton)  AVG (SA)
Tomography Example: Inverted Models

After: Newton

After: SA
Nonlinear Example: Controlled Source Electromagnetic Survey Design

Frequency Sounding

Closed Equations:
\[
\nabla \times E = -i \omega \mu H \\
\nabla \times H = \sigma E + J_s
\]

Data:
\[
E_x(\omega), E_y(\omega), E_z(\omega), H_x(\omega), H_y(\omega), H_z(\omega)
\]
Goal of Survey Design

- What frequencies provide better resolution at the region of interest? with constraints $0.1 \leq f \leq 10000$
Survey Design: Nonlinear CSEM Problem

Before Survey Optimization: Frequencies used: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192 Hz
Survey Design: Nonlinear CSEM Problem

- After Survey Optimization: Frequencies are: 0.88, 2.6, 16.3, 63, 73, 81, 129, 190, 229, 279, 287, 352, 372, 1396, 2251 Hz
More Survey Design Questions…

• Which fields (E or H) and what components are better to acquire?

• Which fields can be combined to give better resolution at region of interest?

• **Methodology:** Compare the PSF from different multi-component surveys at the region of interest.
PSF for Multi-Component Data

ExHyHz
ExHy
Ex

Model

PSF

PSF in ROI
PSF for Multi-Component Data

Model

Hy

Hz

Rho-Phi

PSF

PSF in ROI
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• Point Spread Function for survey design

• Nonlinear Appraisal Analysis and image interpretation

  Nonlinear Point Spread Function
  • Region of Data Influence
  • Funnel Function Analysis (upper and lower bounds)
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• Conclusions and a Recipe
Back to Point Spread Function

\[ p_k = R \Delta_k \]

\[ p_k = \left( G^T W_d^T W_d G + \beta W_m^T W_m \right)^{-1} \left( G^T W_d^T W_d G \right) \Delta_k \]
PSF as Model Construction Process

For a single PSF

\[
(G^T W_d^T W_d G + \beta W_m^T W_m) p_k = (G^T W_d^T W_d G) \Delta_k
\]

PSF construction as an optimization problem

\[
\min \beta \left\| W_m (p_k - \Delta_k) \right\|^2 + \left\| W_d (G p_k - G \Delta_k) \right\|^2
\]

\[
\begin{bmatrix}
W_d G \\
\sqrt{\beta} W_m
\end{bmatrix}
\begin{bmatrix}
p_k
\end{bmatrix}
=
\begin{bmatrix}
W_d G \Delta_k \\
0
\end{bmatrix}
\]
Development of Nonlinear PSF

\[
\min_\beta \left\| W_m (p_k - \Delta_k) \right\|^2 + \left\| W_d \left( Gp_k - G\Delta_k \right) \right\|^2
\]

Linear Operator

\[
\min_\beta \left\| W_m (p_k - \Delta_k) \right\|^2 + \left\| W_d \left( F(p_k) - F(\Delta_k) \right) \right\|^2
\]

Nonlinear Operator

Routh et. al, 2005 (in prep)
Nonlinear Example

\[
d_j = e^{-\int g_j(x) m(x) \, dx} + \epsilon_j
\]

- To compare linear and nonlinear PSF chose a impulse response function.

\[
m(x) = \delta_k
\]

- Solve the nonlinear inverse problem and obtain the model. This is nonlinear PSF

\[
p^{\text{nonlin}}_k(x) = \hat{m}(x)
\]

- Generate the linearized resolution matrix and extract the linearized PSF for the \(k^{\text{th}}\) cell.

\[
p^{\text{lin}}_k(x) = R \delta_k
\]
Comparison of Nonlinear PSF & Linear PSF

(cell 10)

(cell 19)

(cell 20)

(cell 30)

(cell 50)

(cell 75)

(cell 85)

(cell 90)
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• **Conclusions and a Recipe**
Region of Data Influence

- Invert the data with two different reference model \( m_1^{\text{ref}} \), \( m_2^{\text{ref}} \)

\[
\min \phi = \left\| W_d (F(m) - d^{\text{obs}}) \right\|^2 + \beta \left\| W_m (m - m^{\text{ref}}) \right\|^2
\]

- Basic idea: region **not** sensitive to data will recover the reference model \( m_1 \to m_1^{\text{ref}} \), \( m_2 \to m_2^{\text{ref}} \)

\[
RDI = \frac{\left|m_2 - m_1\right|}{\left|m_2^{\text{ref}} - m_1^{\text{ref}}\right|}
\]

- Region **NOT** sensitive to data will indicate RDI \( \to 1 \)

Oldenburg and Li, 1999
Region of Data Influence

RDI Map

Inversion Ref=1.0

Inversion Ref=1.5
Region of Data Influence

RDI Map

Inversion Ref=1.0

Inversion Ref=1.5
Region of Data Influence: DC Resistivity Example
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• **Conclusions and a Recipe**
Funnel Function Approach

- Consider computing average value within a window
- Window is our support volume
- Can we compute unique average value?
- Can we compute upper & lower bound of the average value?

\[ m(x) \]

\[ x_0 - \frac{\Delta}{2} \quad x_0 \quad x_0 + \frac{\Delta}{2} \]

Oldenburg, 1983; Backus & Gilbert, 1970a,b,c
Funnel Function Approach

- Backus-Gilbert Inference theory provides the general mechanism.

\[ d(x_0) = (g(x, x_0), m(x)) \]

\[ e(x_0) = (p(x, x_0), m(x)) \]

Inference theory
Prediction kernel

Oldenburg, 1983
Funnel Function cont…

• What are our choices for $p(x, x_0)$?

$$p(x, x_0) = \delta(x - x_0) \Rightarrow e(x_0) = m(x_0)$$
$$p(x, x_0) = A(x, x_0) \Rightarrow e(x_0) = \left( A(x, x_0), m(x) \right) = \left< m(x_0) \right>_A$$
$$p(x, x_0) = B_\Delta(x, x_0) \Rightarrow e(x_0) = \left( B_\Delta(x, x_0), m(x) \right) = \left< m(x_0) \right>_B$$

• Approach of Backus & Gilbert (1970):
  
  • express inference prediction kernel as linear combination of kernels
    $$A(x, x_0) = \sum_j \beta_j(x_0) g_j(x)$$
  
  • Make prediction kernel close to delta function
    $$\left< m(x_0) \right>_A \Rightarrow \delta(x - x_0)$$

Oldenburg, 1983
Funnel Function Cont…

• In funnel function we chose the support volume

\[ P(x, x_0) = B_\Delta(x_0, x) \]

• Recognize that unique average cannot be obtained since

\[ B_\Delta(x_0, x) \neq \sum_j \alpha_j(x_0) g_j(x) \]

• Determine the bounds on the average value \( \left\langle m(x_0) \right\rangle_B \) by solving two optimization problem (max & min) subject to fitting the data. *(We use interior point Gauss-Newton method to solve).*

Oldenburg, 1983; Routh et. al. 2005 (in prep)
Funnel Function Cont...
Funnel Functions for Tomography Problem

\[ \langle m(r_0) \rangle_B \]

\[ \langle m(x_0) \rangle^{\text{UPPER}} \quad 0.05 \leq m(x) \leq 2 \]

\[ \langle m(x_0) \rangle^{\text{UPPER}} \quad 0.05 \leq m(x) \leq 1.2 \]

\[ \langle m(x_0) \rangle^{\text{LOWER}} \quad 0.05 \leq m(x) \leq 1.2 \]

\[ \langle m(x_0) \rangle^{\text{LOWER}} \quad 0.05 \leq m(x) \leq 2 \]

Routh et. al, (in prep)
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• **Conclusions and a Recipe**
Nonlinear Appraisal Using Inverse Scattering

- 3D Equation for Electric Field ($\nabla \cdot E = 0$): $\nabla^2 E + k^2 E = S(\omega)\delta(r - r_S)$

- Perturbation E Field
  $$\delta E(r) = -i \omega \mu \int G(r, r') \delta\sigma(r') E(r') dv'$$

- Scattering Equation
  $$\delta E = (I - Q)^{-1} Q E_0$$

- For sparse data
  $$\delta E_{\text{OBS}} = P(I - Q)^{-1} Q E_0$$

- The scattering matrix is given by
  $$Q = GC$$
  $$Q_{kj} = -i \omega \mu G(r_k, r_j) C_{jj} \quad j \neq k$$
  $$= 0 \quad j = k$$
Forward Scattering


Nonlinear Appraisal Using Inverse Scattering

- Series expression for the perturbation data
  \[ \delta E^{OBS} = \left( \tilde{G}C + \tilde{G}CGC + \tilde{G}C(\tilde{G}C)^2 + \cdots \right)E_0 \]
  \[ \delta E^{OBS} = \left( \tilde{G}\tilde{C} + \tilde{G}CG\tilde{C} + \tilde{G}CGCG\tilde{C} + \cdots \right) = \left( \tilde{G}\tilde{C} + \tilde{G}E_0^{-1}\tilde{C}G\tilde{C} + \tilde{G}E_0^{-1}\tilde{C}GE_0^{-1}\tilde{C}G\tilde{C} + \cdots \right) \]

- Expand scattering strength
  \[ C = C_1 + C_2 + C_3 + \ldots \]

- Finally, we obtain expression of estimated scattering strength

\[ \tilde{C} = \tilde{G}^{-1}\delta E^{OBS} - \tilde{G}^{-1}\tilde{C}G_1G\tilde{G}^{-1}\delta E^{OBS} + \]
\[ \tilde{G}^{-1}\tilde{C}G_1G\tilde{G}^{-1}\tilde{G}C_1G\tilde{G}^{-1}\delta E^{OBS} + \tilde{G}^{-1}\tilde{G}E_0^{-1}\tilde{G}^{-1}\tilde{G}C_1GE_0C_1G\tilde{G}^{-1}\delta E^{OBS} - \]
\[ \tilde{G}^{-1}\tilde{C}G_1GC_1G\tilde{G}^{-1}\delta E^{OBS} + \ldots \]

where \( \tilde{C} = \hat{C}E_0 \) & \( C_1 = E_0^{-1}\tilde{G}^{-1}\delta E^{OBS} \)

- One choice for the inversion operator

\[ \tilde{G}^{-1} = \left( \tilde{G}^H\tilde{G} + \beta I \right)^{-1}\tilde{G}^H \quad \text{Note: } \tilde{G} = PG \]

Lax, 1951; Weglein et. al., 2003; Cheney & Bonneau, 2004; Malcolm & de Hoop, 2005
Nonlinear Appraisal Using Inverse Scattering

- Substitute \( C_1 = E_0^{-1} \tilde{G}^{-1} \delta E^{obs} \) and \( \delta E^{obs} = \tilde{G} \tilde{C} + \tilde{G} E_0^{-1} \tilde{C} \tilde{G} \tilde{C} + \ldots \)

- Writing terms up to second order

\[
\hat{C} = \tilde{G}^{-1} \tilde{G} \tilde{C} + \tilde{G}^{-1} \tilde{G} E_0^{-1} \tilde{C} \tilde{G} \tilde{C} - \tilde{G}^{-1} \tilde{G} E_0^{-1} \tilde{G}^{-1} \tilde{G} \tilde{C} \tilde{G} \tilde{G}^{-1} \tilde{G} \tilde{C} + \text{HOT}
\]

Writing \( R = \tilde{G}^{-1} \tilde{G} \)

\[
\hat{C} = R \tilde{C} + RE_0^{-1} \tilde{C} \tilde{G} \tilde{C} - RE_0^{-1} R \tilde{C} \tilde{G} \tilde{R} \tilde{C} + \text{HOT}
\]

If \( R = I \) we see higher order nonlinear contribution = 0

\[
\hat{C} = \tilde{C} + E_0^{-1} \tilde{C} \tilde{G} \tilde{C} - E_0^{-1} \tilde{C} \tilde{G} \tilde{C} + \ldots
\]

or, \( \hat{C} = \tilde{C} \) cancel
Conclusions

• Geophysical Data are “inadequate, insufficient and noisy.” Appraisal analysis provides a means to address non-uniqueness in solutions.

• Point Spread Function is particularly useful measure since it is connected to all of the components that affect resolution: physics, prior information, data noise, regularization, model discretization.

• We developed methodology to examine survey design problem.

• Examined funnel functions to get an estimate of bounds for a given support volume.

• Region of data influence method provided a measure where physics have sampled the Earth.

• We arrived at a formalism for nonlinear appraisal using inverse scattering approach, needs more work.
Recipe

Determine the Resolution in Region of Interest with PSF

Determine RDI using two prior

Determine the RDI

Compute Funnel functions
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