New Mathematics and Algorithms for 3-D Image Analysis
IMA, January 9-12, 2006

On mathematics of thermoacoustic imaging

Peter Kuchment
Mathematics Department
Texas A&M University
College Station, TX
http://www.math.tamu.edu/~kuchment

Partially supported by NSF grants

An outline

1. Thermoacoustic Tomography (TAT or TCT)

2. Mathematics of TAT - circular (spherical) Radon transform

3. Uniqueness of reconstruction

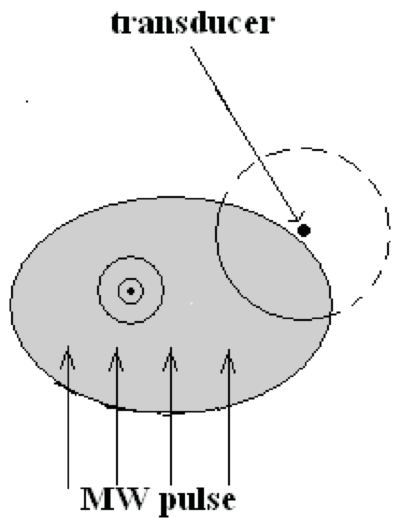
4. Partial data: visible and invisible singularities.

5. Reconstruction: formulas and examples

6. Range conditions

7. Some open problems

1. **Thermoacoustic Tomography (TAT or TCT)** and its sibling photo-acoustic Tomography (PAT)



The integrals of the MW energy absorption function $f(x)$ over circles centered at transducers' locations are measured.

In mammography, cancerous cells absorb 3 – 5 times more MW (or RF) energy than the healthy ones $\Rightarrow$ high contrast. Also high resolution.

2. **Mathematics of TAT - circular (spherical) Radon transform**

Circular Radon transform of f on $\mathbb{R}^d$:

$$Rf(p, r) = \int_{|y-p|=r} f(y) d\sigma(y),$$

Overdetermined.

$S \subset \mathbb{R}^d$ – hypersurface, then

$$R_S f = Rf|_{p \in S}.$$

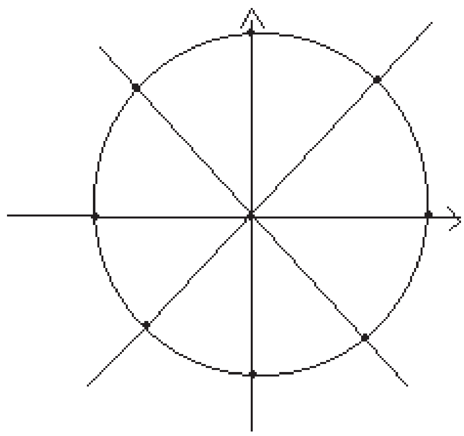
Problems:

- For what sets S is the R_S injective (i.e., f can be uniquely reconstructed from Rf)?
- If R_S is injective, what are inversion formulas?
- How stable is inversion?
- What happens if the data is incomplete?
- Range description for R_S

3. **Uniqueness of reconstruction**

f - compactly supported on $\mathbb{R}^d$. When does $R_S f = 0$ imply $f = 0$? S - **uniqueness set**.

Non-uniqueness sets in $\mathbb{R}^2$
(Lin-Pinkus, '92, Agranovsky-Quinto, '94–'96):
I. $S =$ a straight line. $Rf = 0$ for any odd function.
II. A Coxeter system Σ_N of lines:



III. One can add any finite set of points to a non-uniqueness set (not totally trivial).

Theorem 1. *(Agranovsky-Quinto, '94-'96, conjectured by Lin-Pinkus, '92)* $S \subset \mathbb{R}^2$ *is a non-uniqueness set iff*

$$S \subset \omega\Sigma_N \cup \Phi,$$

Σ_N *- Coxeter system of lines,* ω *rigid motion,* Φ *is finite.*

Proof is based on microlocal analysis and geometry of zeros of harmonic polynomials.

What do harmonic polynomials do here? The answer (that holds in any dimension) is:

Lemma 2. *(various authors '92 – '94) Any non-uniqueness set is a set of zeros of a harmonic polynomials. In particular, any uniqueness set for harmonic polynomials (e.g., any closed hypersurface) is uniqueness set for circular Radon transform.*

Relations with a variety areas of analysis - from approximation theory to PDEs, potential theory, and complex analysis. E.g.:

Theorem 3. *(P. K., Agranovsky & Quinto '93) TFAE:*
1. $S \subset \mathbb{R}^d$ *is a non-uniqueness set.*
2. S *is a nodal set for the wave equation, i.e. there exists a compactly supported* f *such that the solution of the problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u \\ u(0,x) = 0 \\ u_t(0,x) = f(x) \end{cases}$$

vanishes on S *for any moment of time.*
3.S *is a nodal set for the heat equation, i.e. there exists a compactly supported* f *such that the solution of the problem*

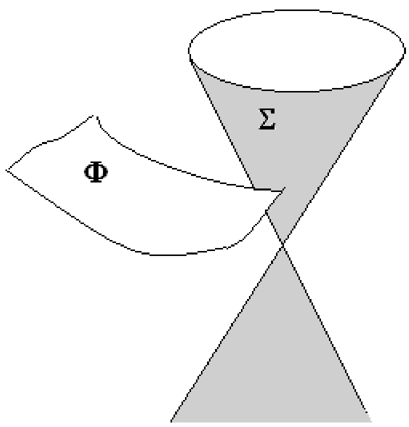
$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ u(0,x) = f(x) \end{cases}$$

vanishes on S *for any moment of time.*

Conjecture: $S \subset \mathbb{R}^d$ is a non-uniqueness set iff

$$S \subset \omega\Sigma \cup \Phi,$$

Σ - the surface of zeros of a homogeneous harmonic polynomial, ω is a rigid motion, Φ is an algebraic surface of codimension at least 2.

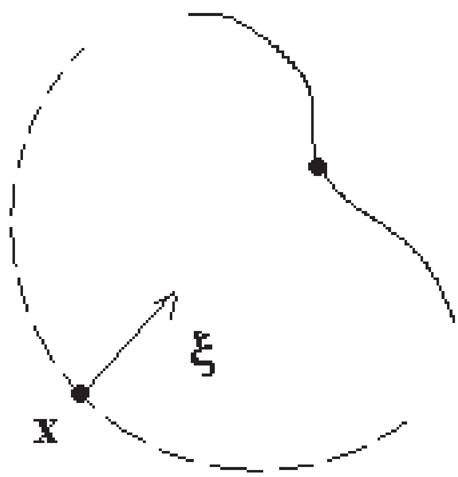


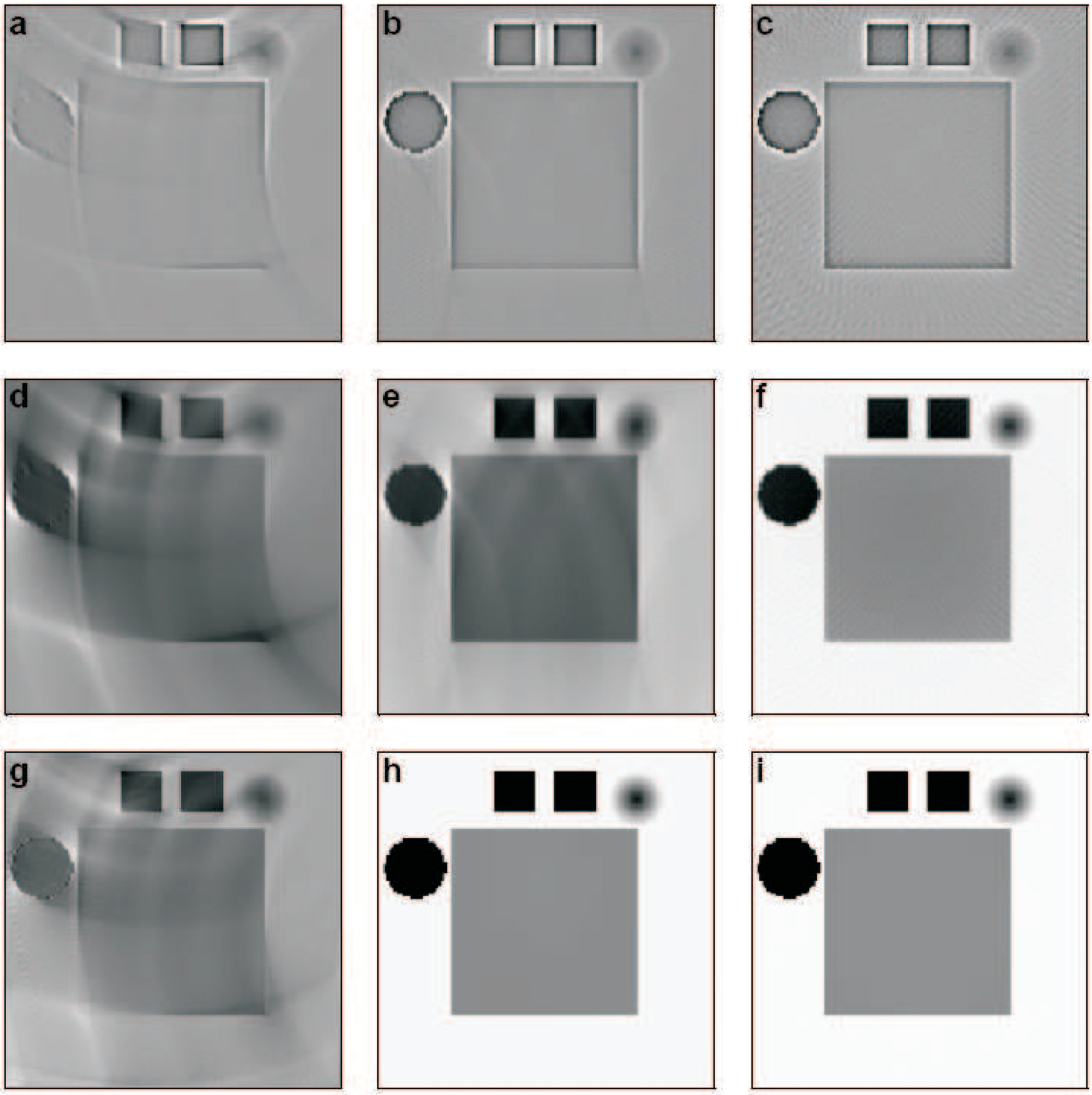
Recent progress based on the wave equation reformulation (Finch & Patch & Rakesh '04, P. K. & Ambartsoumian '05)

4. **Partial data: "visible" and "invisible" singularities**

Uniqueness of reconstruction does not imply practical recoverability: the reconstruction might be severely unstable.

Theorem 4. *(Quinto, Louis & Quinto, Ambartsoumian & P. K.) A wavefront set point* (x, ξ) *of* f *is "stably recoverable" from* $R_S f$ *iff there is a circle centered on* S*, passing through* x*, and normal to* ξ *at this point.*

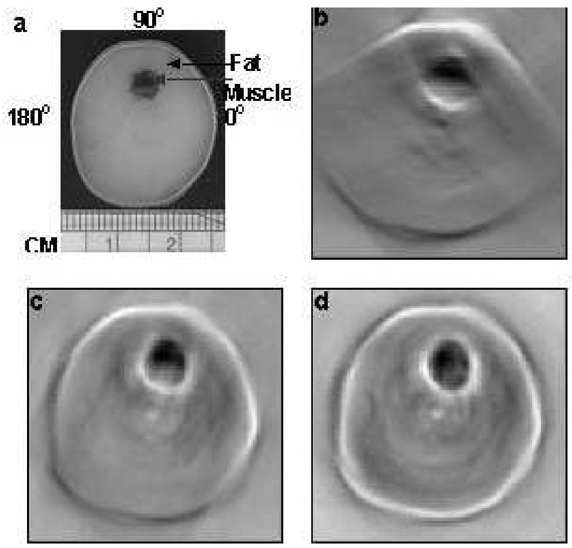



a
b
c
d
e
f
g
h
i

5. **Reconstruction: formulas and examples**

One of the most popular inversion formulas for the standard Radon transform: data $g(\omega, s)$, where the plane of integration is $x \cdot \omega = s$. Inversion consists of *filtration* by taking second s-derivative of the data and *backprojection* by averaging over all planes passing through the point x where the reconstruction is done.

Approximate inversions for the circular Radon R_S mimic this formula. The results are usually good and can be improved by iterations.



Exact inversions of R_S depend on the surface S. Formulas are known for S being a sphere, cylinder, or a plane.

For S - sphere there are inversions through special functions expansions.

Formulas analogous to standard Radon are known for S sphere in odd dimensions only, and for S plane.

One of the $3D$ formulas for S =unit sphere (Finch & Patch & Rakesh '04):

$$f(\mathbf{x}) = \frac{1}{16\pi^3} \Delta_{\mathbf{x}} \left(\int_{|\mathbf{p}|=1} \frac{1}{|\mathbf{x}-\mathbf{p}|} R_{\mathrm{TCT}} f(\mathbf{p}, |\mathbf{x}-\mathbf{p}|) \, d\mathbf{p} \right)$$

Implemented by G. Ambartsoumian & S. Patch ('04–'05)

6. **Range conditions**

Radon's range is of ∞ co-dimensions. Knowing the range is important (error correction, incomplete data completion, etc.).

For standard Radon: for any integer $k \geq 0$,

$$G_k(\omega) = \int_{-\infty}^{\infty} s^k g(\omega, s) ds$$

is a homogeneous polynomial of degree k.

S - sphere. Range conditions for R_S? S. Patch '04 found an incomplete set of such conditions on data $g(p,r) = R_S f(p,r)$ (p - center, r - radius): for any integer $k \geq 0$,

$$G_k(\omega) = \int_0^{\infty} r^{2k} g(p,r) dr$$

is a (non-homogeneous) polynomial of degree at most k.
A complete set is found in $2D$ (Ambartsoumian & P. K '05) and $3D$ (D. Finch & Rakesh)

Theorem 5. *(Ambartsoumian & P. K '05) In order for the function $g(p,r)$ on $S^1 \times \mathbb{R}$ to be representable as $R_S f$ with $f \in C_0^\infty(D)$, it is necessary and sufficient that the following conditions are satisfied:*

(a) $g \in C_0^\infty(S^1 \times (0,2))$.

(b) *For any n, the $2k$-th moment $\int_0^\infty r^{2k} g_n(r)dr$ of the n-th Fourier coefficient of g vanishes for integers $0 \le k < |n|$. (S. Patch's conditions)*

(c) *For any $n \in \mathbb{Z}$, function $\mathcal{H}_0\{g_n(r)/r\}(\sigma) = \int_0^\infty J_0(\sigma r) g_n(r) dr$ vanishes at any zero $\sigma \neq 0$ of Bessel function J_n.*

7. **Some open problems**

1. Inhomogeneities (variable sound speed) in TAT.
2. Uniqueness sets S for non-compactly supported functions.
3. Uniqueness sets in dimensions > 2.
4. Backprojection type inversion formulas in even dimensions.
5. Inversion formulas for more general surfaces S.

8. **Some references**

1. M. Agranovsky, C. Berenstein, and P. Kuchment, Approximation by spherical waves in L^p-spaces, J. Geom. Anal. 6 (1996), no. 3, 365–383.

2. M. Agranovsky and E.T. Quinto, Injectivity sets for the Radon transform over circles and complete systems of radial functions: an announcement, Internat. Math. Research Notes, 11(1994), 467-473.

3. M. Agranovsky and E.T. Quinto, Injectivity sets for the Radon transform over circles and complete systems of radial functions, Journal of Functional Analysis, 139 (1996), 383-414.

4. G. Ambartsoumian and P. Kuchment, On the injectivity of the circular Radon transform, Inverse Problems 21 (2005), 473-485.

5. G. Ambartsoumian and P. Kuchment, A range description for the planar circular Radon transform, preprint arXiv:math.GM/0508082, submitted.

6. G. Ambartsoumian and S. Patch, Thermoacoustic tomography - implementation of exact backprojection formulas, arXiv:math.NA/0510638.

7. L.-E. Andersson, On the determination of a function from spherical averages, SIAM J. Math. Anal. **19** (1988), no. 1, 214–232.

8. R. Courant and D. Hilbert, Methods of mathematical physics, vol. 2, Interscience, NY 1953.

9. A. Denisjuk, Integral geometry on the family of semi-spheres. Fract. Calc. Appl. Anal. **2**(1999), no. 1, 31–46.

10. L. Ehrenpreis, The Universality of the Radon Transform, Oxford Univ. Press 2003.

11. J. A. Fawcett, Inversion of n-dimensional spherical averages, SIAM J. Appl. Math. **45**(1985), no. 2, 336–341.

12. D. Finch and Rakesh, The range of the spherical mean value operator for functions supported in a ball, preprint 2005.

13. D. Finch, Rakesh, and S. Patch, Determining a function from its mean values over a family of spheres, SIAM J. Math. Anal. **35** (2004), no. 5, 1213–1240.

14. I. Gelfand, S. Gindikin, and M. Graev, *Selected Topics in Integral Geometry*, Transl. Math. Monogr. v. 220, Amer. Math. Soc., Providence RI, 2003.

15. F. John, *Plane Waves and Spherical Means, Applied to Partial Differential Equations*, Dover 1971.

16. R. A. Kruger, P. Liu, Y. R. Fang, and C. R. Appledorn, Photoacoustic ultrasound (PAUS)reconstruction tomography, Med. Phys. **22** (1995), 1605-1609.

17. P. Kuchment, Generalized Transforms of Radon Type and Their Applications, to appear In Proc. Symp. Appl. Math., AMS 2006.

18. P. Kuchment and E. T. Quinto, Some problems of integral geometry arising in tomography, chapter XI in [10].

19. V. Lin and A. Pinkus, Fundamentality of ridge functions, J. Approx. Theory, 75(1993), 295-311.

20. V. Lin and A. Pinkus, Approximation of multivariate functions, in Advances in computational mathematics, H. P. Dikshit and C. A. Micchelli, Eds., World Sci. Publ., 1994, 1-9.

21. A. K. Louis and E. T. Quinto, Local tomographic methods in Sonar, in *Surveys on solution methods for inverse problems*, pp. 147-154, Springer, Vienna, 2000.

22. F. Natterer, *The mathematics of computerized tomography*, Wiley, New York, 1986.

23. F. Natterer and F. Wübbeling, *Mathematical Methods in Image Reconstruction*, Monographs on Mathematical Modeling and Computation v. 5, SIAM, Philadelphia, PA 2001.

24. S. Nilsson, Application of fast backprojection techniques for some inverse problems of integral geometry, Linkoeping studies in science and technology, Dissertation 499, Dept. of Mathematics, Linkoeping university, Linkoeping, Sweden 1997.

25. C. J. Nolan and M. Cheney, Synthetic aperture inversion, Inverse Problems **18**(2002), 221–235.

26. S. J. Norton, Reconstruction of a two-dimensional reflecting medium over a circular domain: exact solution, J. Acoust. Soc. Am. **67** (1980), 1266-1273.

27. S. J. Norton and M. Linzer, Ultrasonic reflectivity imaging in three dimensions: exact inverse scattering solutions for plane, cylindrical, and spherical apertures, IEEE Transactions on Biomedical Engineering, 28(1981), 200-202.

28. V. P. Palamodov, Reconstruction from limited data of arc means, J. Fourier Anal. Appl. **6** (2000), no. 1, 25–42.

29. S. K. Patch, Thermoacoustic tomography - consistency conditions and the partial scan problem, Phys. Med. Biol. **49** (2004), 1–11.

30. E. T. Quinto, The dependence of the generalized Radon transform on defining measures, Trans. Amer. Math. Soc. 257(1980), 331–346.

31. M. Xu and L.-H. V. Wang, Time-domain reconstruction for thermoacoustic tomography in a spherical geometry, IEEE Trans. Med. Imag. **21** (2002), 814-822.

32. Y. Xu, D. Feng, and L.-H. V. Wang, Exact frequency-domain reconstruction for thermoacoustic tomography: I. Planar geometry, IEEE Trans. Med. Imag. **21** (2002), 823-828.

33. Y. Xu, M. Xu, and L.-H. V. Wang, Exact frequency-domain reconstruction for thermoacoustic tomography: II. Cylindrical geometry, IEEE Trans. Med. Imag. **21** (2002), 829-833.

34. Y. Xu, L. Wang, G. Ambartsoumian, and P. Kuchment, Reconstructions in limited view thermoacoustic tomography, Medical Physics 31(4) April 2004, 724-733.