Discrete symmetries and Lie algebra automorphisms

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Find all point symmetries $\Gamma : (x, y) \rightarrow (\hat{x}(x, y), \hat{y}(x, y))$ of a given ODE

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\]

The point symmetries are prolonged as follows:

\[
\begin{align*}
\hat{y}' &= \frac{D_x \hat{y}}{D_x \hat{x}}, \\
\hat{y}'' &= \frac{D_x \hat{y}'}{D_x \hat{x}},
\end{align*}
\]

where \( D_x \) is the total derivative w.r.t. \( x \):

\[
D_x = \partial_x + y' \partial_y + y'' \partial_{y'} + \cdots.
\]


A little more detail:
The symmetry condition (2) amounts to

\[
\frac{\{(x + y') (y_{xx} + 2y' y_{xy} + y'^2 y_{yy})\} - \{x \leftrightarrow y\} + (x y_y - y_x y_y) \omega(x, y, y')}{(x + y' y_y)^3}
\]

\[= \omega\left(\hat{x}, \hat{y}, \frac{\hat{y}_x + y' \hat{y}_y}{\hat{x} + y' \hat{y}_y}\right). \tag{3}\]
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\frac{\{ (\hat{x}_x + y'\hat{x}_y)(\hat{y}_{xx} + 2y'\hat{y}_{xy} + y'^2\hat{y}_{yy}) \} - \{ \hat{x} \leftrightarrow \hat{y} \} + (\hat{x}_x\hat{y}_y - \hat{x}_y\hat{y}_x) \omega(x, y, y')}{(\hat{x}_x + y'\hat{x}_y)^3} = \omega \left( \hat{x}, \hat{y}, \frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y} \right). \tag{3}
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Split this condition with respect to \( y' \) to obtain an overdetermined system of PDEs for \( \hat{x}(x, y) \) and \( \hat{y}(x, y) \).
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Problem: The overdetermined system may not be easy to solve (Reid et al. 1993). Things are worse for problems that are not so ‘simple’ (systems, PDEs, higher order, higher symmetries, . . . ).
What can be done (easily)?
Lie point symmetries can be found by linearizing the symmetry condition as follows:

\[ \hat{x} = x + \epsilon \xi(x, y) + O(\epsilon^2), \quad \hat{y} = y + \epsilon \eta(x, y) + O(\epsilon^2). \]
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Splitting the \( O(\epsilon) \) terms w.r.t. \( y' \) gives an overdetermined linear system for the functions \( \xi \) and \( \eta \). Typically, this is easy to simplify and solve.
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For \(|\epsilon|\) sufficiently small, Lie point symmetries are obtained by exponentiation:

\[
\hat{x} = e^{\epsilon X} x, \quad \hat{y} = e^{\epsilon X} y
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where \(X = \xi(x, y) \partial_x + \eta(x, y) \partial_y\). The set of all generators \(X\) is a finite-dimensional Lie algebra \(\mathcal{L}\).
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However, after factoring out Lie symmetries, inequivalent discrete symmetries remain to be found.
First message: Given a differential equation whose Lie point symmetries are known (and nontrivial), one can find the remaining discrete point symmetries with little extra effort.
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For simplicity, restrict attention to real finite-dimensional Lie algebras

\[ \mathcal{L} = \text{Span}(X_1, \ldots, X_R). \]
A useful observation:
Let $\Gamma : (x, y) \to (\hat{x}, \hat{y})$ be a point symmetry. If

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$$

generates the Lie point symmetries $\Gamma_\epsilon = e^{\epsilon X}$ then

$$\hat{X} = \xi(\hat{x}, \hat{y})\partial_{\hat{x}} + \eta(\hat{x}, \hat{y})\partial_{\hat{y}}$$

generates Lie point symmetries $\Gamma \Gamma_\epsilon \Gamma^{-1}$. The adjoint action of $\Gamma$ on the symmetry generators replaces $(x, y)$ by $(\hat{x}, \hat{y})$. 
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Thus $\{\hat{X}_1, \ldots, \hat{X}_R\}$ is a basis for $\mathcal{L}$, so

$$X_i = b_i^l \hat{X}_l; \quad (4)$$

the matrix $B = (b_i^l)$ is constant and nonsingular.
Necessary conditions for discrete symmetries

Some information about $(\hat{x}, \hat{y})$ arises directly from (4):

\[ X_i(\hat{x}) = b_i^l \xi_l(\hat{x}, \hat{y}), \quad X_i(\hat{y}) = b_i^l \eta_l(\hat{x}, \hat{y}). \]  

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Solve the system of PDEs (5) and substitute the result into the symmetry condition (3).
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Problem: What are the constants \(b_i^l\)? (Ignorance is not bliss.)
**Solution:** If $\mathcal{L}$ is nonabelian, examine its structure.
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Each $\hat{X}_i$ has the same coefficients as $X_i$, so

$$[X_i, X_j] = c_{ij}^k X_k \quad \Rightarrow \quad [\hat{X}_i, \hat{X}_j] = c_{ij}^k \hat{X}_k. \quad (6)$$

In other words, $\Gamma$ induces a Lie algebra automorphism.
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Consequently the constants $b^l_i$ satisfy

$$c_{im}^n b^l_i b^m_j = c_{ij}^k b^n_k. \quad (7)$$

Solve (7) to simplify the matrix $B$. 
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$$g_{lm} b_i^l b_j^m = g_{ij}.$$  (8)
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Furthermore, (7) implies the linear constraints

$$\gamma_j b_i^j = \gamma_i, \quad \text{where} \quad \gamma_i = c_{ki}^k. \quad (9)$$
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Some coefficients of $B$ can be simplified by factoring out inner automorphisms at an early stage.
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Some coefficients of $B$ can be simplified by factoring out inner automorphisms at an early stage.

If $X_j$ is not a central element then define the matrix $C(j)$ by $(C(j))_j^k = c_{ij}^k$ and let $A(j) = \exp\{\epsilon_j C(j)\}$. Multiply $B$ by $A(j)$ and choose $\epsilon_j$ to simplify the result.

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Summary of the method

1. Calculate $\mathcal{L}$ and choose a ‘suitable’ basis.
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4. Determine which solutions of (5) satisfy the symmetry condition.
5. Factor out the remaining central Lie symmetries.

The second step is the most difficult, but it depends only on the abstract structure of $\mathcal{L}$. A table of automorphisms would be handy – this would also be useful in other applications.
Decomposability
A Lie algebra $\mathcal{L}$ is \textit{decomposable} if there exist nontrivial subalgebras $\mathcal{M}_i$ such that

$$\mathcal{L} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_S, \quad [\mathcal{M}_i, \mathcal{M}_j] = 0, \quad i \neq j.$$ 

Decompositions can be computed (Rand-Winternitz-Zassenhaus).
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**Theorem**: If $\mathcal{L}$ is decomposable then every Lie algebra automorphism of $\mathcal{L}$ is the product of

- automorphisms of each $\mathcal{M}_i$ (acting trivially on the other components),
- permutations of isomorphic components,
- automorphisms that add a central element to each $X \notin [\mathcal{L}, \mathcal{L}]$. 


Second message: All automorphisms of a decomposable Lie algebra can be constructed from the automorphisms of its components.
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Consequently, the list can be restricted to indecomposable $\mathcal{L}$. 
# Two- and three-dimensional Lie algebras (cf. Patera et al.)

<table>
<thead>
<tr>
<th>Name</th>
<th>Nonzero $c^k_{ij}$ $(i &lt; j)$</th>
<th>Outer Der.</th>
<th>Discrete Gen.</th>
<th>Block Diagonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{2,1}$</td>
<td>$c^1_{12}=1$</td>
<td>–</td>
<td>$p_1$</td>
<td>$(1, 1, 1)$</td>
</tr>
<tr>
<td>$A_{3,1}$</td>
<td>$c^1_{23}=1$ (nilpotent)</td>
<td>–</td>
<td>–</td>
<td>$(\det(G_{23}), G_{23})$</td>
</tr>
<tr>
<td>$A_{3,2}$</td>
<td>$c^1_{13}=c^1_{23}=c^2_{23}=1$</td>
<td>–</td>
<td>–</td>
<td>$(a, a, 1)$</td>
</tr>
<tr>
<td>$A_{3,3}$</td>
<td>$c^1_{13}=c^2_{23}=1$</td>
<td>–</td>
<td>$p_1$</td>
<td>$(S_{12}, 1)$</td>
</tr>
<tr>
<td>$A_{3,4}$</td>
<td>$c^1_{13}=1, c^2_{23}=-1$</td>
<td>–</td>
<td>$p_1, (X_2, X_1, -X_3)$</td>
<td>$(1, a, 1)$</td>
</tr>
<tr>
<td>$A_{3,5}^u$</td>
<td>$c^1_{13}=1, c^2_{23}=u$ $(0 &lt;</td>
<td>u</td>
<td>&lt; 1)$</td>
<td>–</td>
</tr>
<tr>
<td>$A_{3,6}$</td>
<td>$c^2_{13}=-1, c^1_{23}=1$</td>
<td>–</td>
<td>$p_{23}$</td>
<td>$(</td>
</tr>
<tr>
<td>$A_{3,7}^u$</td>
<td>$c^1_{13}=c^2_{23}=u, c^1_{13}=-1, c^2_{23}=1$ $(u &gt; 0)$</td>
<td>–</td>
<td>–</td>
<td>$(T(u), T(u), 1)$</td>
</tr>
<tr>
<td>$A_{3,8}$</td>
<td>$c^1_{12}=c^3_{23}=1, c^2_{13}=-2$ $(\mathfrak{s}l(2, \mathbb{R}))$</td>
<td>–</td>
<td>$p_{13}, (X_3, -X_2, X_1)$</td>
<td>$(1, 1, 1)$</td>
</tr>
<tr>
<td>$A_{3,9}$</td>
<td>$c^3_{12}=c^1_{23}=1, c^2_{13}=-1$ $(\mathfrak{s}o(3))$</td>
<td>–</td>
<td>–</td>
<td>$(1, 1, 1)$</td>
</tr>
</tbody>
</table>

**Code:** $a, b \in \mathbb{R}\{0\}$; $G/S$ – general/special linear block; $p$ – parity switch; $T(u) = e^t$, $t \in [0, 2\pi u)$. 

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**Discrete symmetries and Lie algebra automorphisms**

**Classification of Lie algebra automorphisms**
Example: The Chazy equation,

\[ y''' = 2yy'' - 3y'^2 + \lambda(6y' - y^2)^2, \]

has the Lie algebra \( \mathfrak{sl}(2) \):

\[ X_1 = \partial_x, \quad X_2 = x\partial_x - y\partial_y, \quad X_3 = -x^2\partial_x + (2xy + 6)\partial_y. \]
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From the table, there are four inequivalent automorphisms, generated by

\[ \Gamma_1 : (\hat{X}_1, \hat{X}_2, \hat{X}_3) = (-X_1, X_2, -X_3), \]
\[ \Gamma_2 : (\hat{X}_1, \hat{X}_2, \hat{X}_3) = (X_3, -X_2, X_1). \]
Solving the necessary condition (5) with $\Gamma_1$ gives

$$(\hat{x}, \hat{y}) \in \{(-x, -y), (-x - 6/y, y)\}.$$
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However, only the first of these satisfies the symmetry condition.
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However, only the first of these satisfies the symmetry condition.

Similarly $\Gamma_2$ yields two solutions, of which only

$$(\hat{x}, \hat{y}) = \left(\frac{1}{x}, -x^2y - 6x\right)$$

is a symmetry.
### Four-dimensional Lie algebras

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$A_{4,1}$</td>
<td>$c^1_{24} = c^2_{34} = 1$ (nilpotent)</td>
<td>$E_3^1$, $E_4^3$</td>
<td>$-$</td>
<td>$(ab^2, ab, a, b)$</td>
</tr>
<tr>
<td>$A_{4,2}$</td>
<td>$c^1_{14} = u$, $c^2_{24} = c^3_{34} = 1$ ($u \notin {0, 1}$)</td>
<td>$-$</td>
<td>$-$</td>
<td>$(a, b, b, 1)$</td>
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<tr>
<td>$A_{4,4}$</td>
<td>$c^1_{14} = c^2_{24} = c^3_{34} = 1$</td>
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</tr>
<tr>
<td>$A_{4,5}$</td>
<td>$c^1_{14} = 1$, $c^2_{24} = u$, $c^3_{34} = v$ ($uv \neq 0$, $-1 \leq u &lt; v &lt; 1$)</td>
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<td>$p_1$</td>
<td>$(S_{123}, 1)$</td>
</tr>
<tr>
<td>$A_{4,5}$</td>
<td>$c^1_{14} = c^2_{24} = c^3_{34} = 1$</td>
<td>$-$</td>
<td>$p_{12}$</td>
<td>$(1, T(v), T(v), 1)$</td>
</tr>
<tr>
<td>$A_{4,6}$</td>
<td>$c^1_{14} = u$, $c^2_{24} = c^3_{34} = v$, $c^2_{34} = -1$, $c^2_{34} = 1$ ($u \neq 0$, $v \geq 0$)</td>
<td>$-$</td>
<td>$-$</td>
<td>$(a^2, a, a, 1)$</td>
</tr>
<tr>
<td>$A_{4,7}$</td>
<td>$c^1_{14} = 2$, $c^1_{23} = c^2_{24} = c^3_{34} = 1$</td>
<td>$E_4^1$</td>
<td>$p_{12}$</td>
<td>$(a, 1, a, 1)$</td>
</tr>
<tr>
<td>$A_{4,8}$</td>
<td>$c^1_{23} = c^2_{24} = 1$, $c^3_{34} = -1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{4,9}$</td>
<td>$c^1_{14} = u + 1$, $c^1_{23} = c^2_{24} = 1$, $c^3_{34} = u$ ($0 &lt;</td>
<td>u</td>
<td>&lt; 1$)</td>
<td>$-$</td>
</tr>
<tr>
<td>$A_{4,9}$</td>
<td>$c^1_{14} = c^2_{23} = c^2_{24} = 1$</td>
<td>$E_2^1$</td>
<td>$p_{12}$</td>
<td>$(a, 1, a, 1)$</td>
</tr>
<tr>
<td>$A_{4,9}$</td>
<td>$c^1_{14} = 2$, $c^1_{23} = c^2_{24} = c^3_{34} = 1$</td>
<td>$-$</td>
<td>$p_{12}$</td>
<td>$(1, S_{23}, 1)$</td>
</tr>
<tr>
<td>$A_{4,10}$</td>
<td>$c^1_{23} = c^2_{24} = c^3_{34} = 1$, $c^2_{24} = c^3_{34} = -1$</td>
<td>$E_4^1$</td>
<td>$p_{124}$</td>
<td>$(a^2,</td>
</tr>
<tr>
<td>$A_{4,11}$</td>
<td>$c^1_{14} = 2u$, $c^1_{23} = c^2_{24} = c^3_{34} = 1$, $c^2_{34} = u$, $c^3_{34} = -1$ ($u &gt; 0$)</td>
<td>$-$</td>
<td>$-$</td>
<td>$((T(u))^2, T(u), T(u), 1)$</td>
</tr>
<tr>
<td>$A_{4,12}$</td>
<td>$c^1_{13} = c^2_{23} = c^2_{24} = 1$, $c^2_{14} = -1$</td>
<td>$-$</td>
<td>$p_{24}$</td>
<td>$(1, 1, 1, 1)$</td>
</tr>
</tbody>
</table>
Third message: If you want a list of the automorphisms of five-dimensional indecomposable Lie algebras, see me afterwards!
The End