A Very Basic Mixed Integer Set: Mixing and its Applications

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July 26th, 2005

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Some Basics on Single Item Lot-Sizing: The Problem

$$\min \sum_t (p'_t x_t + h'_t s_t + f_t y_t)$$

$$s_{t-1} + x_t = d_t + s_t \ \ \forall\ t$$

$$x_t \leq C_t y_t \ \ \forall\ t$$

$$s_t, x_t \geq 0, y_t \in \{0, 1\}\ \forall\ t.$$

Classification of Capacity: $CAP = \{CC, U, C\}$
Easy for Constant Capacity CC or Uncapacitated U using DP
Hard for Varying Capacities C: Reduction from 0-1 Knapsack

Some Basics on Single Item Lot-Sizing: Normalized Costs

Using $s_{t-1} + x_t = d_t + s_t$, the objective function can be rewritten either (after elimination of s_t) as

$$\min \sum_t (p_t x_t + 0 s_t + f_t y_t) + CON_1$$

where $p_t = p'_t + \sum_{u=t}^n h'_u$, or (after elimination of x_t) as

$$\min \sum_t (0 x_t + h_t s_t + f_t y_t) + CON_2$$

where $h_t = h'_t + p'_t - p'_{t+1}$.

Some Basics on Single Item Lot-Sizing: Wagner-Whitin Costs

When $h_t \geq 0$ for all t, (or p_t is nonincreasing in t), we say that the costs are **Wagner-Whitin**.

Classification of Problem Type: $PROB = \{LS, WW\}$

Reformulation with Wagner-Whitin Costs - Produce as Late as Possible

$$\min \textstyle\sum_t (h_t s_t + f_t y_t)$$

$$s_{t-1} + \textstyle\sum_{u=t}^{l} C_u y_u \geq d_{tl} \equiv \textstyle\sum_{u=t}^{l} d_u \text{ for } 1 \leq t \leq l \leq n$$

$$s \geq 0, y \in \{0,1\}^n$$

$WW-U,CC$: Valid Inequalities and Convex Hull

$$s_{t-1} \geq \sum_{u=t}^{l} d_u(1 - y_t - \ldots - y_u) \text{ for } 1 \leq t \leq l \leq n$$
$$s \geq 0, 0 \leq y \leq 1$$

This polyhedron gives $\text{conv}(X^{WW-U})$.

Constant Capacity Case?

$$s_{t-1} + C\sum_{u=t}^{l} y_u \geq d_{tl} \equiv \sum_{u=t}^{l} d_u \text{ for } 1 \leq t \leq l \leq n$$
$$s \geq 0, y \in \{0,1\}^n$$

Mixing Sets

Consider the *mixing set* $X^M(s,z,b)$ consisting of the points (s,z) satisfying

$$s + z_l \geq b_l \text{ for } l = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, \; z \in \mathbb{Z}^n_+.$$

Let $f_l = b_l - \lfloor b_l \rfloor$ for all l and suppose without loss of generality that $0 = f_0 \leq f_1 \leq \ldots \leq f_n < 1$.

A tight extended formulation for $\text{conv}(X^M(s,z,b))$ is:

$$s = \sum_{i=1}^n f_i \delta_i + \mu$$
$$z_t + \mu + \sum_{\{i: f_i \geq f_t\}} \delta_i \geq \lfloor b_t \rfloor + 1 \text{ for } t = 1, \ldots, n$$
$$\sum_{i=0}^n \delta_i = 1$$
$$\delta \in \mathbb{R}^{n+1}_+, \mu \in \mathbb{R}^1_+, z \in \mathbb{R}^n_+.$$

Mixing Inequalities

Let $T \subseteq \{1, \ldots, K\}$ with $|T| = t$, and suppose that $i_1, \ldots i_t$ is an ordering of T such that $0 = f_{i_0} \leq f_{i_1} \leq f_{i_2} \leq \ldots \leq f_{i_t} < 1$. Then the $\mathrm{mixing\ inequalities}$

$$s \geq \sum_{\tau=1}^{t} (f_{i_\tau} - f_{i_{\tau-1}})(\lfloor b_{i_\tau} \rfloor + 1 - y_{i_\tau})$$

and

$$s \geq \sum_{\tau=1}^{t} (f_{i_\tau} - f_{i_{\tau-1}})(\lfloor b_{i_\tau} \rfloor + 1 - y_{i_\tau}) + (1 - f_{i_t})(\lfloor b_{i_1} \rfloor - y_{i_1})$$

are valid for X_K^{MIX}.

Example of Mixing Inequalities

Consider the set

$$X = \{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^3 : s + y_1 \geq 1.4, s + y_2 \geq 2.6, s + y_3 \geq 0.7\}.$$

With $|T| = 1$,

$$s \geq 0.4(2 - y_1), s \geq 0.6(3 - y_2), s \geq 0.7(1 - y_3)\ ,$$

then the type 1 mixing inequalities with $|T| > 1$

$$\begin{array}{lllllll}
s & \geq & 0.4(2 - y_1) & + & (0.6 - 0.4)(3 - y_2) & & \\
s & \geq & 0.4(2 - y_1) & & & + & (0.7 - 0.4)(1 - y_3) \\
s & \geq & & & 0.6(3 - y_2) & + & (0.7 - 0.6)(1 - y_3) \\
s & \geq & 0.4(2 - y_1) & + & (0.6 - 0.4)(3 - y_2) & + & (0.7 - 0.6)(1 - y_3)
\end{array}$$

and the type 2 mixing inequalities with $|T| > 1$ have an additional term.

"Generalized" Constant Capacity Lot-Sizing with Wagner-Whitin Costs

Suppose that we are given two vectors $\alpha, \beta \in \mathbb{R}_+^{n(n-1)/2}$ where each coordinate corresponds to an interval $[t, l] \subseteq [1, n]$. In addition suppose that α and β are nondecreasing over intervals and $\beta \leq \alpha$.

Consider now a "generalized" constant capacity lot-sizing set X in which s_t denotes the stock at the end of period t and $y_t \in \{0, 1\}$ is the set-up variable in period t. X is given by:

$$s_{t-1} + C \sum_{u=t}^{l} y_u \geq \alpha_{tl} \text{ for } 1 \leq t \leq l \leq n$$

$$C \sum_{u=t}^{l} y_u \geq \beta_{tl} \text{ for } 1 \leq t \leq l \leq n$$

$$s \in \mathbb{R}_+^n, y \in \{0, 1\}^n.$$

Let $Y^t = (y_t, y_t + y_{t+1}, \ldots, y_t + \ldots + y_n)$ and $\alpha^t = (\alpha_{tt}, \ldots, \alpha_{tn})$. Now the set X can be rewritten as:

$$X = \cap_{t=1}^{n} X^{M}(s_{t-1}/C, Y^{t}, \alpha^{t}/C) \cap \{(y \in \{0,1\}^{n} : \sum_{u=t}^{l} y_u \geq \beta_{tl}/C \text{ for } 1 \leq t \leq l \leq n\}.$$

$$\begin{aligned}\text{conv}(X) \quad = \quad & \bigcap_{t=1}^{n} \text{conv}(X^{M}(s_{t-1}/C, Y^{t}, \alpha^{t}/C)) \\ & \bigcap \{y \in [0,1]^{n} : \sum_{u=t}^{l} y_u \geq \lceil \beta_{tl}/C \rceil \text{ for } 1 \leq t \leq l \leq n\}.\end{aligned}$$

The proof is obtained by taking the proof based on the extended formulation for constant capacity lot-sizing with Wagner-Whitin costs in Pochet and Wolsey , and observing that the addition of a consecutive 1's matrix in the space of the y variables) does not invalidate the proof.

The Mixing Set with Flows

The *mixing set with flows* X^{FM}

$$s + x_t \geq b_t \text{ for } t = 1, \ldots, n$$
$$x_t \leq y_t \text{ for } t = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, x \in \mathbb{R}^n_+, y \in \mathbb{Z}^n_+$$

where $0 = b_0 \leq b_1 \leq \ldots \leq b_n$.

Let $\sigma_k = s + x_k - b_k \geq 0$. Then substituting for s, we get $\sigma_k + x_t - x_k \geq b_t - b_k$, and as a relaxation the set W which is the intersection of m mixing sets

$$\sigma_k + y_t \geq b_t - b_k \text{ for all } 0 \leq k < t \leq n$$
$$\sigma \in \mathbb{R}^{n+1}_+, y \in \mathbb{Z}^n_+.$$

$$W(\sigma, y, B) = \bigcap_k X^M(\sigma_k, Y^k, B^k)$$

$$\mathrm{conv}(W(\sigma, y, B)) = \bigcap_k \mathrm{conv}(X^M(\sigma_k, Y^k, B^k))$$

$$\mathrm{conv}(X^{FM}) = \mathrm{conv}(W(s_k + x_k - b_k, y, B) \cap \{(s, x, y) : 0 \leq x \leq y\}.$$

Generalized Mixing Set: Type I

$$x_i + z_i \geq b_i \text{ for } i \in M$$
$$x \in X(\alpha), z \in X(\beta)$$
$$x \in \mathbb{R}^m, z \in \mathbb{Z}^m$$

where

$$X(\gamma) = \{y \in \mathbb{R}^m : y_i - y_j \leq \gamma_{ij} \; \forall \; i, j, l_i \leq y_i \leq u_i \; \forall \; i\}.$$

The α must be tight: $\alpha_{ik} \leq \alpha_{ij} + \alpha_{jk} \; \forall \; i, j, k$.
The β are integer.

Where are the mixing sets?

Using $x_i - x_j \leq \alpha_{ij}$ gives the mixing set X_j^M:

$$x_j + z_t \geq b_t - \alpha_{tj} \;\forall\; t, z \in X(\beta),\; x_j \in \mathbb{R}_+^1, z \in \mathbb{Z}_+^n.$$

Combining mixing sets:

$$\text{conv}(\bigcap_j X_j^M) = \bigcap(\text{conv}(X_j^M)).$$

Integrality is preserved with the constraints of $x \in X(\alpha)$:

$$\text{conv}(\bigcap_j X_j^M \cap X(\alpha)) = \bigcap(\text{conv}(X_j^M)) \cap X(\alpha).$$

Generalized Mixing Set: Type II

Let $X^{MIX\geq}$ be the set

$$x + z_i \geq b_i \text{ for } i \in M^{\geq}$$
$$x \in \mathbb{R}^1, z \in X(\beta)$$

Let $X^{MIX\leq}$ be the set

$$x + z_i \leq c_i \text{ for } i \in M^{\leq}$$
$$x \in \mathbb{R}^1, z \in X(\beta)$$

Let X^{GMIX} be the set $X^{MIX\geq} \cap X^{MIX\leq}$, or in other words

$$x + z_i \geq b_i \text{ for } i \in M^{\geq}$$
$$x + z_i \leq c_i \text{ for } i \in M^{\leq}$$
$$x \in \mathbb{R}^1, z \in X(\beta) \cap \mathbb{Z}^m.$$

Common x variables

Update $X(\beta)$. Add $z_i - z_j \leq \lfloor c_i - b_j \rfloor$.

$$X^{GMIX} = X^{MIX\geq} \cap X^{MIX\leq}.$$

Integrality is preserved with common x variables:

$$\mathrm{conv}(X^{GMIX}) = \mathrm{conv}(X^{MIX\geq}) \cap \mathrm{conv}(X^{MIX\leq}).$$

Given K such sets

$$\mathrm{conv}(\bigcap_{k=1}^{K} X_k^{GMIX}) = \bigcap_k \mathrm{conv}(X_k^{MIX\geq}) \cap \bigcap_k \mathrm{conv}(X_k^{MIX\leq}).$$

Production Time Windows: The Problem

Problem 1: Distinguishable Orders $PROB - CAP - TWP$
There are K orders, where $K = O(n^2)$.
Order k consists of a quantity $D^k > 0$, and a time window $[b^k, e^k] \subseteq [1, n]$.
Order k must be produced during its time interval and delivered to the client in period e^k.

In the classical problem $LS - CAP$, $K = n$.
For each $k \in \{1, \ldots, n\}$, $D^k = d_k$ and $[b^k, e^k] = [1, k]$.

Brahimi: Formulations

$O(n^2)$ DP Algorithm for $WW - U - TW(P)$

Special Case of Indistinguishable Orders.

Production Time Windows: Notation

Data: Orders D^k and Time Windows $[b^k, e^k]$.

Must be produced in the interval $[t, l]$
$D_{tl} = \sum_{k: t \leq b^k \leq e^k \leq l} D^k$

Order Deliveries in t
$\Delta_t = D_{1t} - D_{1,t-1}$

Order Arrivals in t
$\Gamma_t = D_{tn} - D_{t+1,n}$

Arrivals in $[t, l]$: $\Gamma_{tl} = \sum_{u=t}^{l} \Gamma_u$,
Deliveries in $[t, l]$: $\Delta_{tl} = \sum_{u=t}^{l} \Delta_u$.

Production Time Windows: $LS-CC-TWP$ Formulations

A basic formulation (Brahimi)

$$s_{t-1} + x_t = \Delta_t + s_t$$
$$\sum\nolimits_{u=t}^{l} x_u \geq D_{tl}$$
$$x_t \leq C y_t$$
$$s, x \in \mathbb{R}^n_+, y \in \{0,1\}^n$$

$$\text{Valid Inequality } s_{t-1} + \sum_{u=t}^{l} x_u \geq D_{1l} - D_{1,t-1} = \Delta_{tl}.$$

Wagner-Whitin Costs and Constant Capacities $WW - CC - TWP$

Replace x_u by its upper bound constraints Cy_u gives a relaxation

$$\min \sum_t h_t s_t + \sum_t f_t y_t$$

$$s_{t-1} + C \sum_{u=t}^{l} y_t \geq \Delta_{tl} \text{ for } 1 \leq t \leq l \leq n$$

$$C \sum_{u=t}^{l} y_t \geq D_{tl} \text{ for } 1 \leq t \leq l \leq n$$

$$s \in \mathbb{R}_+^n, y \in \{0,1\}^n.$$

This is a "generalized" constant capacity lot-sizing set X

$$s_{t-1} + C\sum_{u=t}^{l} y_u \geq \Delta_{tl} \text{ for } 1 \leq t \leq l \leq n$$
$$C\sum_{u=t}^{l} y_u \geq D_{tl} \text{ for } 1 \leq t \leq l \leq n$$
$$s \in \mathbb{R}_+^n, y \in \{0,1\}^n.$$

$O(n^2) \times O(n^2)$ tight extended formulation

$$\begin{aligned} \text{conv}(X) \quad = \quad & \bigcap_{t=1}^{n} \text{conv}(X^M(s_{t-1}/C, Y^t, \Delta^t/C)) \\ & \bigcap \{y \in [0,1]^n : \sum_{u=t}^{l} y_u \geq \lceil D_{tl}/C \rceil \text{ for } 1 \leq t \leq l \leq n\}. \end{aligned}$$

Production Time Windows: Indistinguishable Orders

This problem is "easy" by dynamic programming.

It is equivalent to a problem with "non-inclusive" time windows, i.e. the orders and time windows can be ordered so that $b^k \leq b^{k+1}$ and $e^k \leq e^{k+1}$.

It is equivalent to the standard lot-sizing problem with upper bounds on the stocks.

Both in the uncapacitated and constant capacity cases, one has tight extended formulations of the convex hull based on the DP algorithms.

Multi-Item, Family Set-Up Problem

Lot-sizing for $i = 1, \ldots, m$ items with demands d_t^i and storage costs h_t^i
Production is in batches of size C.
The cost per batch in period t is f_t.

$$\min \sum_i \sum_t h_t^i s_t^i + \sum_t f_t Y_t$$
$$s_{t-1}^i + x_t^i = d_t^i + s_t^i \;\forall\; i, t$$
$$\sum_i x_t^i \leq C Y_t \;\forall\; t$$
$$s, x \geq 0, 0 \leq Y_t \leq u_t \;\forall\; t$$

Assumptions: Wagner-Whitin Costs $h_t^i \geq 0 \;\forall\; i, t$
Storage costs are non-increasing over items $h_t^i \geq h_t^{i+1} \;\forall\; i, t$

Create Composite Items and Mixing Sets

Composite Product i is the sum of the i most expensive products:

$$X_t^i = \sum_{j=1}^{i} x_t^j,\ S_t^i = \sum_{j=1}^{i} s_t^j,\ D_t^i = \sum_{j=1}^{i} d_t^j, H_t^i = h_t^i - h_t^{i-1} \geq 0 \text{ and } X_t^i \leq CY_t.$$

Relaxation:

$$\min\{HS + fY : (S,Y) \in \bigcap_{i=1}^{m} X_i^{WW-CC}(S^i, Y, D^i)\}$$

where $X_i^{WW-CC}(S^i, Y, D^i)$ is the Constant Capacity Lot-Sizing Set:

$$S_{t-1}^i + C\sum_{u=t}^{l} Y_u \geq D_{tl}^i \text{ for } 1 \leq t \leq l \leq n$$
$$S^i \in \mathbb{R}_+^n, Y \in \mathbb{Z}_+^n, Y \leq u$$

The Results

The intersection of the convex hulls is integer:

$$\mathrm{conv}(\bigcap_{i=1}^{m} X_i^{WW-CC}(S^i, Y, D^i)) = \bigcap_{i=1}^{m} \mathrm{conv}(X_i^{WW-CC}(S^i, Y, D^i)).$$

The linear program

$$\min \sum_i \sum_t H_t^i S_t^i + \sum_t f_t Y_t$$
$$(S, Y) \in \bigcap_{i=1}^{m} \mathrm{conv}(X_i^{WW-CC}(S^i, Y, D^i))$$
$$s_t^i = S_t^i - S_t^{i-1}, x_t^i = d_t^i + s_t^i - s_{t-1}^i \;\forall\; i, t$$

solves the original problem.

Crucial: $S_{t-1}^i = \max_{l=t,\ldots,n}(D_{tl}^i - CY_{tl})^+$ is nondecreasing in i, and the same for X_t^i.

Stochastic Lot Sizing on a Scenario Tree

Given a rooted directed out-tree $T = (N, A)$, let $D(v)$ be the direct successors of v, $S(v)$ the set of all successors of v and $P(j, k)$ with $k \in S(j)$ the set of nodes on the path from j to k. Node $r = 1 \in N$ is the root. $L = \{v \in N : S(v) = \emptyset\}$ are the leaves. We add a dummy node 0 and an arc $(0, 1)$, and let $p(v)$ be the unique predecessor of v for all $v \in N$.

The lot-sizing problem on a tree $LS - C - TREE$ is:

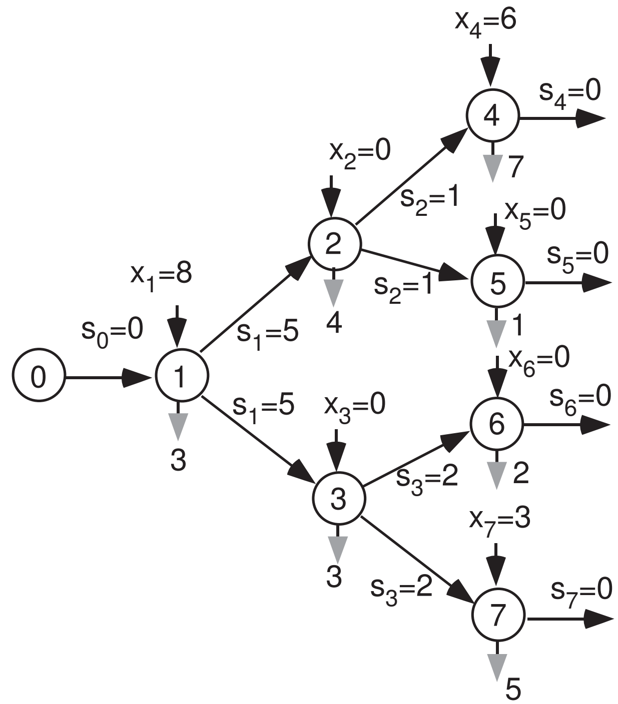
$$\min \sum_{v \in N} (P'_v x_v + Q_v y_v) + \sum_{v \in N \cup \{0\}} H'_v s_v$$

$$s_{p(v)} + x_v = d_v + s_v \text{ for all } v \in N$$

$$x_v \leq C_v y_v \text{ for all } v \in N$$

$$s \in \mathbb{R}_+^{|N|+1}, x \in \mathbb{R}_+^{|N|}, y \in [0, 1]^{|N|},$$

with production costs P'_v, fixed costs Q_v and demands d_v for all $v \in N$, and storage costs H'_v for all $v \in N \cup \{0\}$.



x4=6
s4=0
4
7
x2=0
s2=1
x5=0
2
s5=0
x1=8
s2=1
5
4
1
s0=0
s1=5
x6=0
0
1
s1=5
x3=0
6
s6=0
3
s3=2
2
3
x7=3
3
s3=2
7
s7=0
5

A Relaxation with Constant Capacities

Here we consider the "Wagner-Whitin" relaxation obtained by summing the flow conservation constraints along each directed path $P(v,w)$ and replacing each x_u for $u \in P(v,w)$ by its capacity leading to the set:

$$s_{p(v)} + C \sum_{u \in P(v,w)} y_u \geq \sum_{u \in P(v,w)} d_u \text{ for all } w \in S(v) \cup \{v\}$$

$$\text{and all } v \in N$$

$$s \in \mathbb{R}_+^{|N|+1}, y \in \{0,1\}^{|N|},$$

denoted $X^{WW-CC-TREE}$.

The convex hull of $X^{WW-CC-TREE}$

Specifically for fixed $v \in N$, the set

$$X^{TREE}(v) = \{(s_{p(v)}, y) \in \mathbb{R}^1_+ \times \{0,1\}^{|S(v)|+1} \text{ satisfying}$$

$$s_{p(v)} + C \sum_{u \in P(v,w)} y_u \geq \sum_{u \in P(v,w)} d_u \forall\ w \in S(v) \cup \{v\}\}$$

can be rewritten as a mixing set with network dual constraints, and so its convex hull is known.

$$\text{conv}(X^{WW-CC-TREE}) = \bigcap_{v \in N} \text{conv}(X^{TREE}(v)).$$

Valid Inequalities for $LS - CC - TREE$

- All the mixing inequalities for $\text{conv}(X^{WW-CC-TREE})$.
- Select a subset $T \subset S(v)$. Now we obtain a mixing set

$$s_{p(v)} + \sum_{\tau \in T} x_\tau + C \sum_{u \in P(v,w) \cap T} y_v \geq \sum_{u \in P(v,w)} d_u \text{ for all } w \in S(v)$$
$$s_{p(v)} \in \mathbb{R}^1_+, x \in \mathbb{R}^{|T|}_+, y \in \{0,1\}^{|S(v) \setminus T|+1},$$

  denoted $X^{TREE}(u,T)$ and another family of mixing inequalities.
- If $z, w \in S(v)$ and $\sum_{u \in P(v,w) \setminus \{v\}} d_u > \sum_{u \in P(v,z) \setminus \{v\}} d_u$, the inequality $s_z \geq s_v - \sum_{u \in P(v,z) \setminus \{v\}} d_u$ and $s_v + C \sum_{u \in P(v,w) \setminus \{v\}} y_u \geq \sum_{u \in P(v,w) \setminus \{v\}} d_u$ give

$$s_z + C \sum_{u \in P(v,w) \setminus \{v\}} y_u \geq \sum_{u \in P(v,w) \setminus \{v\}} d_u - \sum_{u \in P(v,z) \setminus \{v\}} d_u$$

Wagner-Whitin Lot-Sizing with Varying Capacities

Consider the set

$$X^{WW-C} = \{(s, y) \in \mathbb{R}_+^{n+1} \times \{0,1\}^n : s_{t-1} + \sum_{u=t}^{l} C_u y_u \geq d_{tl} \text{ for } 1 \leq t \leq l \leq n\}.$$

$$\text{Let } \delta_{tl} = \min\{s_{t-1} + C_t \sum_{u=t}^{l} y_u : (s, y) \in X^{WW-C}\}.$$

- The inequality

$$s_{t-1} + C_t \sum_{u=t}^{l} y_u \geq \delta_{tl}$$

is valid for X^{WW-C} for $1 \leq t \leq l \leq n$.

- The relaxation

$$s_{t-1} + C_t \sum_{u=t}^{l} y_u \geq \delta_{tl} \text{ for } 1 \leq t \leq l \leq n$$
$$s \in \mathbb{R}_+^{n+1}, y \in \{0,1\}^n$$

  is the intersection of n mixing sets.
  The convex hull of this set is:

$$\bigcap_{t=1}^{n} \text{conv}\left(X^{MIX}(s_{t-1}/C_t, (y_t, \ldots, y_n), (\delta_{tt}, \ldots, \delta_{tn})/C_t)\right).$$

- When the C_t are nondecreasing, i.e. $C_t \leq C_{t+1}$ for all t, then the $\{\delta_{tl}\}$ can be calculated in polynomial time.

Questions

- Does mixing tell us how to combine two or three general MIP constraints?
- Optimization over Mixing Sets and Generalizations (Eisenbrand and Conforti)
- Our generalizations of mixing sets are all of the form

$$Ax + By \geq b$$

$$x \in \mathbb{R}^n_+, y \in \mathbb{Z}^p$$

  with A, B both TU matrices. Is there a more general result about MIPs?
- Complexity of Lot-Sizing with Production Time Windows?
- Algorithm (non LP) for Multi-Item Family Set-Up Problem?
- Complete Description for Stochastic Lot-Sizing Polyhedron?

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