Two-step MIR inequalities

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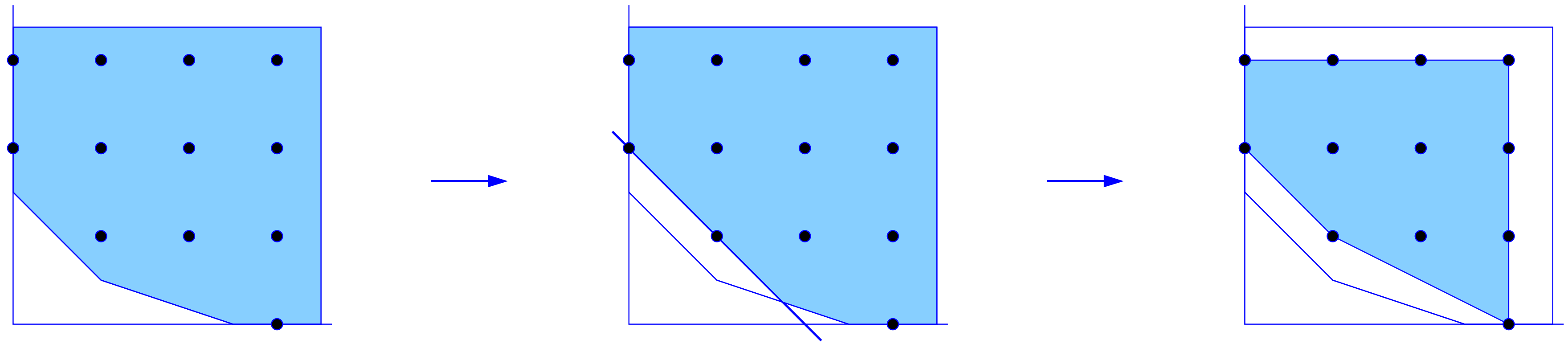
Outline

- *Introduction*
- *Part 1:*
  - *Mixed-integer rounding / GMIC*
  - *Two-step MIR set*
  - *Two-step MIR procedure*
- *Part 2:*
  - *Gomory's Master Cyclic Group Polyhedron*
  - *Shooting experiment*
- *Part 3:*
  - *Separation*
  - *Computation*

Cutting planes for Mixed-Integer Programs

LP Relaxation:	*With Cuts:*	*More Cuts:*
$\min c^T x$	$\min c^T x$	$\min c^T x$
$Ax \geq b$	$Ax \geq b$	$Ax \geq b$
	$\alpha_1^T x \geq d_1$	$\alpha_1^T x \geq d_1$
		$\alpha_2^T x \geq d_2$
		$\vdots$



Single constraint relaxation

- *Let*

$$\sum_{j \in J} c_j v_j + \sum_{i \in I} a_i\, x_i \geq b$$

be an implied constraint of the MIP with $v \in R_+^{|J|}$ and $x \in Z_+^{|I|}$.

- Consider the set

$$W' \;=\; \left\{ v \in R^2, x \in Z^{|I|} \;:\; v_1 - v_2 + \sum_{i \in I} a_i\, x_i \geq b, \;\; x, v \geq 0 \right\}$$

Valid inequalities for W' are also valid for the mixed-integer program.

- Let $\hat{b} = b - \lfloor b \rfloor$, and $\hat{a}_i = a_i - \lfloor a_i \rfloor$. Define $z = \sum_{j \in J} \lfloor a_i \rfloor \; x_i - \lfloor b \rfloor$.
Define the relaxed set:

$$W \;=\; \left\{ v \in R^2, x \in Z^{|I|}, z \in Z \;:\; v_1 - v_2 + \sum_{i \in I} \hat{a}_i\, x_i + z \geq \hat{b}, \;\; x, v \geq 0 \right\}$$

- *Goal: derive cuts for MIP using valid inequalities for W*

Basic mixed-integer rounding set (Wolsey '98)

Let

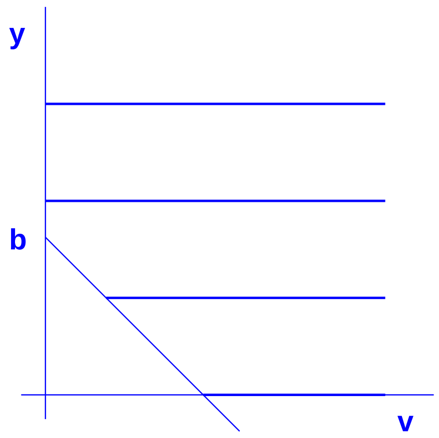
$$Q = \Big\{ v \in R,\ y \in Z\ :\ v + y\ \geq\ b,\ \ v \geq 0 \Big\}$$

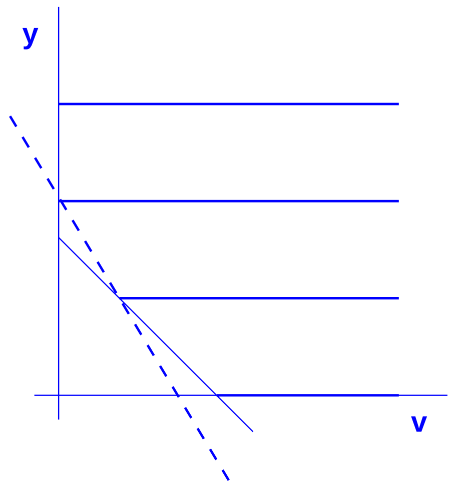
then, MIR Inequality:

$$v \geq \hat{b}(\lceil b \rceil - y)$$

where $\hat{b} = b - \lfloor b \rfloor$*, is valid for* Q *and*

$$conv(Q) = \Big\{ v, y \in R\ :\ v + y\ \geq\ b,\ \ v + \hat{b}y \geq \hat{b}\lceil b \rceil,\ \ v \geq 0 \Big\}.$$





MIR inequalities for W

Base System:

$$v_1 - v_2 + \sum_{\hat{a}_i < \hat{b}} \hat{a}_i x_i + \sum_{\hat{a}_i \geq \hat{b}} \hat{a}_i x_i + z \geq \hat{b}.$$

Relax:

$$v_1 + \sum_{\hat{a}_i < \hat{b}} \hat{a}_i x_i + \sum_{\hat{a}_i \geq \hat{b}} x_i + z \geq \hat{b}.$$

Apply MIR:

$$v_1 + \sum_{\hat{a}_i < \hat{b}} \hat{a}_i x_i + \hat{b} \sum_{\hat{a}_i \geq \hat{b}} x_i + z \geq \hat{b}.$$

- *If the base inequality comes from a row of the simplex tableau:*

MIR inequality = Gomory mixed-integer cut.

(Marchand and Wolsey '98)

Another simple set

- *Let*

$$Q^2 = \Big\{ v \in R,\ y, z \in Z\ :\ v + \alpha y + z\ \geq\ \beta,\ \ v, y, z \geq 0 \Big\},$$

where $0 < \alpha < \beta < 1$.

- *The complete description of* Q^2 *is:*

$$conv(Q^2) = \Big\{ v,\ y,\ z \in R\ :$$

$$\begin{aligned} v + \alpha y + \beta z &\geq \beta \\ \frac{v}{\beta - \alpha \lfloor \beta/\alpha \rfloor} + y + \lceil \beta/\alpha \rceil z &\geq \lceil \beta/\alpha \rceil \\ v, y, z &\geq 0 \quad \Big\} \end{aligned}$$

where the second inequality is called the two-step MIR inequality *(DG'03)*

Two-step MIR

Original:

$$(v + \alpha y) + z \geq \beta.$$

MIR with z integral:

$$(v + \alpha y) + \beta z \geq \beta.$$

Divide by α:

$$\frac{v}{\alpha} + y + \frac{\beta}{\alpha} z \geq \frac{\beta}{\alpha}.$$

Relax coefficient of z:

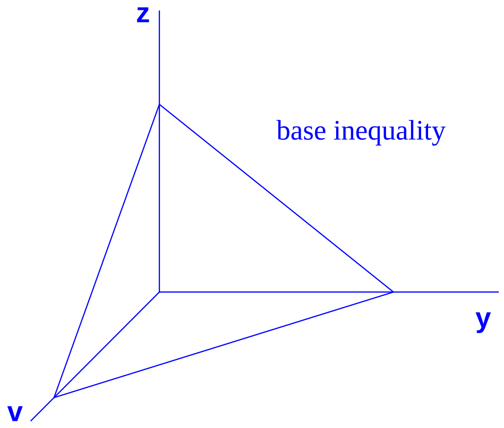
$$\frac{v}{\alpha} + (y + \left\lceil \frac{\beta}{\alpha} \right\rceil z) \geq \frac{\beta}{\alpha}.$$

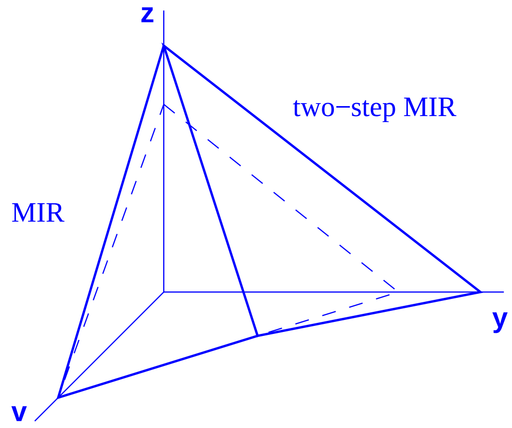
MIR with v/α continuous:

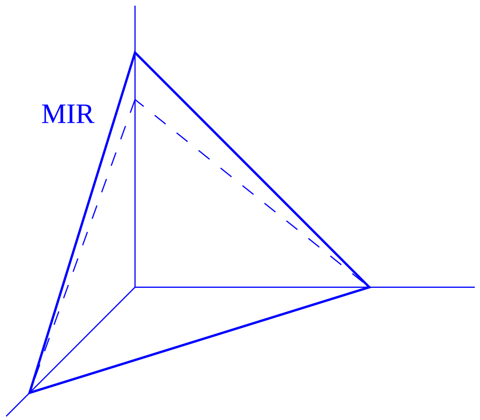
$$\frac{v}{\beta - \alpha \lfloor \beta/\alpha \rfloor} + y + \lceil \beta/\alpha \rceil z \geq \lceil \beta/\alpha \rceil$$

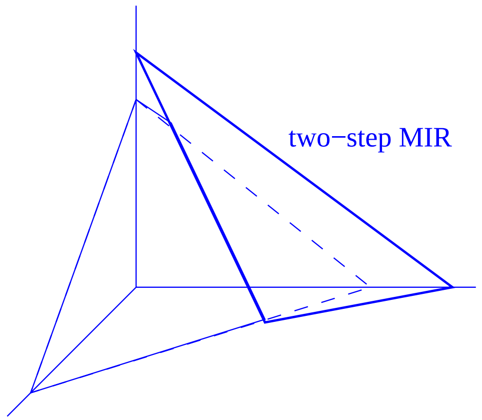
Facets of Q^2

$$Q^2 = \left\{ v \in R,\ y, z \in Z\ :\ v + \alpha y + z\ \geq\ \beta,\ \ v, y, z \geq 0 \right\}$$









A slightly more general set

What if $z \geq 0$ does not hold?

$$Q^{2R} = \Big\{ v \in R,\ y, z \in Z \ :\ v + \alpha y + z \ \geq \ \beta, \ \ v, y \geq 0 \Big\}.$$

- This set is more useful for generating cuts.

Theorem (DG'03) If $1/\alpha \geq \lceil \beta/\alpha \rceil$, then the two-step MIR inequality

$$\frac{v}{\beta - \alpha \lfloor \beta/\alpha \rfloor} + y + \lceil \beta/\alpha \rceil z \quad \geq \quad \lceil \beta/\alpha \rceil$$

is valid for Q^{2R}. Furthermore, it is facet defining provided that $\beta/\alpha \notin Z$.

- Not a complete description of Q^{2R}.
- Agra and Constantino (2003) and Atamtürk and Rajan (2003) provide a polynomial algorithm to enumerate all facets of Q^{2R}.

Derivation of two-step MIR for Q^{2R}

Original:

$$(v + \alpha y) + z \geq \beta.$$

Applying MIR with z integral:

$$(v + \alpha y) + \beta z \geq \beta.$$

Therefore

$$\frac{v}{\alpha} + y + \frac{\beta}{\alpha} z \geq \frac{\beta}{\alpha} \quad \textit{and} \quad \frac{v}{\alpha} + y + \frac{1}{\alpha} z \geq \frac{\beta}{\alpha}$$

are valid for Q^{2R}.

When $1/\alpha \geq \lceil \beta/\alpha \rceil$, convex combination gives:

$$\frac{v}{\alpha} + (y + \left\lceil \frac{\beta}{\alpha} \right\rceil z) \geq \frac{\beta}{\alpha}$$

Applying MIR with $(y + \lceil \beta/\alpha \rceil z)$ integral gives the two-step MIR inequality.

Example

Base:

$$v + 0.4y + z \geq 1.7$$

MIR with z integral:

$$(v + 0.4y) + 0.7z \geq 1.4$$

divide by 0.4:

$$2.5v + y + 1.75z \geq 3.5$$

Relaxation | Positive combination with base

Relaxation:

$$2.5v + (y + 2z) \geq 3.5$$

Positive combination with base:

$$2.5v + y + 1.75(z - 1) \geq 1.75$$

$$2.5v + y + 2(z - 1) \geq 1.75$$

$$2.5v + (y + 2z) \geq 3.75$$

MIR with $(y + 2z)$ integral:

$$2.5v + 0.5(y + 2z) \geq 2 \qquad 2.5v + 0.75(y + 2z) \geq 3$$

Applying two-step MIR's to W

We know that if $v + \alpha y + z \geq \beta$ is valid, and $v, y \geq 0$, then

$$\frac{v}{\beta - \alpha \lfloor \beta/\alpha \rfloor} + y + \lceil \beta/\alpha \rceil z \quad \geq \quad \lceil \beta/\alpha \rceil$$

is also valid if $1/\alpha \geq \lceil \beta/\alpha \rceil$.

To apply to $v_1 - v_2 + \sum_{i \in I} \hat{a}_i x_i + z \geq \hat{b}$, where $v, x \geq 0$ and x, z integer,

- Chose an α such that $1/\alpha \geq \lceil \hat{b}/\alpha \rceil > \hat{b}/\alpha$
- Relax the inequality to obtain an inequality of the form

$$\underbrace{(\ v_1 + \ldots\ )}_{\geq 0} \quad + \quad \alpha \underbrace{(\quad \ldots \quad)}_{\in Z_+} \quad + \quad \underbrace{(\quad \ldots \quad + z\ )}_{\in Z} \quad \geq \quad \hat{b}$$

- Apply two-step MIR inequality to the relaxed inequality

Applying two-step MIR's – cont'd

- Depending on $\hat{a}_i$, we can rewrite/relax it as
  - $\hat{a}_i \to 1$, or,
  - $\hat{a}_i \to \alpha(\lceil \hat{a}_i/\alpha \rceil)$, or,
  - $\hat{a}_i = \alpha(\lfloor \hat{a}_i/\alpha \rfloor) + (\hat{a}_i - \alpha \lfloor \hat{a}_i/\alpha \rfloor)$
- Partitioning I into I_1, I_2 and I_3 in a certain way gives the "best" relaxed inequality

$$v_1 + \sum_{j \in I_1} (\hat{a}_i - \alpha \lfloor \hat{a}_i/\alpha \rfloor)\, x_i \; + \alpha \sum_{j \in I_1} \lfloor \hat{a}_i/\alpha \rfloor \; x_i + \sum_{j \in I_2} \lceil \hat{a}_i/\alpha \rceil \; x_i \; + \; \sum_{j \in I_3} x_i + z \; \geq \hat{b}.$$

- Applied to $v_1 - v_2 + \sum_{i \in I} a_i\, x_i \geq b$ this leads to the two-step MIR inequality

$$v_1 + \sum_{j \in I} \; \rho\, \tau \; \lfloor a_i \rfloor + \min\{\rho\, \tau \;,\; k_i \rho \; + \hat{a}_i - k_i \alpha \;,\; l_i \rho \;\} \; x_i \geq \rho\, \tau \,\lceil b \rceil,$$

where $\tau = \lceil \hat{b}/\alpha \rceil$, $\rho = \hat{b} - \alpha \lfloor \hat{b}/\alpha \rfloor$, $k_i = \lfloor \hat{a}_i/\alpha \rfloor$ *and* $l_i = \lceil \hat{a}_i/\alpha \rceil$.

Gomory's Master Cyclic Group Polyhedron for $n, r \in Z$ and $n > r > 0$:

$$P(n,r) \;=\; conv\{w \in Z_+^{n-1} : \sum_{i=1}^{n-1} i\, w_i = r\ (mod)\ n\}$$

or, equivalently,

$$P(n,r) \;=\; conv\Big\{w \in Z_+^{n-1} : \exists z \in Z\ s.t.\ \sum_{i=1}^{n-1} (\frac{i}{n})\, w_i - z = \frac{r}{n}\Big\}$$

Introduced by Gomory (1969). Also studied by GJ '72, GJE '03, AGJE '03.

For rational data, W, without the continious variables, looks like a restriction of $P(n,r)$
.

$$Y \;=\; \Big\{x \in Z_+^{|J|}, z \in Z\ :\ \sum_{j \in J} \hat{a}_i\, x_i + z = \hat{b}\Big\}$$

For some large $\bar{n}$, Y is a face of $P(\bar{n}, \bar{n} \cdot \hat{b})$.

Sub-additive functions on $(0,1)$

- For all facets of $P(n,r)$, there is a sub-additive function that generates it.
- Ex: Gomory mixed-integer cut on $(\sum_{j\in J} \hat{a}_j x_j + z = \hat{b})$:

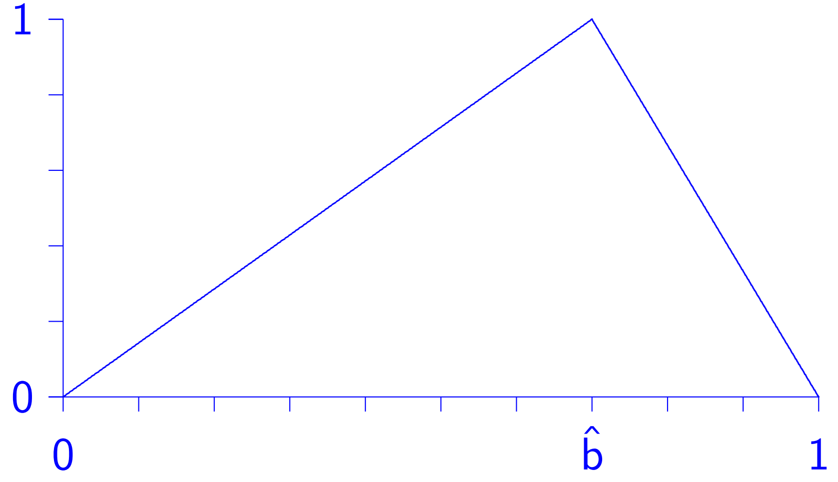
$$\sum_{\hat{a}_j<\hat{b}} \frac{\hat{a}_j}{\hat{b}} x_j + \sum_{\hat{a}_j\geq\hat{b}} \frac{1-\hat{a}_j}{1-\hat{b}} x_j \ \geq \ 1.$$

can also be written as

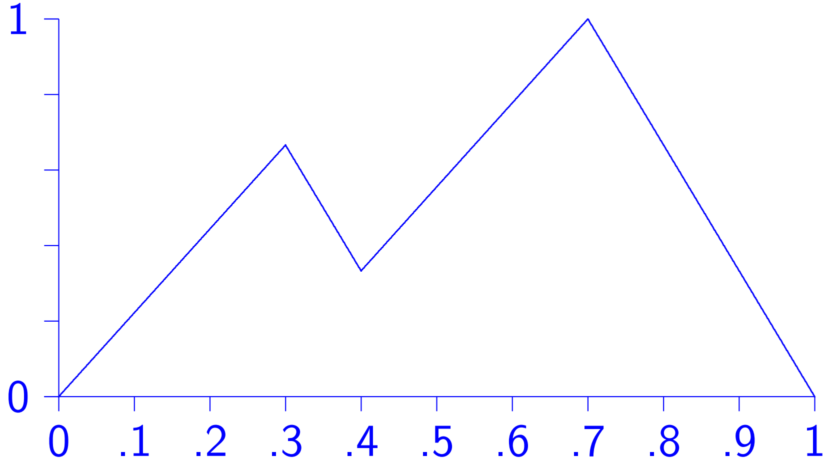
$$\sum_{j\in J} f^{\hat{b}}(\hat{a}_j) x_j \geq 1$$

where

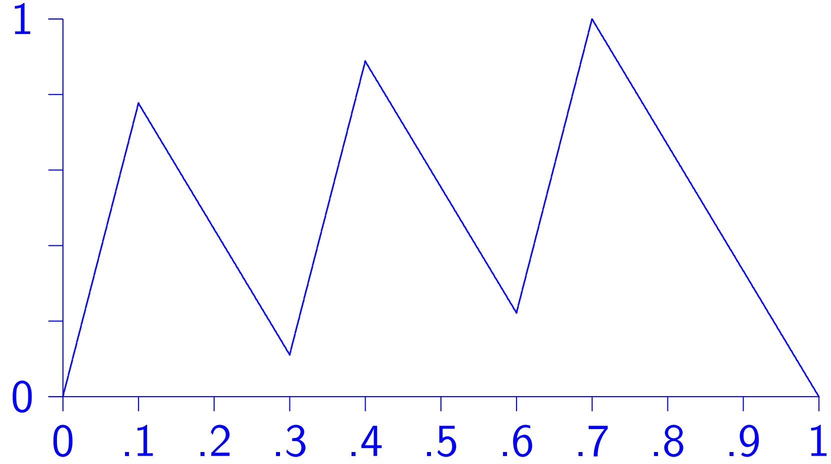
$$f^{\hat{b}}(\hat{v}) = \begin{cases} \hat{v}/\hat{b} & \text{if } \hat{v} < \hat{b}, \\ (1-\hat{v})/(1-\hat{b}) & \text{if } \hat{v} \geq \hat{b}, \end{cases}$$



Two-step MIR functions



$\hat{b} = .7,\ \alpha = .4,\ \left\lceil \hat{b}/\alpha \right\rceil = 2$



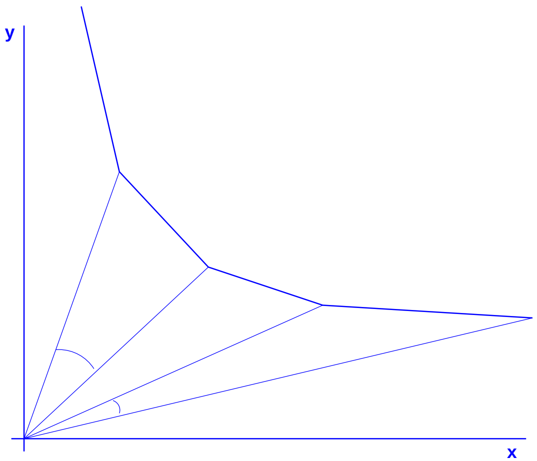
$\hat{b} = .7,\ \alpha = .3,\ \left\lceil \hat{b}/\alpha \right\rceil = 3$

- The two-step MIR function defines a facet of $P(n, r)$ provided that α is a multiple of $1/n$ and $1/\alpha > \lceil \hat{b}/\alpha \rceil$. (DG'03)
- Scaled two-step MIR functions also define a facet of $P(n, r)$. (DG'03)
- These inequalities generalize the 2slope inequalities of Aroaz, Evans, Gomory, and Johnson (2003)
- Gomory and Johnson (1972) call these functions 2-slope functions.

Gomory's Shooting Experiment

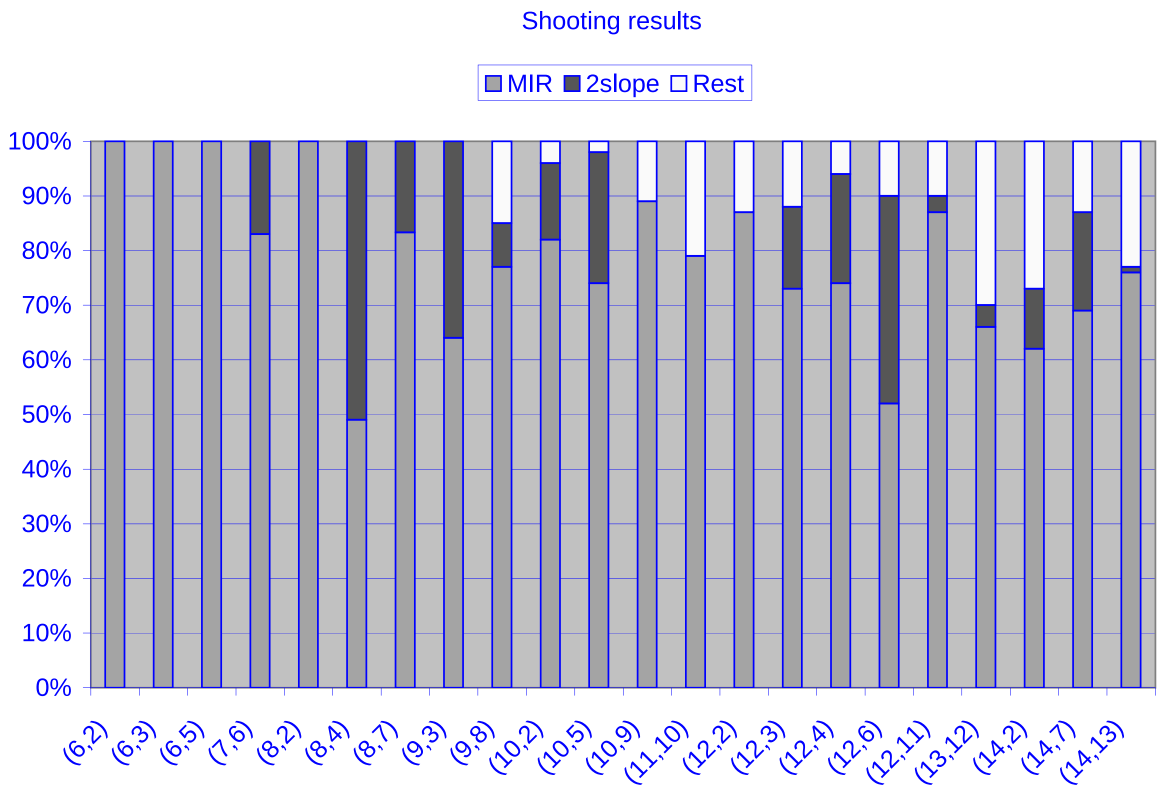
- Repeat the following experiment a number of times:
  - Choose a random direction $d \in R_+^{n-1}$
  - Find the facet of $P(n, r)$ first encountered along the ray $\{\lambda d \mid \lambda \geq 0\}$
  - Record the facet.
- If a facet is hit many times, then consider it as an important facet.

- Random means $d/||d||_2$ is uniformly distributed over the surface of the unit-sphere.
- Hit count is proportional to the size of the "solid angle"
- The experiment assumes that
  - Large facets are more important then the rest
  - Large facets subtend large angles at the origin.



Shooting results of Gomory, Johnson, Evans (2003)

- For small n (< 30), they repeat the experiment 10,000 times
- Observe that a small number of facets absorb significant number of hits



Our Shooting Results

- For larger n (< 200), repeat the experiment 100,000 times
- Focus on MIR-based facets

- Main observation: MIR-based facets still appear important for large n.

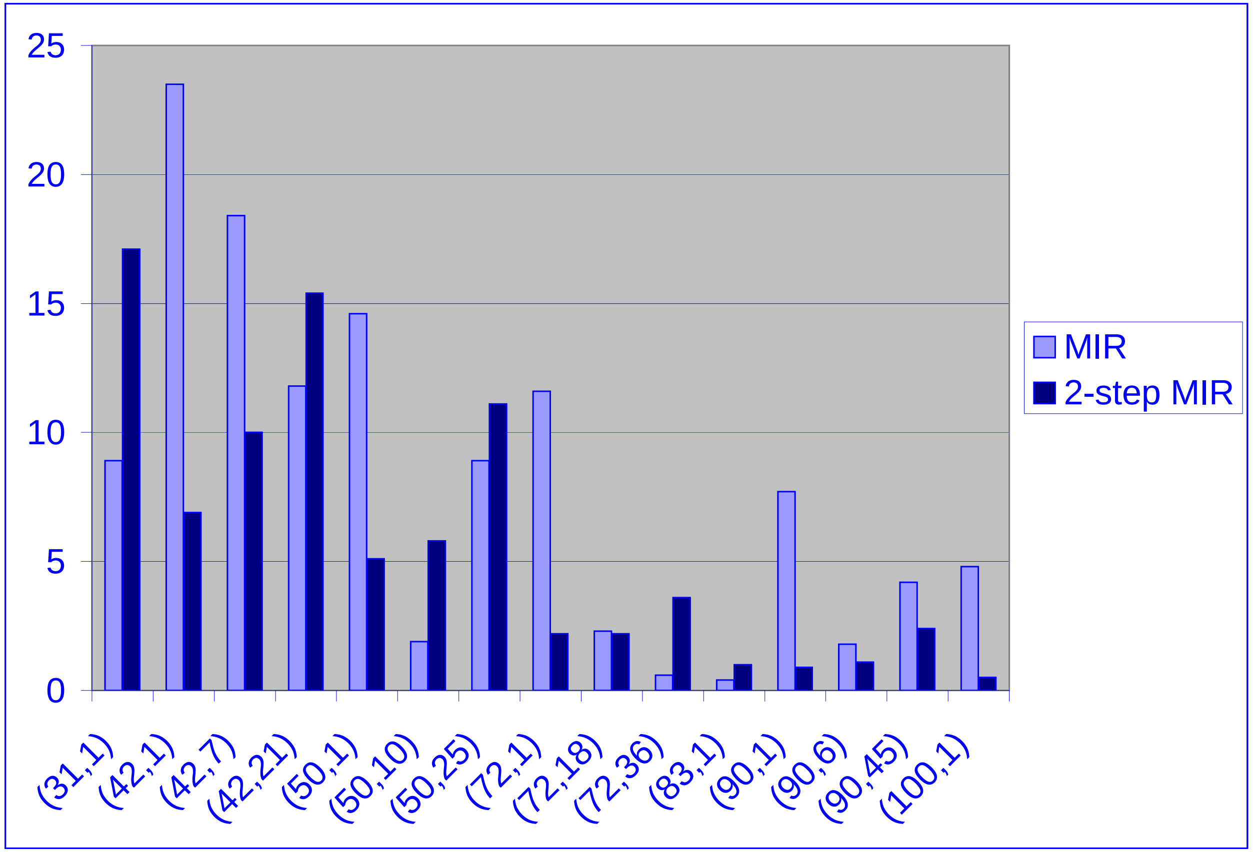
the group	*total MIRs*	*total facets hit*	*% all mirs*	*% top 10*	*% top 10 MIRs*
P(31,1)	*194*	*29,617*	*26.0*	*8.6*	*8.6*
P(42,1)	*207*	*46,407*	*30.4*	*21.9*	*21.8*
P(42,7)	*216*	*47,918*	*28.4*	*19.3*	*19.2*
P(42,21)	*247*	*53,754*	*27.2*	*15.5*	*15.5*
P(50,1)	*378*	*65,346*	*19.7*	*13.7*	*13.7*
P(50,10)	*370*	*74,736*	*7.6*	*1.4*	*1.3*
P(50,25)	*465*	*68,202*	*20.0*	*11.2*	*11.2*

Table 1: Shots absorbed by MIR-based facets for selected $P(n, r)$

- but the hit rate becomes less dramatic as n increases.

Shooting results and two-step MIR's

- Two-step MIR facets appear to be the second most important class.
- There are at most $(n/2)$ MIR facets for $P(n, r)$
- There are at most $(n^2/2)$ two-step MIR facets for $P(n, r)$



- Applied to $v_1 - v_2 + \sum_{i\in I} a_i\, x_i \geq b$ two-step MIR inequality

$$\frac{v_1}{\rho\,\tau} + \sum_{j\in I} \lfloor a_i \rfloor\, x_i + \frac{1}{\rho\,\tau} \sum_{j\in I} \min\{\rho\,\tau\ ,\ k_i\rho\ + \hat{a}_i - k_i\alpha\ ,\ l_i\rho\ \} x_i \geq \lceil b \rceil,$$

- *MIR inequality*

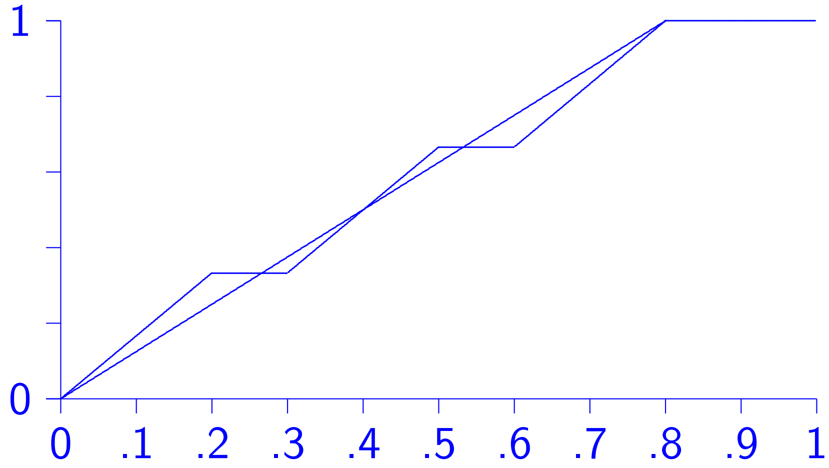
$$\frac{v_1}{\hat{b}} + \sum_{j\in I} \lfloor a_i \rfloor\, x_i + \frac{1}{\hat{b}} \sum_{j\in I} \min\{\hat{b}, \hat{a}_i\} x_i \geq \lceil b \rceil,$$

where $\hat{b} = \rho + (\tau - 1)\alpha \ \Rightarrow\ \hat{b} - \rho\tau = (\alpha - \rho)(\tau - 1)$

Example:

$\hat{b} = 0.8, \alpha = 0.3$

$\rho = 0.2,\ \tau = 3$



Given

$$(\bar{v}, \bar{x}) \in W^R \; = \; \Big\{ v, x \in R^{2+|I|} \; : \; v_1 - v_2 + \sum_{i \in I} a_i \, x_i \geq b, \;\; x, v \geq 0 \Big\},$$

how to chose an α *that gives a violated inequality?*

The violation of the two-step MIR ineqiality generated by

$$\alpha \in \mathcal{I}^\tau = \begin{cases} (\hat{b}/\tau, 1/\tau] & \text{if } 2 \leq \tau < \left\lceil 1/(1-\hat{b}) \right\rceil \\ (\hat{b}/\tau, \hat{b}/(\tau - 1)) & \text{if } \tau \geq \left\lceil 1/(1-\hat{b}) \right\rceil . \end{cases} \tag{1}$$

is

$$f(\alpha) = \rho^\alpha \tau \lceil b \rceil - \bar{v}_1 - \sum_{j \in I^+} h^{a_i}(\alpha) \bar{x}_i$$

where $\rho^\alpha = \hat{b} - \alpha(\tau - 1)$ and

$$h^a(\alpha) = \; \rho^\alpha \tau \lfloor a \rfloor + \min\{\rho^\alpha \tau \; , \; \lfloor \hat{a}/\alpha \rfloor \rho^\alpha \; + \hat{a} - \lfloor \hat{a}/\alpha \rfloor \alpha \; , \; \lceil \hat{a}/\alpha \rceil \rho^\alpha \; \} \;\; .$$

It is possible to show the following for $\alpha \in \mathcal{I}^\tau$:

1. *The cut coefficient function $h^a(\alpha)$ is continuous.*
2. *The violation function*

$$f(\alpha) = \rho^\alpha \tau \lceil b \rceil - \bar{v}_1 - \sum_{j \in I^+} h^{a_i}(\alpha) \bar{x}_i$$

   is continuous.

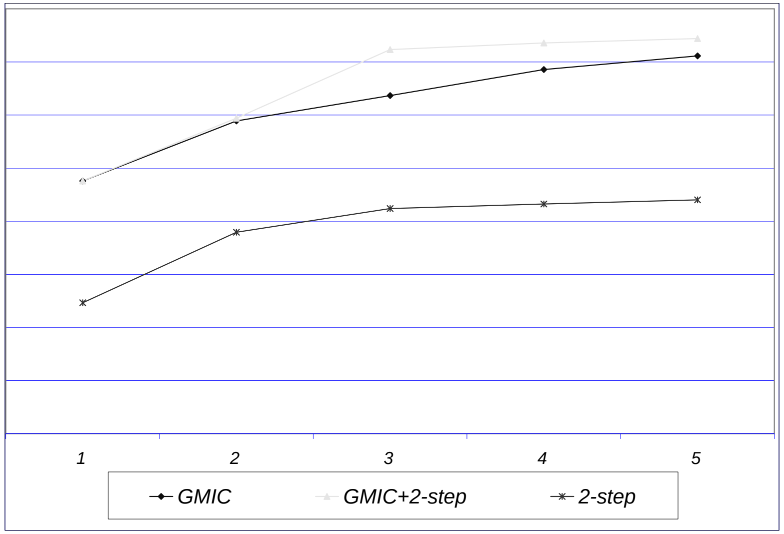
3. The violation function $f(\alpha)$ is piecewise linear.
4. Breakpoints of the function $f(\alpha)$ happen at $\alpha = \hat{a}_i / t$ for some $i \in I^+$ and $t \in Z_+$.
5. There are at most $|I^+|$ such α's and they can be identified by inspection.
6. The best $\alpha \in \mathcal{I}^\tau$ can be identified in linear-time.
7. The best $\alpha \in \cup_{2 \le \tau \le p} \mathcal{I}^\tau$ can be identified in $O(p \cdot |I|)$ time.

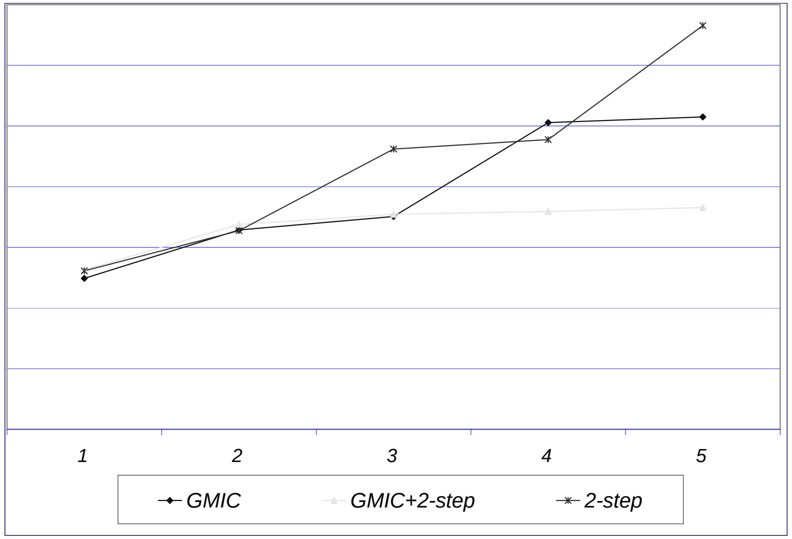
Computational experiments

- *How to obtain base inequalties?*
  - *Rows of the simplex tableau*
  - *Rows of the formulation (work-in-progress)*
- *Problem instances:*
  - *Unbounded Atamturk instances (random, dense, mixed-integer)*
  - *Bounded Atamturk instances (random, dense, mixed-integer)*
  - *Cornuejols, Li and Vandenbussche type instances (random, sparse, pure integer)*
  - *Dash, Kalagnanam, Reddy and Song steel instances (real, sparse, pure integer)*
  - *MIPLIB (work-in-progress)*

How to measure effectiveness?

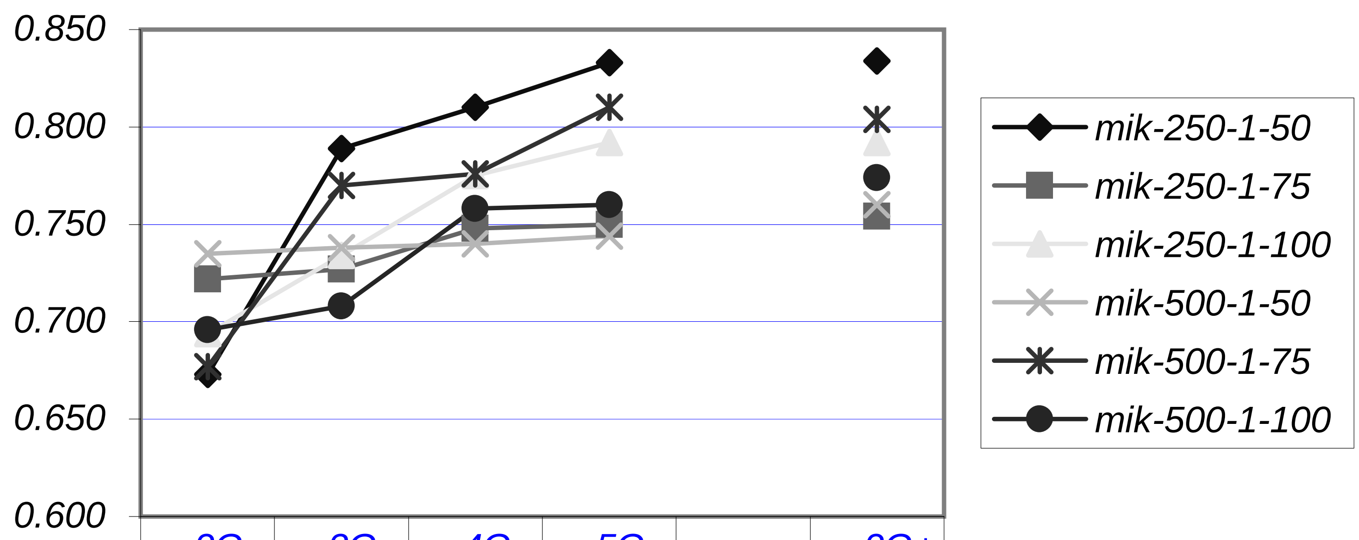
- *Two randomly generated problems with 20 rows and 100 cols:*





- *List cuts:*
  - *Remember the base inequalities for MIR's*
  - *Add 2-steps MIR's at the end only.*

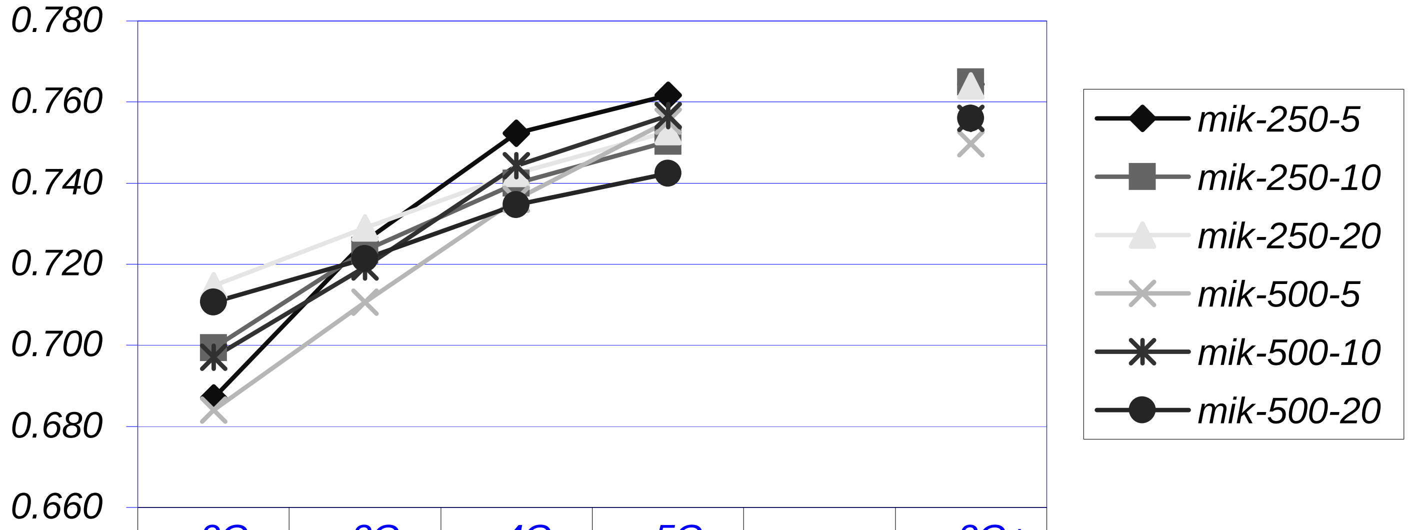
Unbounded Atamtürk Instances



	2G	3G	4G	5G		2G+
mik-250-1-50	0.673	0.789	0.810	0.833		0.834
mik-250-1-75	0.722	0.727	0.748	0.750		0.754
mik-250-1-100	0.694	0.734	0.775	0.792		0.792
mik-500-1-50	0.735	0.738	0.740	0.744		0.760
mik-500-1-75	0.677	0.770	0.776	0.810		0.804
mik-500-1-100	0.696	0.708	0.758	0.760		0.774

- *$2G+$: 2 rounds of GMIC followed by 2-step list cuts.*

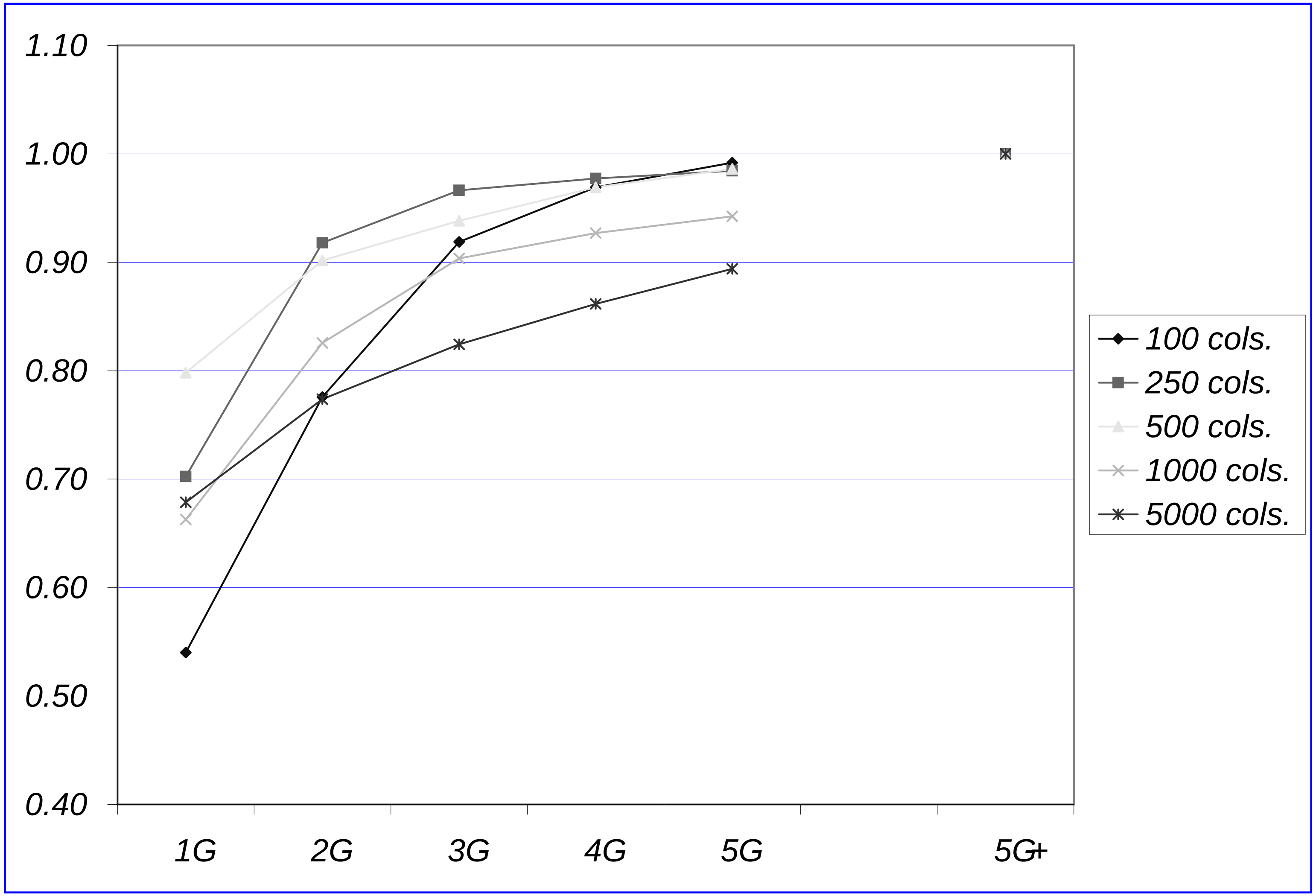
Bounded Atamtürk Instances



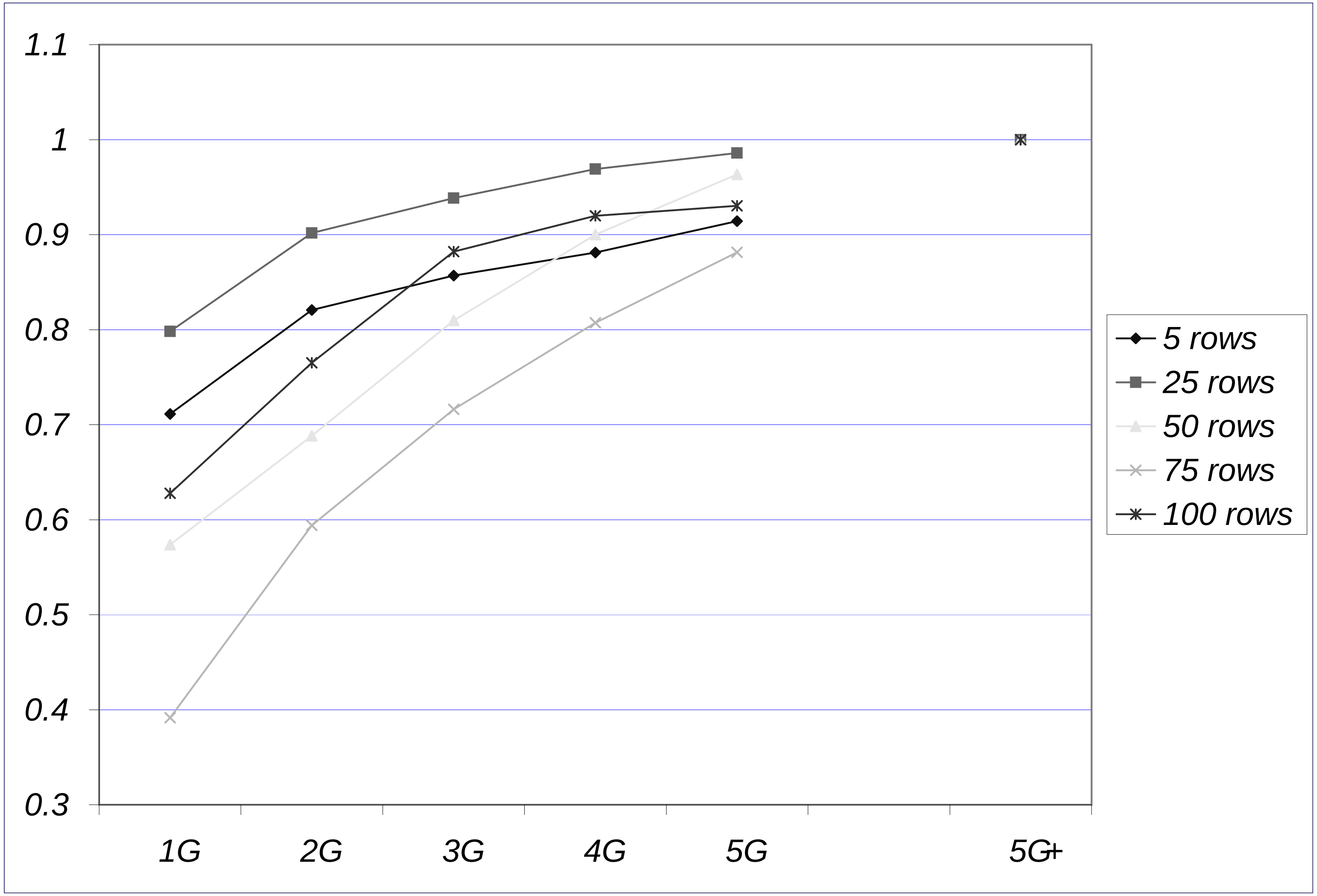
	2G	3G	4G	5G		2G+
mik-250-5	0.687	0.726	0.752	0.762		0.764
mik-250-10	0.699	0.723	0.740	0.750		0.765
mik-250-20	0.715	0.729	0.742	0.753		0.764
mik-500-5	0.684	0.711	0.736	0.755		0.750
mik-500-10	0.697	0.719	0.744	0.757		0.756
mik-500-20	0.711	0.721	0.735	0.742		0.756

- *$2G+$: 2 rounds of GMIC followed by 2-step list cuts.*

Random Instances with 25 rows

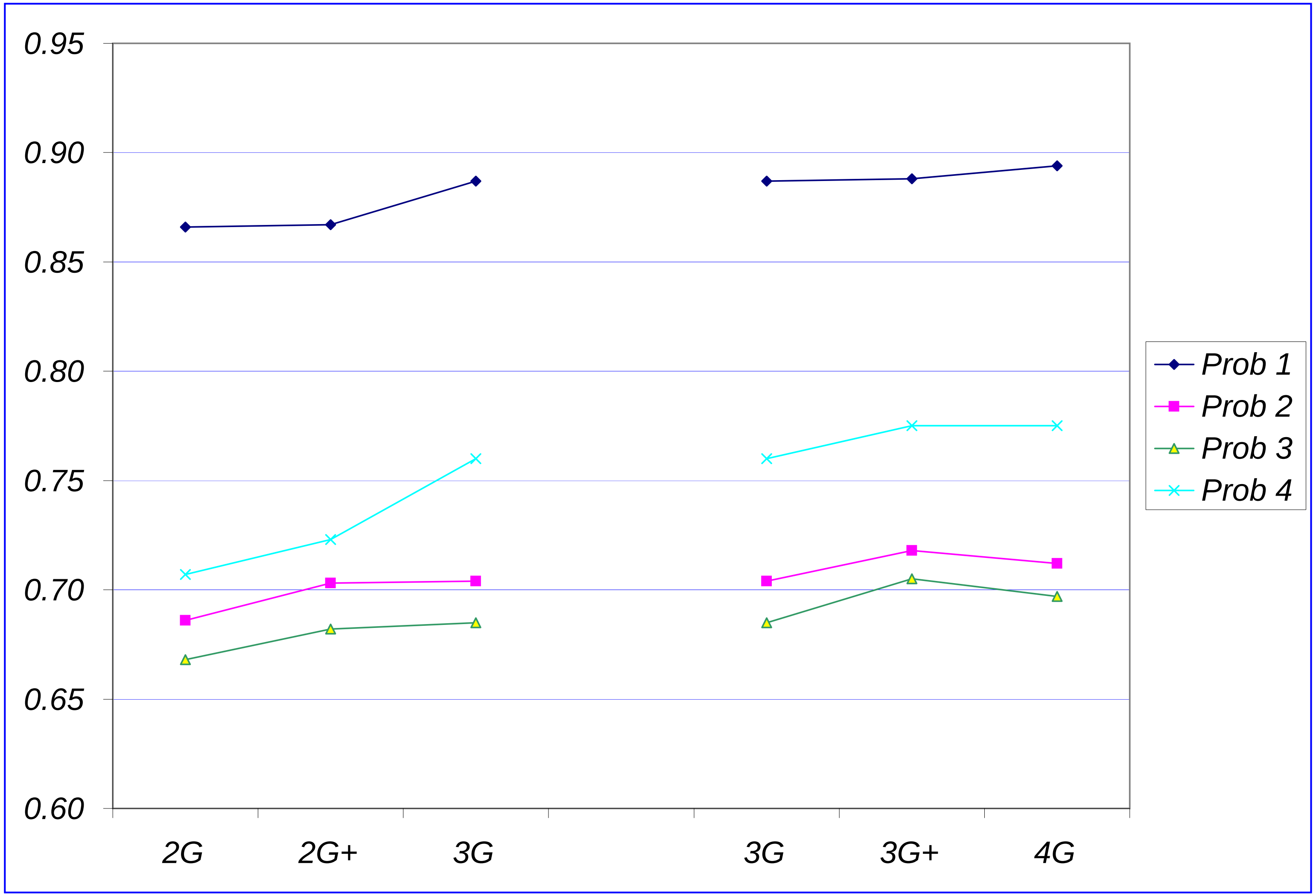


Random Instances with 500 cols



Steel Instances

Cutting stock problem instances with 5000 cols and 3000 rows



Thank You