LABORATOIRE d'ANALYSE et d'ARCHITECTURE des SYSTEMES

INTEGER PROGRAMMING, DUALITY and SUPERADDITIVE FUNCTIONS

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The integer program

$$\max_{x} \{ c' x \mid \quad A x = b; \quad x \in \mathbf{N}^n \}$$

with $A = [A_1, \ldots, A_n] \in \mathbf{Z}^{m \times n}, b \in \mathbf{Z}^m$ has a dual problem

$$D: \quad \min_{f \in \Gamma} \{ f(b) \mid \quad f(A_j) \geq c_j, \quad j = 1, \ldots, n \}$$

where Γ is the space of functions $f : \mathbf{R}^m \to \mathbf{R}$ that are superadditive $(f(a+b) \geq f(a) + f(b))$, and with $f(0) = 0$. (Jeroslow, Wolsey, etc ...)

Elegant but rather abstract and not very practical; however, used to derive valid inequalities. Moreover, the celebrated Gomory cuts used in Solvers (CPLEX, XPRESS-MP, ..) can be interpreted as such superadditive functions.

In addition, the optimization problem

$$D: \quad \min_{f \in \Gamma} \{ f(b) \mid \quad f(A_j) \geq c_j, \quad j = 1, \ldots, n\}$$

is too rich, and somehow, a *rephrasing* of the problem: Indeed, the set Γ contains in particular the optimal value function $f : \mathbf{Z}^m \to \mathbf{R} \cup \{-\infty\}$

$$b \mapsto f(b) := \max_x \{ c'x \quad \mid \quad Ax = b; \quad x \in \mathbf{N}^n\}$$

Similarly, the standard LP problem $\max_x \{c'x \mid A\,x = b;\ x \geq 0\}$ has an abstract dual problem

$$D_L: \quad \min_{f \in \Delta} \{ f(b) \mid \quad f(A_j) \geq c_j, \quad j = 1, \ldots, n\}$$

where Δ is now the set fo functions $f : \mathbf{R}^m \to \mathbf{R} \cup \{-\infty\}$ that are concave (or even concave piecewise linear).

But the simpler linear program

$$D_L^* : \quad \min_{f \text{ linear}} \{ f(b) \mid \quad f(A_j) \geq c_j, \quad j = 1, \ldots, n \}$$

that is,

$$D_L^* : \quad \min_{\lambda} \{ b'\lambda \mid \quad A_j'\lambda \geq c_j, \quad j = 1, \ldots, n \}$$

is a valid dual ... for a *fixed* value of b. It does not use concave piecewise linear functions (but does not contain the optimal value function $b \mapsto f(b) := \max_{x \geq 0} \{ c'x \mid A\,x = b \}$).

Hence, for integer programs, one should also obtain a dual simpler than the superadditive dual ....

CONTINUOUS OPTIM. | **DISCRETE OPTIM.**

$$f(b,c) := \max c'x \quad \text{s.t.} \left[\begin{array}{rcl} Ax & \leq & b \\ x & \in & \mathbf{R}^n_+ \end{array}\right] \longleftrightarrow f_d(b,c) := \max c'x \quad \text{s.t.} \left[\begin{array}{rcl} Ax & \leq & b \\ x & \in & \mathbf{N}^n \end{array}\right]$$

$\updownarrow$ | $\updownarrow$

INTEGRATION | **SUMMATION**

$$\widehat{f}(b,c) := \int_\Omega e^{c'x} dx, \quad \Omega := \left[\begin{array}{rcl} Ax & \leq & b \\ x & \in & \mathbf{R}^n_+ \end{array}\right] \longleftrightarrow \widehat{f}_d(b,c) := \sum_\Omega e^{c'x}, \quad \Omega := \left[\begin{array}{rcl} Ax & \leq & b \\ x & \in & \mathbf{N}^n \end{array}\right]$$

$$\mathrm{e}^{f(b,c)} = \lim_{r\to\infty} \widehat{f}(b, rc)^{1/r}; \quad \mathrm{e}^{f_d(b,c)} = \lim_{r\to\infty} \widehat{f}_d(b, rc)^{1/r}.$$

or, equivalently

$$f(b, c) = \lim_{r\to\infty} \frac{1}{r} \ln \widehat{f}(b, rc); \quad f_d(b, c) = \lim_{r\to\infty} \frac{1}{r} \ln \widehat{f}_d(b, rc).$$

CONTINUOUS OPTIM.	|	DISCRETE OPTIM.
Legendre-Fenchel Duality	$\longleftrightarrow$	??
$\updownarrow$		$\updownarrow$
INTEGRATION	|	SUMMATION
Laplace-Transform Duality	$\longleftrightarrow$	Z-Transform Duality

Legendre-Fenchel duality : $f : \mathbf{R}^n \to \mathbf{R}$ convex; $f^* : \mathbf{R}^n \to \mathbf{R}$.

$$\lambda \mapsto f^*(\lambda) = \mathcal{F}(f)(\lambda) := \sup_y \{\lambda' y - f(y)\}.$$

(One-sided) Laplace-Transform: $f : \mathbf{R}^n_+ \to \mathbf{R}$; $F : \mathbf{C}^n \to \mathbf{C}$.

$$\lambda \mapsto F(\lambda) = \mathcal{L}(f)(\lambda) := \int_{\mathbf{R}^n_+} \mathrm{e}^{-\lambda' y} f(y)\, dy.$$

(One-sided) Z-Transform: $f : \mathbf{Z}^n_+ \to \mathbf{R}$; $F : \mathbf{C}^n \to \mathbf{C}$.

$$\lambda \mapsto F(z) = \mathcal{Z}(f)(z) := \sum_{m \in \mathbf{Z}^n_+} z^{-m} f(m).$$

with $\Omega := \{x \in \mathbf{R}^n \mid \quad A\,x \leq y; \quad x \geq 0\}$

Fenchel-duality:

$$f(y,c) := \max_{x \in \Omega} c'x$$

$$f^*(\lambda,c) := \min_{y} \{\lambda' y - f(y,c)\}$$

with : $A'\lambda - c \geq 0;\ \lambda \geq 0$

Laplace-duality

$$\widehat{f}(y,c) := \int_{\Omega} e^{c'x}\, dx$$

$$\widehat{F}(\lambda,c) := \int e^{-\lambda y}\, \widehat{f}(y,c)\, dy$$

$$= \int_{x \geq 0} e^{c'x} \left[\int_{Ax \leq y} e^{-\lambda y}\, dy \right] dx$$

$$= \frac{1}{\prod_{j=1}^{m} \lambda_j \prod_{k=1}^{n} (A'\lambda - c)_k}$$

with : $\Re(A'\lambda - c) > 0;\ \Re(\lambda) > 0$

In standard form, with $\Omega := \{x \in \mathbf{R}^n \mid \quad A\,x \,=\, b; \quad x \geq 0\}$

$$\widehat{F}(\lambda, c) = \frac{1}{\prod_{k=1}^{n} (A'\lambda - c)_k}$$

and one retrieves $\widehat{f}(y, c)$ by

$$\widehat{f}(y,c) \,=\, \int_{\Gamma} \mathrm{e}^{\,y'\,\lambda}\,\widehat{F}(\lambda, c)\,d\lambda \,=\, \int_{\Gamma} \frac{\mathrm{e}^{\,y'\,\lambda}}{\prod_{j=1}^{n}(A'\,\lambda - c)_j}\,d\lambda$$

Integration by Cauchy's residue techniques $\longrightarrow$

• The (multidimensional) poles λ of $\widehat{F}$ solve $\pi^{\sigma}\,A_{\sigma} \,=\, c_{\sigma}$ for all bases A_{σ} of the continuous LP, ... which yields ....

Brion and Vergne 's continuous formula

Terminology of LP in standard form:

Let $A_\sigma := [A_{\sigma_1}|\ldots|A_{\sigma_m}]$ be a **basis** of $\max\{c'x|Ax = b; x \geq 0\}$, with $x(\sigma)$ the corresponding **vertex**, $\pi^\sigma := c'_\sigma A_\sigma^{-1}$ the associated **dual variable** and the **reduced cost** vector $c_k - \pi^\sigma A_k$, $k \notin \sigma$. Then :

$$\widehat{f}(b,c) = \sum_{x(\sigma):\text{ vertex of }\Omega(b)} \frac{e^{c'x(\sigma)}}{\det(A_\sigma) \prod_{k\notin\sigma} (-c_k + \pi^\sigma A_k)}$$

from which it easily follows that

$$\log\left[\lim_{r\to\infty} \widehat{f}(b,rc)\right]^{1/r} = \max_{x(\sigma):\text{ vertex of }\Omega(b)} c'\,x(\sigma).$$

Discrete $\mathbb{Z}$-duality

Let $\Omega(y) := \{x \in \mathbf{R}^n \mid \quad A\,x \,=\, y; \quad x \geq 0\}$.

$$\widehat{f}_d(y,c) \;:=\; \sum_{x \,\in\, \Omega(y) \,\cap\, \mathbf{Z}^n} \mathrm{e}^{c'x}$$

Then the *generating function* or $\mathbb{Z}$-transform of $\widehat{f}_d$ reads

$$z \mapsto \quad \widehat{F}_d(z,c) := \sum_{y \in \mathbf{Z}^m} z^{-y}\, \widehat{f}_d(y,c),$$

and, by simple algebra

$$\widehat{F}_d(z,c) = \prod_{j=1}^{n} \frac{1}{1-\mathrm{e}^{c_j}\, z^{-A_j}} \quad (= \prod_{j=1}^{n} \frac{1}{1-\mathrm{e}^{c_j}\, {z_1}^{-A_{1j}} \ldots {z_m}^{-A_{mj}}})$$

$$\text{with } |z^{A_j}| > \mathrm{e}^{c_j} \quad \forall j = 1,\ldots,n.$$

To compute $f_d(y, c)$ it then suffices to invert the generating function, that is:

$$f_d(y, c) = \int_{|z|=\gamma} z^{y-1} \widehat{F}_d(z, c) \, dz$$

which can be done by repeated application of Cauchy' s residue Theorem, which in turn requires computing the poles of the generating function $\widehat{F}_d$.

• The Poles of $\widehat{F}_d$ are also associated with the bases of the LP $\max \{ c' x \mid A x = y, \; x \geq 0 \}$!!

Let $\sigma := [A_{\sigma_1}|\cdots|A_{\sigma_m}]$ be a feasible basis of the LP $\max_x\{c'\,x\,|\,A\,x = y,\ x \geq 0\}$, with $\mu(\sigma) := \det(A_\sigma)$, and associated "dual" variables π^σ, which are solutions of $\pi A_\sigma = c_\sigma$.

Indeed, each basis σ provides $\mu(\sigma)$ complex poles $z = e^\lambda$ in $\mathbf{C}^m$, which are solutions of the polynomial equations $e^{A'_\sigma \lambda} = e^{c_\sigma}$. (Compare with $\pi^\sigma A_\sigma = c_\sigma$)

The $\mu(\sigma)$ solutions $z = e^\lambda$ are of the form

$$\lambda \;=\; \pi^\sigma + 2i\pi\frac{v}{\mu(\sigma)}; \qquad v \in V_\sigma \subset \mathbf{Z}^m$$

$$V_\sigma = \{\, v \in \mathbf{Z}^m \,|\quad v'A_\sigma \;=\; 0 \quad \mathrm{mod}\,\mu(\sigma)\}$$

Brion and Vergne 's discrete formula

Re-interpreted with these data, Brion and Vergne 's original discrete formula reads (Lasserre)

$$\widehat{f}_d(y,c) \;=\; \sum_{x(\sigma):\ \text{vertex of } \Omega(y)} e^{c' x(\sigma)} \times \frac{1}{\mu(\sigma)}\left[\sum_{v\in V_\sigma} \frac{e^{2i\pi v' y/\mu(\sigma)}}{\prod_{k\notin\sigma}\left(1-e^{-(2i\pi v' A_k/\mu(\sigma))}\, e^{(c_k-\pi^\sigma A_k)}\right)}\right]$$

Back to **optimization :** $f_d(b,c) = \max\{ c'x \,|\, A\,x = b;\ x \in \mathbf{N}^n\}$.

Theorem . Assume that

$$\max_{\sigma} \mathrm{e}^{c'x(\sigma)} \times \lim_{r\to\infty} \left[\sum_{v\in V_\sigma} \frac{\mathrm{e}^{2i\pi v'b/\mu(\sigma)}}{\prod_{k\notin\sigma} (1-\mathrm{e}^{-(2i\pi v'A_k/\mu(\sigma))}\,\mathrm{e}^{r(c_k-\pi^\sigma A_k)})} \} \right]^{1/r}$$

is attained at a unique basis σ^*. Then :

$$f_d(b,c) = c'\,x(\sigma^*) + \sum_{k\notin\sigma^*} (c_k - \pi^{\sigma^*} A_k)\, x_k^* = \sum_j c_j\, x_j^*.$$

σ^* is an optimal basis of the linear program, and $x(\sigma^*)$ (resp. x^*) is an optimal solution of the linear (resp. integer) program.

In this case :

$$f_d(b,c) = c'x(\sigma^*) + \left\{ \begin{array}{llll} \max & & \sum_{k \notin \sigma^*} (c_k - \pi^{\sigma^*} A_k) x_k & \\ & A_{\sigma^*} u & + \sum_{k \notin \sigma^*} A_k x_k & = b \\ & u \in \mathbf{Z}^m; & x_k \in \mathbf{N} \quad \forall k \notin \sigma^* & \end{array} \right.$$

$f_d(b,c) = c'x(\sigma^*) + \rho :=$ **opt. value of GOMORY relaxation!**

$$\lim_{r\to\infty}\left[\sum_{v\in V_\sigma}\frac{e^{2i\pi v'b/\mu(\sigma)}}{\prod_{k\notin\sigma}(1-e^{-(2i\pi v'A_k/\mu(\sigma))}\, e^{r(c_k-\pi^\sigma A_k)})}\}\right]^{1/r}$$

is the same as detecting the leading term u^ρ of the rational fraction

$$u\mapsto\left[\sum_{v\in V_\sigma}\frac{e^{2i\pi v'b/\mu(\sigma)}}{\prod_{k\notin\sigma}(1-e^{-(2i\pi v'A_k/\mu(\sigma))}\, u^{(c_k-\pi^\sigma A_k)})}\right]$$

as $u\to\infty$.

With $u := \mathrm{e}^r$, $\widehat{f}_d(y, rc)$ is the rational fraction

$$u \mapsto \sum_{x(\sigma):\ \text{vertex of } \Omega(y)} u^{c' x(\sigma)} \times$$

$$\frac{1}{\mu(\sigma)} \left[\sum_{v \in V_\sigma} \frac{\mathrm{e}^{2i\pi v' y/\mu(\sigma)}}{\prod_{k \notin \sigma} \left(1 - \mathrm{e}^{-(2i\pi v' A_k/\mu(\sigma))}\, u^{(c_k - \pi^\sigma A_k)}\right)} \right]$$

$$= \sum_{\text{bases } \sigma \text{ of } \Omega(y)} g_\sigma(u)$$

When Gomory relaxation is not exact, the leading term as $u \to \infty$ is not unique and cancellations occur.

A Discrete Farkas Lemma

Let $A \in \mathbf{N}^{m \times n}, b \in \mathbf{N}^m$ and consider the problem deciding **whether or not** $Ax = b$ has a solution $x \in \mathbf{N}^n$, or, equivalently, deciding **whether or not** $\widehat{f}_d(b, c) \geq 1$.

Theorem: (i) $Ax = b$ *has a solution* $x \in \mathbf{N}^n$ *if and only if the polynomial* $b \mapsto z^b - 1$ *in* $\mathbf{R}[z_1, \ldots, z_m]$ *can be written*

$$z^b - 1 = \sum_{j=1}^{n} Q_j(z)(z^{A_j} - 1) \quad (= \sum_{j=1}^{n} Q_j(z)(z_1^{A_{1j}} \ldots z_m^{A_{mj}} - 1)$$

for some polynomials $z \mapsto Q_j(z)$ *with* **nonnegative coefficients.**

(ii) *The degree of the* Q_j*'s is bounded by* $\sum_{j=1}^{m} b_j - \max_k \sum_{j=1}^{m} A_{jk}$.

A single LP to solve with $n \times \binom{b^*+m}{b^*}$ variables, $\binom{b^*+m}{m}$ constraints and a (sparse) matrix of coefficients in $\{0, \pm 1\}$

One also retrieves the classical **Farkas Lemma** in $\mathbf{R}^n$, that is,

$$\{x \in \mathbf{R}^n \,|\, A\,x = b; x \geq 0\} \neq \emptyset \qquad \Leftrightarrow \qquad \left[A'\,u \,\geq\, 0 \Rightarrow \quad b'\,u \,\geq\, 0 \right]$$

Indeed, if $A\,x = b$ has a solution $x \in \mathbf{N}^n$, then with $u = \ln z$,

$$\mathrm{e}^{b'u} - 1 \,=\, \sum_{j=1}^{n} Q_j(e^{u_1}, \ldots, e^{u_m})(\mathrm{e}^{(A'u)_j} - 1).$$

Therefore,

$$A'u \geq 0 \,\Rightarrow\, \mathrm{e}^{(A'\,u)_j} - 1 \geq 0 \,\Rightarrow\, \mathrm{e}^{b'u} - 1 \geq 0 \quad \Rightarrow b'u \geq 0,$$

and so $A\,x = b$ has a nonnegative solution $x \in \mathbf{R}^n_+$.

The general case $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$.

Let $\Omega := \{x \in \mathbf{R}^n | Ax = b; \quad x \geq 0\}$ be a **polytope.**

Let $\alpha \in \mathbf{N}^n$ be such that for every column A_j of A,

$A_{kj} + \alpha_j \geq 0 \; \forall\, k = 1, \ldots, m$; let $\mathbf{N} \ni \beta \geq \rho(\alpha) := \max\{\alpha' x \,|\, x \in \Omega\}$.

Theorem: (i) *$Ax = b$ has a solution $x \in \mathbf{N}^n$ if and only if the polynomial $z \mapsto z^b (zy)^\beta - 1$ in $\mathbf{R}[z_1, \ldots, z_m]$ can be written*

$$z^b (zy)^\beta - 1 = Q_0(z,y)(zy - 1) + \sum_{j=1}^{n} Q_j(z,y)(z^{A_j}(zy)^{\alpha_j} - 1),$$

for some polynomials $Q_j(z)$, all with **nonnegative coefficients.**

(ii) *The degree of the Q_j's is bounded by $b^* := (m+1)\beta + \sum_{j=1}^{m} b_j$.*

Back to standard Farkas lemma

$$\{x \in \mathbf{R}^n \mid A\,x = b;\ \ x \geq 0\} \neq \emptyset \quad \Leftrightarrow \quad \left[A'\lambda \geq 0 \Rightarrow b'\lambda \geq 0 \right]$$

But, **equivalently** $\{x \in \mathbf{R}^n \mid A\,x = b,\ \ x \geq 0\} \neq \emptyset$ if and only if the polynomial $\lambda \mapsto b'\lambda$ can be written

$$b'\,\lambda = \sum_{j=1}^{n} Q_j(\lambda)(A'\,\lambda)_j,$$

for some polynomials $\{Q_j\} \subset \mathbf{R}[\lambda_1, \ldots, \lambda_m]$, all with **nonnegative** coefficients.

In this case, each Q_j is necessarily a constant, that is, $Q_j \equiv Q_j(0) = x_j \geq 0$, and $A\,x = b$!

$\mathsf{P} = \{x \in \mathbf{R}^n \mid Ax = b,\ x \geq 0\}$	$\mathsf{P} \cap \mathbf{Z}^n$
$x \in \mathsf{P}$ $\Leftrightarrow x = Q(0,\ldots,0)$ with $Q \in \mathbf{R}[\lambda_1,\ldots,\lambda_m]$ $b'\lambda = \langle Q, A'\lambda\rangle$ $Q \succeq 0$	$x \in$ integer hull (P) $\Leftrightarrow x = Q(1,\ldots,1)$ with $Q \in \mathbf{R}[\mathrm{e}^{\lambda_1},\ldots,\mathrm{e}^{\lambda_m}]$ $\mathrm{e}^{b'\lambda} - 1 = \langle Q, \mathrm{e}^{A'\lambda} - 1_n\rangle$ $Q \succeq 0$

Comparing continuous and discrete Farkas lemma

An equivalent Linear program

Let $0 \le q = \{q_{j\alpha}\} \in \mathbf{R}^{ns}$ be the coefficients of the Q_j's in

$$z^b - 1 = \sum_{j=1}^{n} Q_j(z)\,(z^{A_j} - 1)$$

They are solutions of a linear system

$$\mathsf{M}\, q = r, \quad q \ge 0$$

for some matrix M and vector r, both with $0, \pm 1$ coefficients.

** M and r are easily obtained from A, b with **no** computation

Write $q = (q_1, q_2, \ldots, q_n)$ with each $q_j = \{q_{j\alpha}\} \in \mathbf{R}^s$, and let $\widehat{c}_{j\alpha} := c_j$ for all α

Theorem : *Let* $A \in \mathbf{N}^{m\times n}, b \in \mathbf{N}^m, c \in \mathbf{R}^n$.

(i) The integer program $\mathbf{P} \;\to\; \max_x \{ c'x \mid A\,x = b,\; x \in \mathbf{N}^n\}$ has same value as the linear program

$$\mathbf{Q} \to \max_q \{ \sum_{j=1}^{n} \widehat{c}'_j \, q_j \quad | \quad \mathsf{M}\, q = r; \quad q \geq 0\}.$$

(ii) Let q^* be an optimal vertex, and let

$$x_j^* := \sum_\alpha q^*_{j\alpha} = Q_j(1) \quad j = 1, \ldots, n.$$

Then $x^* \in \mathbf{N}^n$ and x^* is an optimal solution of $\mathbf{P}$.

The link with superadditive functions

The LP-dual $\mathbf{Q}^*$ of the linear program $\mathbf{Q}$ reads

$$\mathbf{Q}^* \to \min_{\pi} \{\pi' r \mid \quad \mathsf{M}' \pi \geq \widehat{c}\}.$$

More precisely, with $\mathcal{D} := \prod_{j=1}^{n} \{0, 1, \ldots, b_j\} \subset \mathbf{N}^m$,

$$\mathbf{Q}^* \to \begin{cases} \qquad \min_{\pi} \quad \pi(b) - \pi(0) \\ \\ \text{s.t.} \quad \pi(A_j + \alpha) - \pi(\alpha) \quad \geq c_j, \quad \alpha + A_j \in \mathcal{D},\ j = 1, \ldots, n \end{cases}$$

Let $\Pi := \{\pi : \mathcal{D} \to \mathbf{R}\}$, and for every $\pi \in \Pi$, let $f_\pi : \mathcal{D} \to \mathbf{R}$ be the function

$$f_\pi(x) := \inf_{x + \alpha \in \mathcal{D}} \pi(x + \alpha) - \pi(\alpha), \qquad x \in \mathcal{D}$$

For every $\pi \in \Pi$, the function f_π is superadditive and $f_\pi(0) = 0$.

With $\mathbf{Q}^*$ one may then associate the optimization problem

$$\mathbf{S}^* \rightarrow \begin{cases} \min\limits_{\pi \in \Pi} & f_\pi(b) \\ \text{s.t.} \quad f_\pi(A_j) & \geq c_j, \quad j = 1, \ldots, n. \end{cases}$$

Thus, $\mathbf{S}^*$ is a *simplified and explicit form* of the abstract superadditive dual, as we only consider *finite* superadditive functions $f : \mathcal{D} \rightarrow \mathbf{R}$, instead of $f : \mathbf{N}^m \rightarrow \mathbf{R} \cup \{-\infty\}$.

It is the analogue for IP with $A\,x = b$ of Wolsey's dual for IP with $A\,x \leq b$. Note that $\mathbf{Q}^*$ is simpler than $\mathbf{S}^*$.

Moreover $\mathbf{Q}^*$ is simpler than $\mathbf{S}^*$ as in the LP $\mathbf{Q}^*$, the function $\pi : \mathcal{D} \to \mathbf{R}$ is NOT required to be super additive!! In $\mathbf{S}^*$ on has to write the $O(|\mathcal{D}|^2)$ *superadditivity* constraints

$$\pi(x+\alpha) - \pi(\alpha) \geq \pi(x), \qquad x, \alpha \in \mathcal{D}, \text{ with } x+\alpha \in \mathcal{D},$$

and the n additional constraints $\pi(A_j) \geq c_j$, $j = 1, \ldots, n$.

On the other hand, in $\mathbf{Q}^*$ one only has the $n\, O(|\mathcal{D}|)$ constraints

$$\pi(A_j+\alpha) - \pi(\alpha) \geq c_j \qquad \alpha \in \mathcal{D}, \text{ with } A_j+\alpha \in \mathcal{D},$$

For instance, in the unbounded knapsack problem

$$\max_x \{c'\, x \mid \sum_{j=1}^{n} a_j\, x_j = b; \quad x \in \mathbf{N}^n\}$$

One has $n + b^2/2$ constraints in $\mathbf{S}^*$, and $nb - \sum_{j=1}^n a_j$ in $\mathbf{Q}^*$.

With $\mathsf{P} = \{x \in \mathbf{R}^n_+ \mid A\,x = b\}$, the integer hull $\mathsf{co}(\mathsf{P} \cap \mathbf{Z}^n)$ reads

$$\mathsf{co}(\mathsf{P} \cap \mathbf{Z}^n) \;=\; \{x \in \mathbf{R}^n \mid \quad -\sum_{j=1}^{n} \lambda_j^* \, x_j \;\leq\; b'\pi^*\},$$

for finitely many (π^*, λ^*), generators of the convex cone $\Omega \subset \mathbf{R}^{|\mathcal{D}|} \times \mathbf{R}^n$

$$\Omega := \Big\{(\pi, \lambda) \in \mathbf{R}^{|\mathcal{D}|} \times \mathbf{R}^n \;\;|\;\; \pi(\alpha + A_j) - \pi(\alpha) + \lambda_j \;\geq\; 0, \\ \alpha + A_j \in \mathcal{D},\; j = 1, \ldots n\Big\}$$

Equivalently, again with

$$x \mapsto \quad f_{\pi^*}(x) := \inf_{\alpha + x \in \mathcal{D}} \pi^*(x + \alpha) - \pi^*(\alpha), \quad x \in \mathcal{D}$$

$$\mathsf{co}(\mathsf{P} \cap \mathbf{Z}^n) \;=\; \{x \in \mathbf{R}^n \mid \quad \sum_{j=1}^{n} f_{\pi^*}(A_j)x_j \;\leq\; f_{\pi^*}(b)\},$$

CONCLUSION

Generating functions permit to exhibit a natural **duality** for integer programming, an IP-analogue of **LP duality**.

This duality permits to simplify the abstract **superadditive** dual, and so, might help providing **efficient Gomory cuts** in MIP solvers like CPLEX, or XPRESS-MP.

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Another dual problem

Let $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m, c \in \mathbf{R}^n$. Let $y \mapsto f(y,c) = \max\{c'x \,|\, Ax = y;\ x \geq 0\}$. The Fenchel transform $(-f)^*$ of the convex function $-f(.,c)$ is the convex function

$$\lambda \mapsto \quad (-f)^*(\lambda, c) = \sup_{y\in\mathbf{R}^m} \lambda' y + f(y,c).$$

The dual problem of the linear program is obtained from Fenchel duality as

$$\begin{aligned} f(b,c) &= \inf_{\lambda\in\mathbf{R}^m} b'\lambda + (-f)^*(-\lambda, c) \\ &= \inf_{\lambda\in\mathbf{R}^m} b'\lambda + \sup_{x\in\mathbf{R}^m_+} (c - A'\lambda)'x = \min_{\lambda} \{b'\lambda \,|\, A'\lambda \geq c\} \end{aligned}$$

Equivalently

$$\mathrm{e}^{f(b,c)} = \inf_{\lambda\in\mathbf{R}^m} \sup_{x\in\mathbf{R}^m_+} \mathrm{e}^{(b-A\,x)'\lambda} \mathrm{e}^{c'\,x}$$

Define

$$\rho^* := \inf_{z \in \mathbf{C}^m} \sup_{x \in \mathbf{N}^n} \Re\left(z^{b-Ax}\,\mathrm{e}^{c'x}\right) = \inf_{z \in \mathbf{C}^m} f_d^*(z,c).$$

Hence,

$$f_d^*(z,c) = \Re\left(z^b \prod_{j=1}^{n} (z^{-A_j}\mathrm{e}^{c_j})^{x_j}\right) < \infty \quad \text{if } |z^{A_j}| \geq \mathrm{e}^{c_j} \quad \forall j$$

(that is, $A' \ln|z| \geq c$). Next, (writing $z \in \mathbf{C}$ as $\mathrm{e}^{\lambda}\mathrm{e}^{i\theta}$)

$$\rho^* \leq \inf_{z \in \mathbf{R}^m} \sup_{x \in \mathbf{R}^n_+} \Re\left(z^{b-Ax}\,\mathrm{e}^{c'x}\right) = \inf_{\lambda \in \mathbf{R}^m} \sup_{x \in \mathbf{R}^n_+} \mathrm{e}^{(b-Ax)'\lambda}\,\mathrm{e}^{c'x} = \mathrm{e}^{f(b,c)}$$

Finally, with $z \in \mathbf{C}^m$ arbitrary fixed

$$\sup_{x \in \mathbf{N}^n} \Re\left(z^{b-Ax}\,\mathrm{e}^{c'x}\right) \geq \mathrm{e}^{c'x^*} = \mathrm{e}^{f_d(b,c)}$$

Hence $\mathbf{f_d(b,c)} \leq \ln \rho^* \leq \mathbf{f(b,c)}$.

Theorem: Let σ^* be an optimal basis of the linear program. Under uniqueness of the "$\max_\sigma$" in Brion and Vergne 's formula, and an additional technical condition

$$e^{f_d(b,c)} = \rho^* = \max_{x \in \mathbf{N}^n} \Re\left(\widehat{z}^{b-Ax} e^{c'x}\right) = f_d^*(\widehat{z}, c)$$

where $\widehat{z}^{A_j} = \gamma\, e^{c_j} \quad \forall j \in \sigma^*$ for some real $\gamma > 1$.

$\widehat{z}$ is an optimal solution of the dual problem

$$\inf_{z \in \mathbf{C}^m} f_d^*(z, c)$$