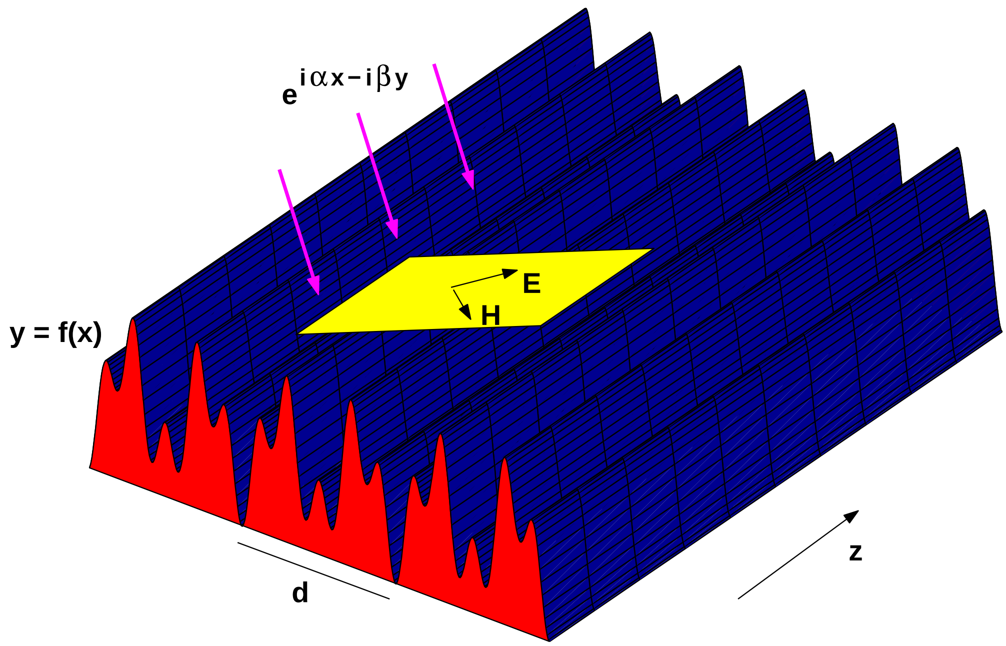


$\Longrightarrow$ **2 scalar problems:** $\begin{cases} u = E_z & \textbf{(TE)} \\ u = H_z & \textbf{(TM)} \end{cases}$

$$\begin{aligned} \Delta u^{\pm} + k^{\pm} u^{\pm} &= 0 && \text{in } \pm y > \pm f(x) \\ u^{+} - u^{-} &= -e^{i\alpha x - i\beta y} && \text{on } y = f(x) \\ \frac{\partial u^{+}}{\partial \vec{n}} - C\,\frac{\partial u^{-}}{\partial \vec{n}} &= -\frac{\partial}{\partial \vec{n}}\left(e^{i\alpha x - i\beta y}\right) && \text{on } y = f(x) \end{aligned}$$

$\bigoplus$ **Radiation Conditions**

$$\left[C = \begin{cases} 1 & \text{(TE)} \\ \left(\dfrac{k^{+}}{k^{-}}\right)^{2} & \text{(TM)} \end{cases} \right].$$

Periodicity of the structure $\Longrightarrow$ **u is quasi-periodic**, i.e.

$$u(x+d,y) = e^{i\alpha d}u(x,y) \qquad (e^{-i\alpha x}u(x,y) \text{ is } d\text{-periodic}).$$

$\therefore$ (separation of variables) **RAYLEIGH SERIES**

$$u^{\pm}(x,y) = \sum_{r=-\infty}^{\infty} \underbrace{B_r^{\pm}}_{\text{UNKNOWNS!}} e^{i\alpha_r x + i\beta_r^{\pm} y} \qquad \begin{pmatrix} \alpha_r = \alpha + 2\pi r/d \\ \alpha_r^2 + (\beta_r^{\pm})^2 = (k^{\pm})^2 \end{pmatrix}$$

Note: $\beta_r^{\pm} = \sqrt{(k^{\pm})^2 - (\alpha_r)^2}$
$\Rightarrow$ only **finitely many** propagating modes

Simplest case: $y < f(x)$ contains a **perfect conductor**
$\Rightarrow u = u^+$, $k = k^+$ and

$$\Delta u^+ + k^+ u^+ = 0 \text{ in } y > f(x)$$

with boundary conditions, on $y = f(x)$,

$$u^+ = -e^{i\alpha x - i\beta y} \quad \text{(TE)} \qquad \text{or} \qquad \frac{\partial u^+}{\partial \vec{n}} = -\frac{\partial}{\partial \vec{n}}\left(e^{i\alpha x - i\beta y}\right) \quad \text{(TM)}$$

$$u(x,y) = \sum_{r=-\infty}^{\infty} B_r \, e^{i\alpha_r x + i\beta_r y}$$

Perfectly conducting grating (TE):

$$u(x,y) = \sum_{r=-\infty}^{\infty} B_r \ e^{i\alpha_r x + i\beta_r y}$$

Rayleigh's method:

$$u(x,f(x)) = \sum_{r=-\infty}^{\infty} B_r \ e^{i\alpha_r x + i\beta_r f(x)}$$

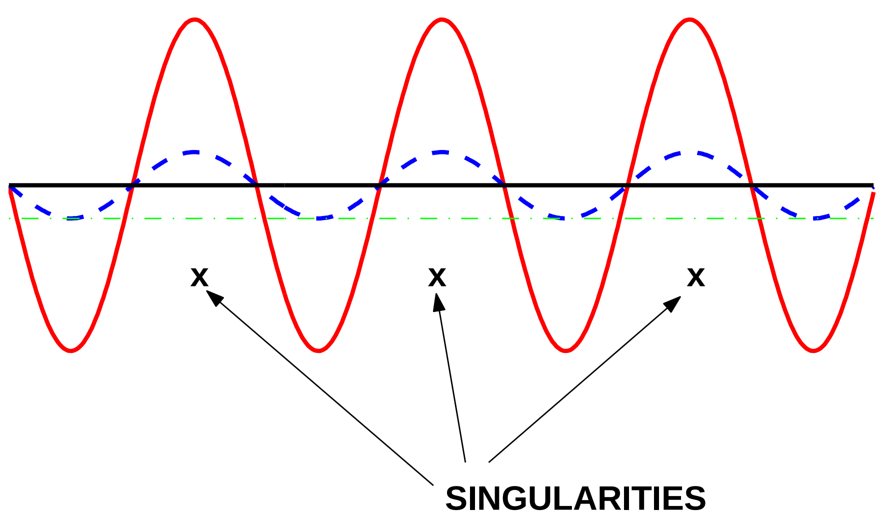
Permissible?: **"Rayleigh's hypothesis"**

In general, NO (**YES for sufficiently shallow gratings**)

e.g. $f(x) = \frac{h}{2} \cos\left(\frac{2\pi}{d} x\right)$

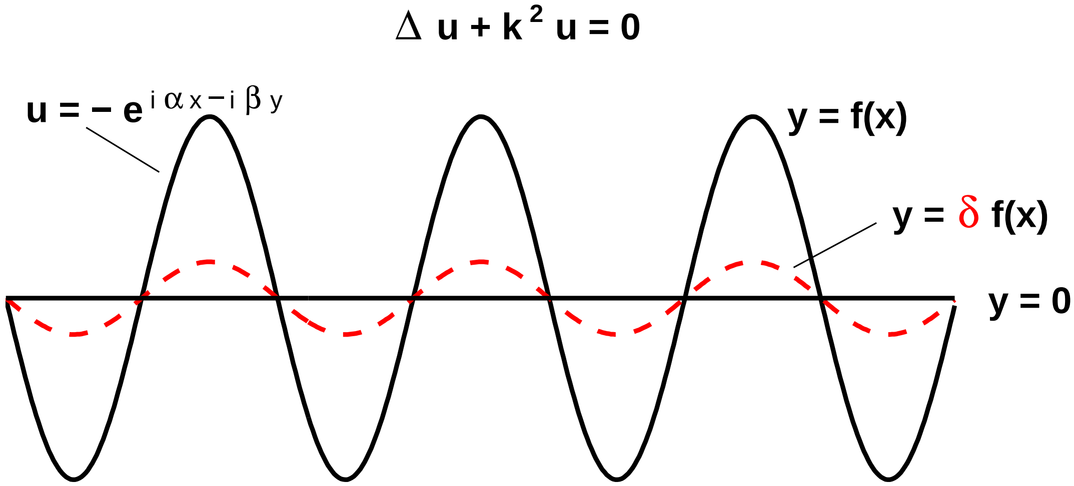
Rayleigh's hypothesis holds $\Longleftrightarrow h/d < 0.142$

$$\begin{bmatrix} \text{Petit and Cadilhac, 1966} & \Rightarrow \\ \text{Millar, 1971} & \Leftarrow \end{bmatrix}$$



• **VARIATION OF BOUNDARIES**:

• Singly-periodic gratings
• Perfect conductor, TE



$$u = u(x, y; \delta) \ : \ \begin{cases} \Delta u + k^2 u = 0 & \text{on } y > \delta f(x) \\ u(x, \delta f(x); \delta) = -e^{i\alpha x - i\beta\delta f(x)} \end{cases}$$

1. $\delta = 0$:

$$u(x, 0, \underbrace{0}_{\text{PLANAR INTERFACE}}) = -e^{i\alpha x} \ \underbrace{\Longrightarrow}_{\text{LAW OF REFLECTION}} \ u(x, y, 0) = e^{i\alpha x + i\beta y}$$

2. $\dfrac{d}{d\delta}$ at $\delta = 0$:

$$\begin{aligned} f(x) u_y(x, 0, 0) + u_\delta(x, 0, 0) &= i\beta f(x) e^{i\alpha x} \\ u_\delta(x, 0, 0) &= 2i\beta f(x) e^{i\alpha x} \\ \Longrightarrow \qquad u_\delta(x, y, \underbrace{0}_{\text{PLANAR INTERFACE}}) &= \cdots \end{aligned}$$

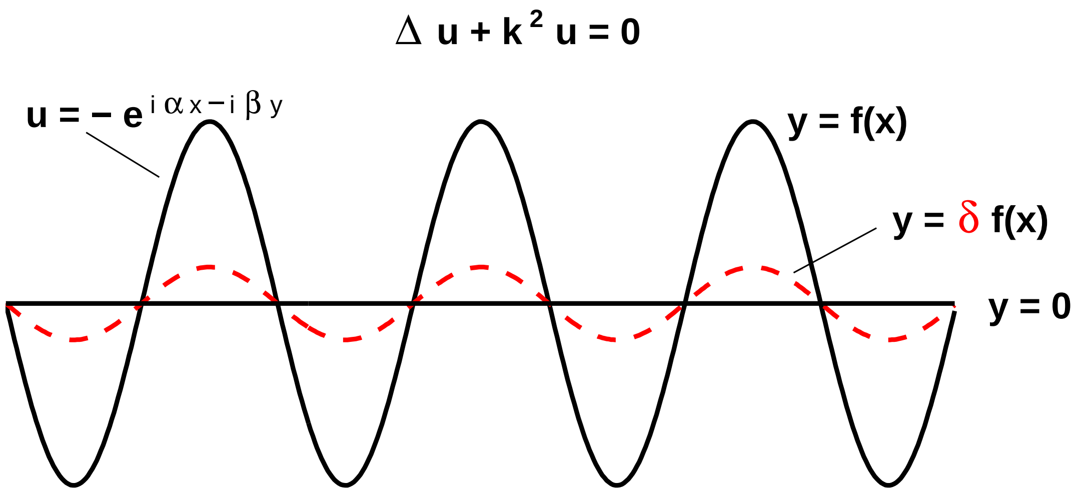
3. $\dfrac{d^2}{d\delta^2}$ at $\delta = 0$: ...

⋮

$\Longrightarrow$ RECURSIVE FORMULAS

• **RECURSIVE FORMULAS**:

• Singly-periodic gratings
• Perfect conductor, TE



Let

$$f(x) = \sum_{r=-F}^{F} C_{1,r} e^{iKrx}, \quad \frac{f(x)^l}{l!} = \sum_{r=-lF}^{lF} C_{l,r} e^{iKrx} \quad \left(K = \frac{2\pi}{d}\right)$$

and

$$u(x,y;\delta) = E_z^{scat}(x,y,\delta) = \sum_{n=0}^{\infty} u_n(x,y)\delta^n = \sum_{r=-\infty}^{\infty} B_r(\delta) e^{i\alpha_r x + i\beta_r y}$$

where

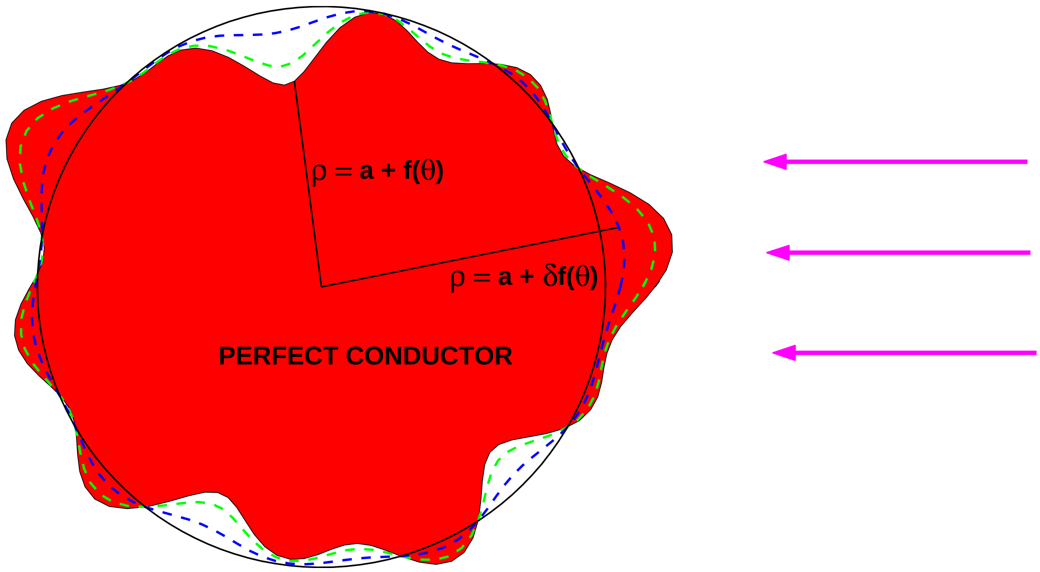
$$u_n(x,y) = \sum_{r=-\infty}^{\infty} d_{n,r} e^{i\alpha_r x + i\beta_r y} \quad \text{and} \quad B_r(\delta) = \sum_{n=0}^{\infty} d_{n,r}\delta^n$$

Then, the recursive formulas take the form

SIMPLE!

$$d_{n,r} = -(-i\beta)^n C_{n,r} - \sum_{l=0}^{n-1} \sum_q C_{n-l,r-q} (i\beta_q)^{n-l} d_{l,q}$$

- **RECURSIVE FORMULAS**:
- 2-d bounded obstacles
- Perfect conductor, TE



Let

$$f(\theta) = \sum_{r=-F}^{F} C_{1,r} e^{ir\theta}, \quad \frac{f(\theta)^l}{l!} = \sum_{r=-lF}^{lF} C_{l,r} e^{ir\theta}$$

and

$$u(\rho,\theta;\delta) = E_y^{scat}(\rho,\theta,\delta) = \sum_{n=0}^{\infty} u_n(\rho,\theta)\delta^n = \sum_{r=-\infty}^{\infty} B_r(\delta)(-i)^r H_r^{(1)}(k\rho)$$

where

$$u_n(\rho,\theta) = \sum_{r=-\infty}^{\infty} d_{n,r}(-i)^r H_r^{(1)}(k\rho) \quad \text{and} \quad B_r(\delta) = \sum_{n=0}^{\infty} d_{n,r}\delta^n$$

Then, the recursive formulas take the form

$$\begin{aligned} d_{n,r} &= -k^n \sum_q C_{n,r-q}(-i)^{q-r} \frac{d^n J_q}{dz^n}(ka)/H_r^{(1)}(ka) \\ &\quad - \sum_{l=0}^{n-1} k^{n-l} \sum_q d_{l,q} C_{n-l,r-q}(-i)^{q-r} \frac{d^{n-l} H_q^{(1)}}{dz^{n-l}}(ka)/H_r^{(1)}(ka) \end{aligned}$$

- **RECURSIVE FORMULAS**:

  - 3-d (biperiodic) dielectric/metallic gratings

$$\alpha_r d^{1,+}_{n,(r,s)} + \beta_s d^{2,+}_{n,(r,s)} + \gamma^+_{r,s} d^{3,+}_{n,(r,s)} = 0$$

$$\alpha_r d^{1,-}_{n,(r,s)} + \beta_s d^{2,-}_{n,(r,s)} + \gamma^-_{r,s} d^{3,-}_{n,(r,s)} = 0\,.$$

$$\begin{aligned} d^{2,+}_{n,(r,s)} - d^{2,-}_{n,(r,s)} &= -\left(A^2(-i\gamma)^n + A^3(-i\gamma)^{n-1}(iK_2 s)\right) C_{n,(r,s)} \\ - \textstyle\sum_{k=0}^{n-1} \sum_{l=\max(-kF,r-(n-k)F)}^{\min(kF,r+(n-k)F)} \sum_{m=\max(-kF,s-(n-k)F)}^{\min(kF,s+(n-k)F)} & \Big[(i\gamma^+_{l,m})^{n-k} d^{2,+}_{k,(l,m)} - (-i\gamma^-_{l,m})^{n-k} d^{2,-}_{k,(l,m)} \\ & + \Big((i\gamma^+_{l,m})^{n-1-k} d^{3,+}_{k,(l,m)} - (-i\gamma^-_{l,m})^{n-1-k} d^{3,-}_{k,(l,m)} \Big) (iK_2(s-m)) \Big] C_{n-k,(r-l,s-m)} \end{aligned}$$

$$\begin{aligned} & \gamma^+_{r,s} d^{1,+}_{n,(r,s)} + \gamma^-_{r,s} d^{1,-}_{n,(r,s)} - \alpha_r d^{3,+}_{n,(r,s)} + \alpha_r d^{3,-}_{n,(r,s)} \\ &= -\left[-(\gamma A^1 + \alpha A^3)(-i\gamma)^n + (\alpha A^2 - \beta A^1)(-i\gamma)^{n-1}(iK_2 s)\right] C_{n,(r,s)} \\ &- \textstyle\sum_{k=0}^{n-1} \sum_{l=\max(-kF,r-(n-k)F)}^{\min(kF,r+(n-k)F)} \sum_{m=\max(-kF,s-(n-k)F)}^{\min(kF,s+(n-k)F)} \Big[\gamma^+_{l,m} (i\gamma^+_{l,m})^{n-k} d^{1,+}_{k,(l,m)} \\ &+ \gamma^-_{l,m}(-i\gamma^-_{l,m})^{n-k} d^{1,-}_{k,(l,m)} - (i\gamma^+_{l,m})^{n-k} \alpha_r d^{3,+}_{k,(l,m)} + (-i\gamma^-_{l,m})^{n-k} \alpha_r d^{3,-}_{k,(l,m)} \\ &+ \Big((i\gamma^+_{l,m})^{n-k-1} \alpha_r d^{2,+}_{k,(l,m)} - (-i\gamma^-_{l,m})^{n-k-1} \alpha_r d^{2,-}_{k,(l,m)} - (i\gamma^+_{l,m})^{n-k-1} \beta_s d^{1,+}_{k,(l,m)} \\ &+ (-i\gamma^-_{l,m})^{n-k-1} \beta_s d^{1,-}_{k,(l,m)} \Big) (iK_2(s-m)) \Big] C_{n-k,(r-l,s-m)}\,. \end{aligned}$$

$$\begin{aligned} d^{1,+}_{n,(r,s)} - d^{1,-}_{n,(r,s)} &= -\left(A^1(-i\gamma)^n + A^3(-i\gamma)^{n-1}(iK_1 r)\right) C_{n,(r,s)} \\ - \textstyle\sum_{k=0}^{n-1} \sum_{l=\max(-kF,r-(n-k)F)}^{\min(kF,r+(n-k)F)} \sum_{m=\max(-kF,s-(n-k)F)}^{\min(kF,s+(n-k)F)} & \Big[(i\gamma^+_{l,m})^{n-k} d^{1,+}_{k,(l,m)} - (-i\gamma^-_{l,m})^{n-k} d^{1,-}_{k,(l,m)} \\ & + \Big((i\gamma^+_{l,m})^{n-1-k} d^{3,+}_{k,(l,m)} - (-i\gamma^-_{l,m})^{n-1-k} d^{3,-}_{k,(l,m)} \Big) (iK_1(r-l)) \Big] C_{n-k,(r-l,s-m)} \end{aligned}$$

$$\begin{aligned} & \beta_r d^{3,+}_{n,(r,s)} - \beta_r d^{3,-}_{n,(r,s)} - \gamma^+_{r,s} d^{2,+}_{n,(r,s)} - \gamma^-_{r,s} d^{2,-}_{n,(r,s)} \\ &= -\left[(\gamma A^2 + \beta A^3)(-i\gamma)^n + (\alpha A^2 - \beta A^1)(-i\gamma)^{n-1}(iK_1 r)\right] C_{n,(r,s)} \\ &- \textstyle\sum_{k=0}^{n-1} \sum_{l=\max(-kF,r-(n-k)F)}^{\min(kF,r+(n-k)F)} \sum_{m=\max(-kF,s-(n-k)F)}^{\min(kF,s+(n-k)F)} \Big[(i\gamma^+_{l,m})^{n-k} \beta_s d^{3,+}_{k,(l,m)} \\ &- (-i\gamma^-_{l,m})^{n-k} \beta_s d^{3,-}_{k,(l,m)} - \gamma^+_{l,m} (i\gamma^+_{l,m})^{n-k} d^{2,+}_{k,(l,m)} - \gamma^-_{l,m}(-i\gamma^-_{l,m})^{n-k} d^{2,-}_{k,(l,m)} \\ &+ \Big((i\gamma^+_{l,m})^{n-k-1} \alpha_r d^{2,+}_{k,(l,m)} - (-i\gamma^-_{l,m})^{n-k-1} \alpha_r d^{2,-}_{k,(l,m)} - (i\gamma^+_{l,m})^{n-k-1} \beta_s d^{1,+}_{k,(l,m)} \\ &+ (-i\gamma^-_{l,m})^{n-k-1} \beta_s d^{1,-}_{k,(l,m)} \Big) (iK_1(r-l)) \Big] C_{n-k,(r-l,s-m)}\,. \end{aligned}$$

BUT ...

Are the formal differentiations justified?

THEOREM (O. Bruno and FR)

Assume the boundary of the scatterer is analytic. Then,

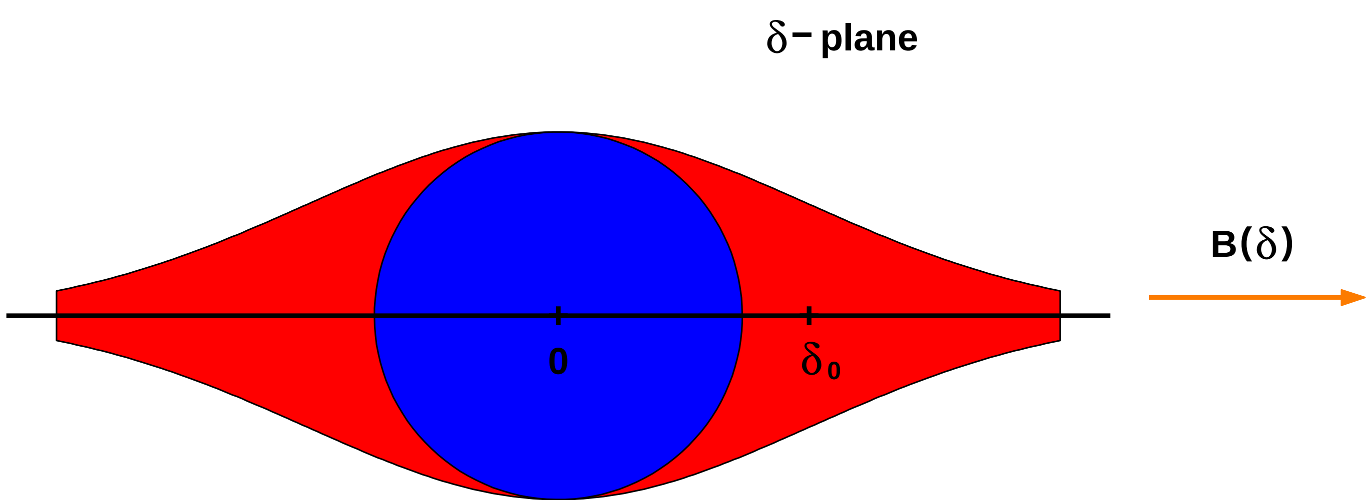
(1) *the fields are jointly analytic functions of the spatial variables and perturbation parameters;*

(2) *the fields can be uniformly continued beyond their domain of definition (i.e. beyond scatterering interfaces); and*

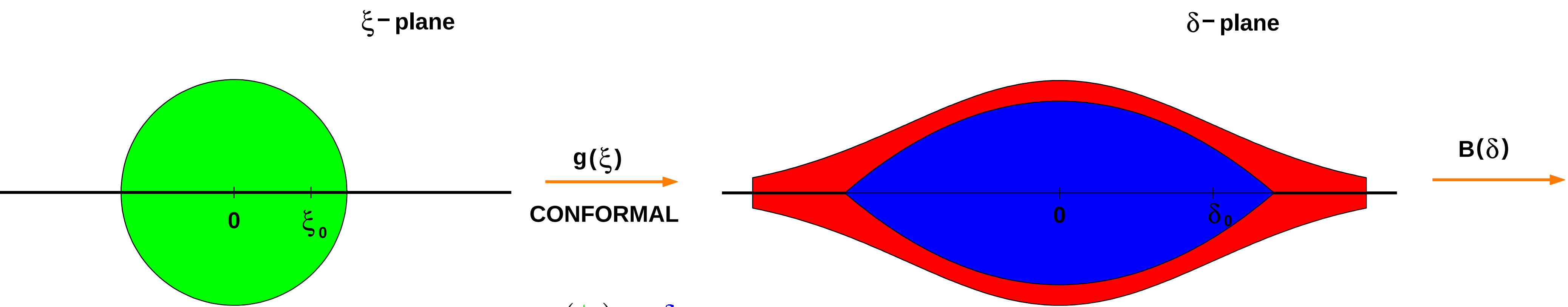
(3) *the domain of analyticity in the pertubation parameter extends to all values for which the deformed scatterer does not self-intersect.*

In the case of diffraction gratings, for instance,

(3) $\Longrightarrow$ the region of analyticity of the fields in δ contains a neighborhood of the whole real line!



• ENHANCED CONVERGENCE



$$g(\xi_0) = \delta_0$$
$$g(0) = 0$$

ξ_0 is inside the disk of convergence of $B(g(\xi))$ about $\xi = 0$ (!)