Pricing under stochastic volatility: complete solution via decoupled system of Monge–Ampère and Black–Scholes PDEs

Srdjan D. Stojanovic

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.

http://math.uc.edu/~srdjan/

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Abstract

We have found, at least from the practical point of view, the complete solution of the option pricing problem for underlying securities obeying stochastic volatility price dynamics. Although partial solutions have existed, and in spite of a considerable attention to it, this problem has been open for about 20 years. The pricing problem is reduced to solving an uncoupled system of a Monge–Ampère type PDE and a Black–Scholes type PDE.

Résumé

Tarifer sous l’hypothèse de volatilité stochastique : solution complète par l’utilisation d’un système découplé d’EDP de Monge-Ampère et de Black-Scholes. Nous trouvons au moins d’un point de vue pratique, la solution complète au problème de tarification des options sur les titres sous-jacents, dont les prix obéissent à une dynamique de volatilité stochastique. Bien que des solutions partielles existent et aient fait l’objet d’une attention certaine, ce problème est resté ouvert depuis 20 ans. Le problème de la tarification se réduit à résoudre un système découplé d’EDP de type Monge-Ampère et Black Scholes.

Version française abrégée

Considérons un « marché boursier à volatilité stochastique », consistant d’un titre unique au prix \( Y(t) \) à la date \( t \) et vérifiant le système suivant d’équations différentielles stochastiques de type Itô

\[
\begin{align*}
    dY(t) &= Y(t) (\alpha(t, Y(t), \nu(t)) - \mathcal{D}_1(t, Y(t), \nu(t))) dt + Y(t) p(t, Y(t), \nu(t)) dB_1(t) \\
    d\nu(t) &= q(t, Y(t), \nu(t)) dt + w(t, Y(t), \nu(t)) \left( \rho(t, Y(t), \nu(t)) dB_1(t) + \sqrt{1 - \rho(t, Y(t), \nu(t))^2} dB_2(t) \right)
\end{align*}
\]

(0.1)

où \( B_1 \) et \( B_2 \) représentent des mouvements browniens indépendants, \( \alpha \) est le taux d’appréciation du titre, \( \mathcal{D}_1 \) est le dividende, \( p \geq 0 \) est la volatilité, et \( \nu \) est un facteur scalaire arbitraire (par exemple, la volatilité : \( p(t, Y, \nu) = \nu \)) à direction \( q \), à diffusion \( w \geq 0 \) et où \( -1 \leq \rho \leq 1 \) est le coefficient de corrélation entre prix et facteur.

Soit également un marché à options constitué d’une option unique sur le titre ci-dessus, de gain fixe \( \nu(Y) \) à la date de maturité \( T \), et au prix \( V(t, Y, \nu(t)) \) à la date \( t < T \), et où \( V(t, Y, \nu) \) est une fonction inconnue a priori.

Soit \( X > 0 \), la richesse. L’utilité associée à la richesse est mesurée par une fonction d’utilité de type HARA : \( \psi_Y(X) = X^{1-\gamma} / (1-\gamma) \) pour \( \gamma \in (0, \infty) \), \( \gamma \neq 1 \), et \( \psi_1(X) = \log(X) \).

Considérons une stratégie de portefeuille par autofinancement \( \Pi(t, X, Y, \nu) = [\Pi_1, \Pi_2] \), où \( \Pi_1 \) est la valeur monétaire de l’investissement dans l’action et \( \Pi_2 \) est la valeur de l’investissement dans l’option. Etant donné (0.1), et pour une stratégie fixe \( \Pi \), il existe une EDS qui caractérise l’évolution de la richesse \( X(t) = X^{\Pi}(t) \). La stratégie de portefeuille \( \gamma \)-optimale (le problème de Merton), notée \( \Pi^*_\gamma = [\Pi^*_\gamma, \Pi^*_\gamma^2] \) est telle que :
\[ \sup \Pi E_{t,X,Y,*} \psi_\gamma(X^{\Pi}(T)) = E_{t,X,Y,*} \psi_\gamma(X^{\Pi^*}(T)). \] (0.2)

**Définition 1.** Pour tout \( \gamma \in (0, \infty) \), la fonction du prix de l’option \( V_\gamma(t, Y, \nu) \) est définie comme la solution d’une équation Black-Scholes (abrégée) \( \Pi_\gamma^* = 0 \).

**Théorème 1.** Pour tout \( \gamma \in (0, \infty), \gamma \neq 1 \), le prix de l’option correspondant \( V_\gamma(t, Y, \nu) \) est (formellement) solution de l’EDP Black-Scholes généralisée

\[ V_\gamma^{(1,0,0)} + \frac{1}{2} Y^2 V_\gamma^{(0,2,0)} \rho^2 + Y w \rho V_\gamma^{(0,1,1)} + \frac{1}{2} w^2 V_\gamma^{(0,0,2)} + Y (r - \mathbb{D}_1) V_\gamma^{(0,1,0)} - r V_\gamma + \left( q - \frac{(a - r) w \rho}{p} + w^2 (1 - \rho^2) \frac{f_\gamma^{(0,0,1)}}{f_\gamma} \right) V_\gamma^{(0,0,1)} = 0 \] (0.3)

de condition terminale \( V_\gamma(T, Y, \nu) = \nu(Y) \), et où \( f_\gamma \) est une solution appropriée de l’EDP de type Monge-Ampère

\[ -\left( \frac{(a - r)^2}{2} + \gamma \rho \right) f_\gamma^2 + \left( q \gamma - 1 \right) \frac{w(a - r) \rho}{p} f_\gamma^{(0,0,1)} f_\gamma + \frac{w^2 Y}{2(\gamma - 1)} f_\gamma^{(0,0,2)} f_\gamma + \frac{Y (a - r + \gamma (r - \mathbb{D}_1)))}{\gamma - 1} f_\gamma^{(1,0,0)} f_\gamma + \frac{p w Y \gamma \rho}{\gamma - 1} f_\gamma^{(0,1,1)} f_\gamma + \frac{p^2 Y^2 \gamma}{2(\gamma - 1)} f_\gamma^{(2,0,0)} f_\gamma + \frac{\gamma}{\gamma - 1} f_\gamma^{(1,0,0)} f_\gamma + \frac{1}{2} w^2 \rho^2 (f_\gamma^{(0,0,1)})^2 - \frac{1}{2} w^2 Y^2 (f_\gamma^{(0,1,0)\gamma})^2 -pw Y \rho f_\gamma^{(0,0,1)} f_\gamma^{(0,1,0)} = 0 \] (0.4)
de condition terminale \( f_\gamma(T, Y, \nu) = 1 \).

Pour \( \gamma = 1 \), le prix de l’option correspondant \( V_1 = V_1(t, Y, \nu) \) est solution de l’EDP Black-Scholes généralisée (0.3) où \( f_1 = 1 \), soit pour \( f_1^{(0,0,1)} / f_1 = 0 \) ((0.4) ne sont pas nécessaires dans ce cas).

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1 Statement of problem and it's solution

Consider a "stochastic volatility stock market," consisting of a single security with price \( Y(t) \) at time \( t \), obeying Itô SDE system

\[ dY(t) = Y(t) (a(t, Y(t), \nu(t)) - \mathbb{D}_1(t, Y(t), \nu(t))) dt + Y(t) \rho(t, Y(t), \nu(t)) dB_1(t) \]
\[ d\nu(t) = q(t, Y(t), \nu(t)) dt + w(t, Y(t), \nu(t)) (\rho(t, Y(t), \nu(t)) dB_1(t) + \sqrt{1 - \rho(t, Y(t), \nu(t))^2} dB_2(t)) \] (1.1)

where \( B_1 \) and \( B_2 \) are independent Brownian motions, and where \( a \) is the appreciation rate, \( \mathbb{D}_1 \) is the dividend rate, \( \rho \) is the volatility, and \( \nu \) is an arbitrary scalar factor (for example, volatility: \( \rho(t, Y, \nu) = \nu \)), with a drift \( q \), diffusion \( w \), and where \( \rho \) is the price/factor correlation. Of course, it is not essential that \( \nu \) is scalar.

Consider also an associated option price system, consisting of a single option on the above underlying, with a fixed payoff \( \nu(Y) \) (for example, \( \nu_{call}(Y) = \text{Max}(0, Y - k) \)) at some strike price \( k \) at a fixed expiration time \( T \), with a price \( V(t, Y, \nu(t)) \) at time \( t < T \), where function \( V(t, Y, \nu(t)) \) is a priori unknown. Assuming (1.1), it is not difficult to write down an Itô SDE characterizing the evolution of \( V(t, Y, \nu(t)) \).

The problem is to characterize, and be able to compute, "a fair option price function" \( V(t, Y, \nu) \).

Let \( X > 0 \) denotes the wealth. We measure utility of wealth via HARA class of utility functions \( \psi_\gamma(X) = X^{1-\gamma} / (1 - \gamma) \), for \( \gamma \in (0, \infty), \gamma \neq 1 \), and \( \psi_1(X) = \log(X) \). Parameter \( \gamma \) is called risk-aversion.

Consider a self-financing portfolio hedging strategy \( \Pi(t, X, Y, \nu) = [\Pi_1(t, X, Y, \nu), \Pi_2(t, X, Y, \nu)] \), where \( \Pi_1 \) is the cash value of the investment into the underlying stock, and \( \Pi_2 \) is the cash value of the investment into the option. Assuming (1.1), and for a fixed strategy \( \Pi \), it is not difficult to write down an Itô SDE characterizing corresponding evolution of the wealth \( X(t) = X^{\Pi}(t) \) (see (2.3) below). The \( \gamma \)-optimal portfolio strategy (Merton’s problem) is a strategy \( \Pi_\gamma^* = [\Pi_\gamma^{*1}, \Pi_\gamma^{*2}] \), such that

\[ \sup \Pi E_{t,X,Y,*} \psi_\gamma(X^{\Pi}(T)) = E_{t,X,Y,*} \psi_\gamma(X^{\Pi^*}(T)). \] (1.2)
DEFINITION 1. For every $\gamma \in (0, \infty)$, the fair option price function $V_{\gamma}(t, Y, \psi)$ is defined as a solution of the (abstract) Black–Scholes equation $\Pi_{\psi}^{2} = 0$.

Intuitively, "a fair option price" is such a price for which it is not rational to speculate by investing, long or short, into options. Variants of this definition have been present in the literature already (see, e.g., [1] and references given there). We shall use alternative notations for partial derivatives: for example, $\partial^{2}V(t, Y, \psi)/\partial \psi^{2} = V^{(0,0,2)}(t, Y, \psi)$.

THEOREM 1. For every $\gamma \in (0, \infty)$, $\gamma \neq 1$, the corresponding fair option price $V_{\gamma} = V_{\gamma}(t, Y, \psi)$ is characterized (formally) as a solution of a generalized Black–Scholes PDE

$$V_{\gamma}^{(1,0,0)} + \frac{1}{2} Y^{2} V_{\gamma}^{(0,2,0)} p^{2} + Y w_{\gamma} V_{\gamma}^{(0,1,1)} p + \frac{1}{2} w^{2} V_{\gamma}^{(0,0,2)} + Y (r - D_{1}) V_{\gamma}^{(0,1,0)} - r V_{\gamma}$$

$$+ \left( q - \frac{(a - r) w_{\gamma}}{p} + w^{2} (1 - \rho^{2}) \frac{f_{\gamma}^{(0,0,1)}}{f_{\gamma}} \right) V_{\gamma}^{(0,0,1)} = 0$$

with the terminal condition

$$V_{\gamma}(T, Y, \psi) = \nu(Y)$$

and where $f_{\gamma} = f_{\gamma}(t, Y, \psi)$ is an appropriate solution of a Monge–Ampère type PDE

$$-\left( \frac{(a - r)^{2}}{2} r^{2} + r \gamma \right) f_{\gamma}^{2} + \left( \frac{q \gamma}{\gamma - 1} - \frac{w (a - r) \rho}{p} \right) f_{\gamma}^{(0,0,1)} f_{\gamma} + \frac{w^{2} \gamma}{2 (\gamma - 1)} f_{\gamma}^{(0,0,2)} f_{\gamma}$$

$$+ \frac{Y (a - r + \gamma (r - D_{1}))}{\gamma - 1} f_{\gamma}^{(0,1,0)} f_{\gamma} + \frac{p w Y \gamma \rho}{f_{\gamma}^{(0,1,1)} f_{\gamma} + \frac{p^{2} Y^{2}}{2 (\gamma - 1)} f_{\gamma}^{(0,2,0)} f_{\gamma}$$

$$+ \frac{\gamma}{\gamma - 1} f_{\gamma}^{(1,0,0)} f_{\gamma}^{2} = \frac{1}{2} w^{2} \rho^{2} (f_{\gamma}^{(0,0,1)})^{2} - \frac{1}{2} p^{2} Y^{2} (f_{\gamma}^{(0,1,0)})^{2} - p w Y \rho f_{\gamma}^{(0,0,1)} f_{\gamma}^{(0,1,0)} = 0$$

with the terminal condition

$$f_{\gamma}(T, Y, \psi) = 1.$$ For $\gamma = 1$, the corresponding fair option price $V_{1} = V_{1}(t, Y, \psi)$ is characterized as a solution of the generalized Black–Scholes PDE (1.3) with $f_{1} = 1$, i.e., with $f_{1}^{(0,0,1)} / f_{1} = 0 ((1.5)-(1.6) is not needed in that case).

REMARK 1. Comparing (1.3) with the literature (see, e.g., equation (15) in [6]) we can see that Theorem 1 implies that, for any $\gamma \in (0, \infty)$, the so-called "market price of volatility risk," which we denote $M_{\gamma}$, is given by

$$M_{\gamma} = w \sqrt{1 - \rho^{2}} \frac{f_{\gamma}^{(0,0,1)}}{f_{\gamma}}$$

where $f_{\gamma}$ is an appropriate solution of (1.5)–(1.6). The exact expression for the market price of volatility risk was not known so far. In particular, $M_{1} = 0$. Analysing (1.3), we can see that fair option price is unique, i.e., it is same for all $\gamma \in (0, \infty)$, iff $\partial M_{\gamma} / \partial \gamma = 0$, and therefore iff $M_{\gamma} = 0$, and assuming furthermore $w^{2} (1 - \rho^{2}) > 0$, iff $f_{\gamma}^{(0,0,1)} = 0$, for all $\gamma \in (0, \infty)$. It was claimed recently in the literature (see [3, 4]) that for uniqueness of fair option prices it suffices that the Sharpe ratio $(a - r)/p$ is constant with respect to $\gamma$. This does not seem to be correct, since although the Sharpe ratio may have a dominant effect on the solution of (1.5), and therefore indeed the "uniqueness" of prices under the above condition may hold approximately, $-(a - r) \rho (2 p^{2}) f_{\gamma}^{2}$ is not the only term in (1.5) that may cause $f_{\gamma}^{(0,0,1)} = 0$. Indeed, suppose that $f_{\gamma}^{(0,0,1)} = 0$; terms $p^{2} Y^{2} \gamma f_{\gamma}^{(0,2,0)} f_{\gamma} / (2 (\gamma - 1))$ and $- p^{2} Y^{2} (f_{\gamma}^{(0,1,0)})^{2}/2$ remain in (1.5), and unless, which is trivial, the volatility $p = p(t, Y, \psi)$ does not depend on the factor $\psi$, we cannot expect that $f_{\gamma}^{(0,1,0)} = 0$. Numerical computations confirm my findings.

REMARK 2. Multiplying equation (1.5) by $(\gamma - 1)$, sending $\gamma \to 1$, and dividing by $f_{1}$, one arrives (at least formally) at

$$\frac{1}{2} f_{1}^{(0,0,2)} w^{2} + p Y \rho f_{1}^{(0,1,1)} w + q f_{1}^{(0,0,1)} + Y w f_{1}^{(0,1,0)} - Y D_{1} f_{1}^{(0,1,0)} + \frac{1}{2} p^{2} Y^{2} f_{1}^{(0,2,0)} + f_{1}^{(1,0,0)} = 0$$

which (is a linear equation, and) together with (1.6), is solved, obviously, by $f_{1} = 1$. This argument is not necessary to derive the above result in the case $\gamma = 1$—it follows from the portfolio theory below.

REMARK 3. System (1.3)–(1.6) is uncoupled, which is very useful. Furthermore, $f_{\gamma}$ does not depend on a particular option payoff $\nu$. 

REM runk 4. If \( w = 0 \), or if \( \rho = \pm 1 \), then \( \mathbb{M}_\gamma = 0 \), and the same fair option price \( V = V_\gamma = V_\gamma(t, Y, \gamma) \) holds for all \( \gamma \in (0, \infty) \). For example, if \( w = 0 \), the price is characterized as a solution of a (hypoelliptic (see [7], and also [15, 16])) Black–Scholes PDE

\[
V_\gamma^{(1,0,0)} + \frac{1}{2} Y^2 V_\gamma^{(0,2,0)} + Y (r - \mathbb{D}) V_\gamma^{(0,1,0)} - r V_\gamma + q V_\gamma^{(0,0,1)} = 0
\]

(1.9)

with the terminal condition (1.4). If also \( q = 0 \), then (1.9) simplifies further to the (usual) Black–Scholes PDE.

2 Preliminary results in general portfolio theory

Consider, quite generally, a set of \( m \) factors \( A(t) = \{ A_1(t), \ldots, A_m(t) \} \) obeying the Itô SDE dynamics

\[
dA(t) = b(t, A(t)) \, dt + c(t, A(t)) \, dB(t)
\]

(2.1)

where \( B(t) = \{ B_1(t), \ldots, B_n(t) \} \) is the \( n \)-dimensional Brownian motion (so, \( b(t, A) \), is an \( m \)-vector valued function, and \( c(t, A) \) is an \( m \times n \)-matrix valued function). In addition to the factors, consider a set of \( k \) tradable assets with prices \( S(t) = \{ S_1(t), \ldots, S_k(t) \} \), obeying (non-linear in \( A \)) Itô SDE dynamics

\[
dS(t) = S(t) (a(t, A(t)) - \mathbb{D}(t, A(t))) \, dt + S(t) \sigma(t, A(t)) \, dB(t)
\]

(2.2)

where the vector-valued function \( a(t, A) \) is the \( k \)-vector of appreciation rates, \( \mathbb{D}(t, A) \) is the \( k \)-vector of dividend rates of the corresponding assets, \( \sigma(t, A) \) is the (volatility) \( k \times n \)-matrix valued function. If factor is tradable, then it can be represented also as one of the equations in system (2.2), and vice-versa. We assume \( \sigma, \sigma^T > 0 \). It can be shown that

\[
dX(t) = (\Pi(t, X(t), A(t))) \sigma(t, A(t)) \, dt + \Pi(t, X(t), A(t)) \, dB(t)
\]

(2.3)

and (2.1) and (2.3) form a closed SDE system, to be controlled. In particular, \( \mathbb{D} \) is eliminated, while components of \( \mathbb{D} \) may still be hidden in (2.1) and (2.3), and more specifically in \( a, b \) (see (3.1)). We shall refer to functions \( a, \sigma, \sigma^T > 0 \) as the market coefficients.

The objective of the investor is to, for a given utility function \( \psi \) (not necessarily HARA), maximize the expected value of the utility of the final total wealth, i.e., to find an optimal hedging strategy \( \Pi^*(t, X, A) \) such that:

\[
\varphi(t, X, A) = \sup_{\Pi} E_{t, X, A} \psi(X(t)) = E_{t, X, A} \psi(X^{\Pi^*}(T)).
\]

(2.4)

The standard formalism for solving the stochastic control problem (2.4) is to attempt to associate the Hamilton–Jacobi–Bellman (HJB) PDE characterizing the value function \( \varphi = \varphi(t, X, A) \):

\[
\begin{align*}
\max \left[ & \frac{\partial \varphi}{\partial t} + \Pi(a(t), X(t), \sigma(t), \sigma^T(t), \Pi) \left( \frac{\partial \varphi}{\partial X} \right)^T + b(t, X(t), \Pi) \right], \\
& + \frac{1}{2} \left( \frac{\partial \varphi}{\partial X} \right)^T \Pi \left( \frac{\partial \varphi}{\partial X} \right) \right] = 0
\end{align*}
\]

(2.5)

with the terminal condition \( \varphi(T, X, A) = \psi(X) \). After some simplifications, one can see that solving the HJB PDE (2.5) is equivalent to finding an appropriate solution of the Monge–Ampère type PDE:

\[
\begin{align*}
\frac{\partial^2 \varphi}{\partial X^2} & \left( \frac{\partial \varphi}{\partial X} \right)^2 \left( a(t, X(t), \sigma(t), \sigma^T(t), \Pi) - r X \right) \frac{\partial \varphi}{\partial X} - \frac{1}{2} \left( \frac{\partial \varphi}{\partial X} \right)^T \sigma(t, X(t), \sigma^T(t), \Pi) \frac{\partial \varphi}{\partial X} \\
& + \frac{1}{2} \left( \frac{\partial \varphi}{\partial X} \right)^T \sigma(t, X(t), \sigma^T(t), \Pi) \frac{\partial \varphi}{\partial X} = 0
\end{align*}
\]

(2.6)

with the same terminal condition, while the optimal hedging strategy is given by

\[
\Pi^*(t, X, A) = - 1 \left( \frac{\partial \varphi}{\partial X} \right)^T (a(t, X(t), \sigma(t), \sigma^T(t), \Pi) - r X) \frac{\partial \varphi}{\partial X} - \frac{1}{2} \left( \frac{\partial \varphi}{\partial X} \right)^T \sigma(t, X(t), \sigma^T(t), \Pi) \frac{\partial \varphi}{\partial X}.
\]

(2.7)
As in Merton's case (see also [14, 15]), in the case of HARA utility \( \psi_\gamma \), we seek the solution of (2.6), in the form 
\[
\varphi(t, X, A) = X^{1-\gamma} f_\gamma(t, A) \big( 1 - \gamma \big), \quad \text{if } \gamma \in (0, \infty), \gamma \neq 1, \text{ and in the form } \varphi(t, X, A) = f_1(t, A) + \log(X) \text{ if } \gamma = 1. \]
In the first case \( f = f_\gamma \) solves
\[
- \frac{\gamma}{1-\gamma} f \frac{\partial f}{\partial t} - \frac{1}{2} f^2 \big( (a_s - r) \sigma_s \sigma_s^T \big)^{-1} (a_s - r) + \frac{1}{2} \gamma b \big( 1 - \gamma \big)^{1/2} \nabla_A f \\
- f \left( (a_s - r) \sigma_s \sigma_s^T \big)^{-1} \sigma_s \sigma_s^T \nabla_A f \right) - \frac{1}{2} \sigma_s \sigma_s^T \sigma_s \sigma_s^T \nabla_A f \left( \frac{\gamma}{1-\gamma} \right)^{1/2} \nabla_A f \left( \frac{\gamma}{1-\gamma} \right)^{1/2} \nabla_A f = 0
\]
(2.8)

together with the terminal condition \( f(T, A) = 1 \), while the optimal hedging strategy is now given by
\[
\Pi_\gamma^*(t, X, A) = X \left( a_s - r \right) \left( \sigma_s \sigma_s^T \right)^{-1} = X P_\gamma^*(t, A).
\]

(2.9)

In the case \( \gamma = 1 \), \( f_1 \) solves instead a simple linear PDE, which is even irrelevant, since the optimal hedging strategy \( \Pi_1^* \) does not depend on \( f_1 \):
\[
\Pi_1^*(t, X, A) = X (a_s - r) \left( \sigma_s \sigma_s^T \right)^{-1} = X P_1^*(t, A).
\]

(2.10)

We refer to equation (2.8) as the reduced Monge–Ampère PDE of optimal portfolio hedging.

Remark 5. Of course, (2.6), (2.7), (2.8), (2.9), and (2.10) have much broader significance then being a tool for proving Theorem 1.

3 A sketch of the proof of Theorem 1

We shall consider only the case \( \gamma \in (0, \infty), \gamma \neq 1 \). We apply the general portfolio theory of Section 2 in the case of a "stochastic volatility market" of Section 1, and in particular (2.8) and (2.9). To that end we identify factors \( A(t) = \{ Y(t), \psi(t) \} \), tradable assets \( S(t) = \{ Y(t), V(t), Y(t), \psi(t) \} \), and the market coefficients
\[
\begin{align*}
\alpha_s &= \left\{ a, \frac{1}{V} \left[ \frac{1}{2} Y^2 V^{(0,2)} p^2 + Y w p V^{(0,1)} + q Y^2 V^{(0,2)} + Y \left( a - D_1 \right) V^{(0,1)} + Y V^{(0,1)} \right] \right\} \\
\sigma_s &= \left\{ \frac{p}{w p V^{(0,1)}} \left[ 0 \right], \frac{0}{w \sqrt{1-\rho^2} V^{(0,1)}} \right\}, \quad b = \left\{ Y \left( a - D_1 \right), q, c \right\} = \left\{ Y p \left[ 0 \right], w p \left[ 0 \right] \right\}. \quad \gamma \end{align*}
\]

(3.1)

So, we extract the "abstract Black–Scholes" equation \( \Pi_\gamma^* = 0 \). Indeed, we compute, using (2.9),
\[
P_\gamma^* = \left\{ \left[ V^{(1,0)} \right], \left[ \frac{1}{2} Y^2 V^{(0,2)} p^2 + Y w p V^{(0,1)} + \left[ \frac{1}{2} w^2 V^{(0,2)} \right] \right] \right\}
\]
\[
+ Y \left( r - D_1 \right) V^{(1,0)} - r V + \left\{ q + \left( \frac{r-a}{w p} \right) Y^2 \left( 1 - \rho^2 \right) f_{\gamma y}^{(0,1)} f_{\gamma y} \left( \frac{V}{\gamma} \right) \right\} \left( \frac{V}{\gamma} \right)^{1/2} \left( \frac{\gamma - 1}{\gamma} \right) \frac{V}{w^2 \left( 1 - \rho^2 \right)^{1/2} V^{(0,1)/2}}
\]

(3.2)

and therefore (1.3) follows. Equation (1.3) is not closed, since \( f_\gamma = f \) is not known and in fact may depend on \( V \). Equation (1.3) is therefore coupled with the equation (2.8), which now reads as
\[
\begin{align*}
\alpha_1 & \left( f \right) f^{(0,1)} f + \frac{\gamma Y}{2} f^{(0,2)} f + \frac{\gamma Y}{\gamma-1} \left( r-a \right) Y f^{(0,1,0)} f \\
+ \left( \frac{Y p w p}{\gamma} \right) f^{(0,1)} f + \frac{\gamma Y^2}{2 r - 2} f^{(0,2,0)} f + \frac{Y}{\gamma - 1} f^{(1,0,0)} f - \frac{1}{2} w^2 f^{(0,0,1)} f = 0
\end{align*}
\]

(3.3)

where the remaining coefficients \( a_1(V) \) and \( a_2(V) \) depend on \( V \), and are given by
\[ a_1(V) = \frac{1}{2} \left( -2 r \gamma + ((Y^2 V^{(0,2,0)})^2 + 2Yw\rho V^{(0,1,1)} + 2YqV^{(0,0,1)} + w^2 V^{(0,0,2)} 
+ 2Ya V^{(0,1,0)} + 2YD_1V^{(0,1,0)} + 2V^{(1,0,0)})p^2 
+ 2w(r - a)\rho V^{(0,0,1)})/ \left( (a - r) \left( r - a \right) \left( w^2 V^{(0,0,1)} - 2YD_1V^{(0,1,0)} + 2V^{(1,0,0)} \right) \right) \right) \]

and

\[ a_2(V) = \frac{1}{2} \left( 2 \left( (Y^2 V^{(0,2,0)} - (Y - 1)) Y V^{(0,0,1)} - (Y - 1) Y^2 V^{(0,2,0)} \left( (Y - 1) \left( Y^2 V^{(0,2,0)} \right) - \right) \right) \]

making (1.3) and (3.3) a coupled system. Moreover, the coupling (3.4) and (3.5) appears quite challenging (high derivatives of \( V \) are involved, and \( V^{(0,1,1)} \) is in the denominator). Now, using (1.3), we express

\[ V^{(1,0,0)} + \frac{1}{2} Y^2 V^{(0,2,0)} \left( Y V^{(0,1,1)} - 2Y (r - D_1) V^{(0,1,0)} - r V \right) \]

and using (3.6) in (3.4) and (3.5), we get (uncoupled)

\[ a_1(V) = a_1 = \frac{1}{2} \left( - (a - r)^2 / p^2 + w^2 (p^2 - 1) f^{(0,0,1)} / f^2 - 2 r y \right) \]

and

\[ a_2(V) = a_2 = (f (Y p q + (Y - 1) w (r - a) \rho) - (Y - 1) p w^2 (p^2 - 1) f^{(0,0,1)}) / ((Y - 1) f p) \].

Plugging (3.7) and (3.8) into (3.3), and simplifying, we get (1.5).

4 Computational Example

Let \( a = (3 \sqrt{\gamma} + 0.1) \sqrt{\gamma} \), \( D_1 = 0 \), \( p = \sqrt{\gamma} \), \( q = 16(0.12 - \gamma) \), \( \rho = 1/2 \), \( w = \sqrt{\gamma} \), \( r = .025 \), \( T = .5 \). One of the (Monte-Carlo generated, see, e.g., [15]) price-Y/volatility-\( \sqrt{\gamma} \) trajectories looks like:

For \( \gamma_1 = 1/10 \), and some time \( t < T \), \( f_y \) and corresponding \( f_y^{(0,0,1)}/f_y \) look like:
while for $\gamma_2 = 100$, and same time $t < T$, $f_\gamma$ and corresponding $f_\gamma^{(0,0,1)} / f_\gamma$ look like

The spread $V(t, Y, \nu) - V_{100}(t, Y, \nu)$ between two corresponding price-functions, for a call-option with strike price $k = 60$ (and expiration $T$), as well as the computed option prices $V_{1/10}(t, Y, \nu)$ and $V_{1/100}(t, Y, \nu)$, look like:

REMARK 6. Once the pricing problem is solved, i.e., $V$ is computed, the hedging problem can be settled quickly either via (constrained) portfolio rule $\Pi^* = \{ -w \rho V^{(0,0,1)} / p - Y V^{(0,1,0)}, V \}$, which is an appropriate analogue of the Black–Scholes hedging, or via (constrained) portfolio rule $\Pi^*_T = \{ (a - r) / p^2 - w \rho V^{(0,0,1)} / p - Y V^{(0,1,0)}, V \}$, which is the "log-utility" (constrained) portfolio rule. A full scale of analogous (constrained) portfolio rules corresponding to $\gamma \in (0, \infty), \gamma \neq 1$, are possible as well, but they are beyond the scope of this note, and will be discussed elsewhere.

5 References


Stoianovic S., Optimal momentum hedging via hypoelliptic reduced Monge–Ampère PDEs, to appear in SIAM Journal on Control & Optimization.