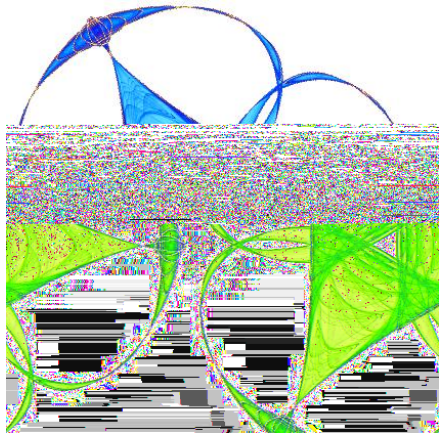
Approximation by quadrilateral finite elements

Douglas N. Arnold



Institute for Mathematics and its Applications

D. Arnold, D. Boffi, R. Falk, L. Gastaldi, *Finite element approximation on quadrilateral meshes*, Comm. in Num. Meth. Engrg. 17 (2001).

D. Arnold, D. Boffi, R. Falk, *Approximation by quadrilateral finite elements*, Math. Comp., to appear 2002.

http://ima.umn.edu/~arnold

For many PDE problems, the solution is characterized as a stationary point of and appropriate energy functional $\mathcal{L}$ over an appropriate function space S.

Poisson's equation:

$$\text{minimize} \quad \mathcal{L}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega f u\,dx \quad \text{over } \mathring{H}^1(\Omega)$$

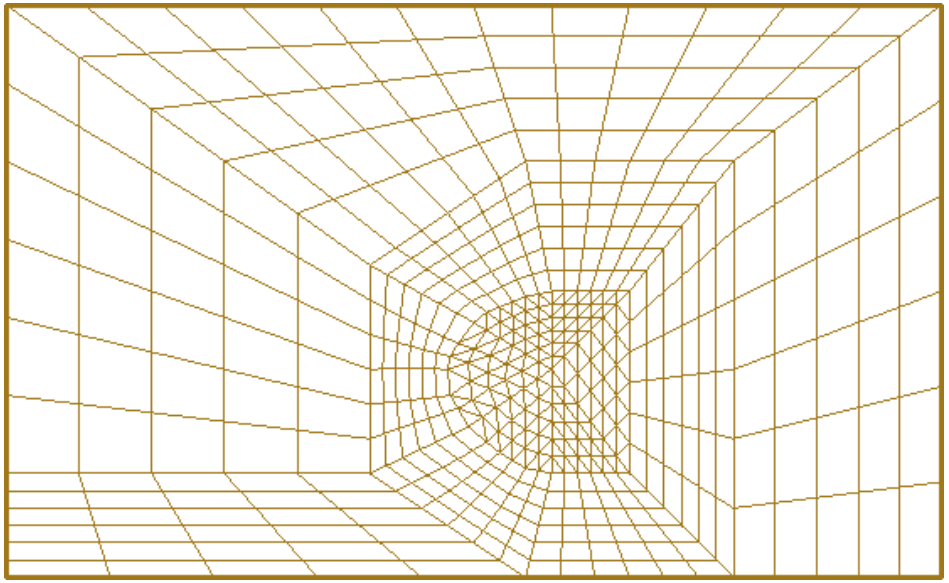
Stokes equations: $(\boldsymbol{u}, p)$ is a saddle-point of:

$$\mathcal{L}(\boldsymbol{u}, p) = \frac{1}{2}\int_\Omega |\nabla \boldsymbol{u}|^2\,dx + \int_\Omega p \operatorname{div} \boldsymbol{u}\,dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,dx$$

over $\mathring{H}^1(\Omega, \mathbb{R}^n) \times L^2(\Omega)$.

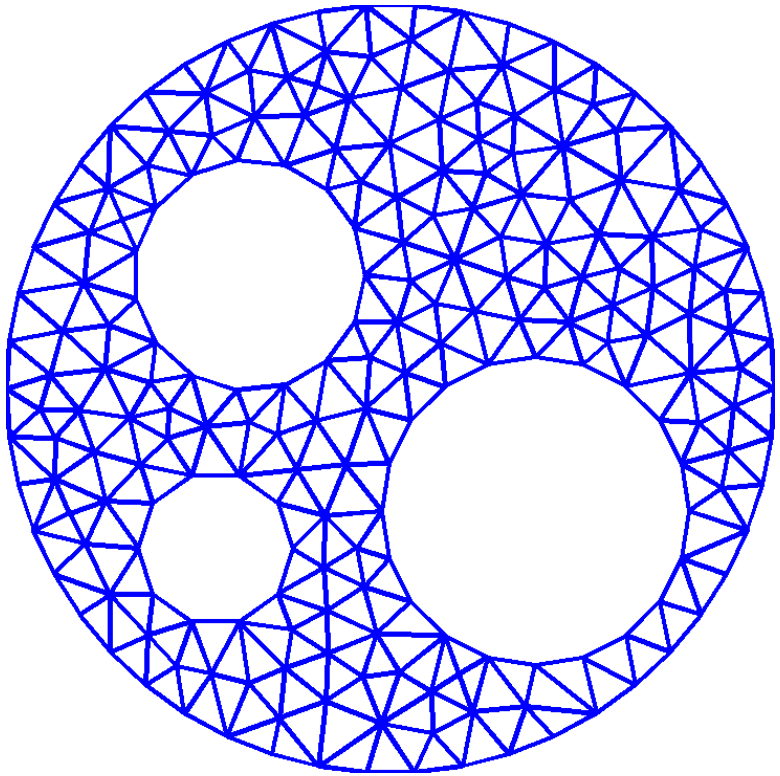
Given a subspace S_h of S we obtain an approximation to the solution u by seeking a critical point u_h of $\mathcal{L}$ over S_h.

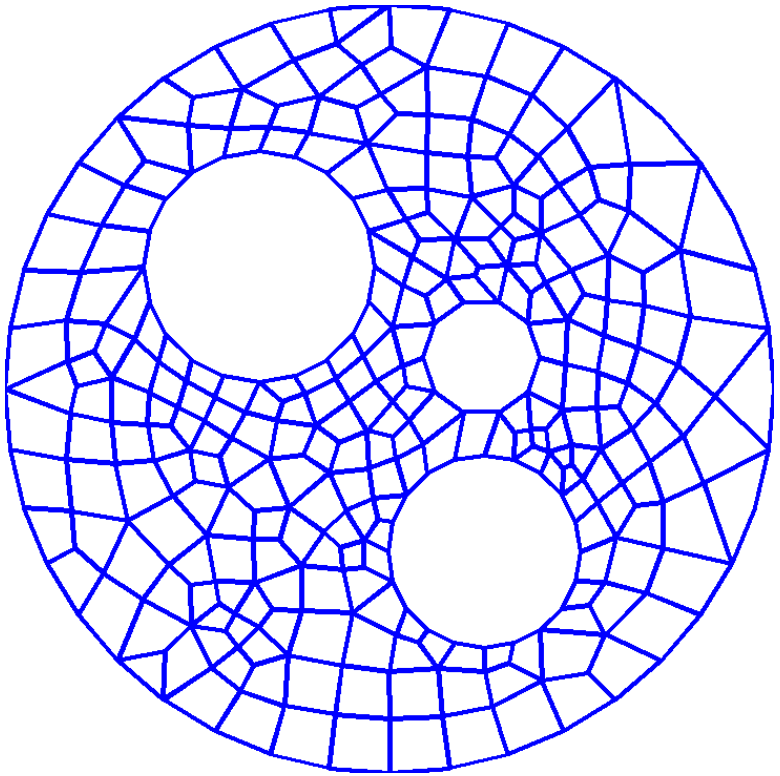
When the space S_h is constructed piecewise with respect to some triangulation of the domain Ω, this is a *finite element method*.



Finite element meshes in 2D

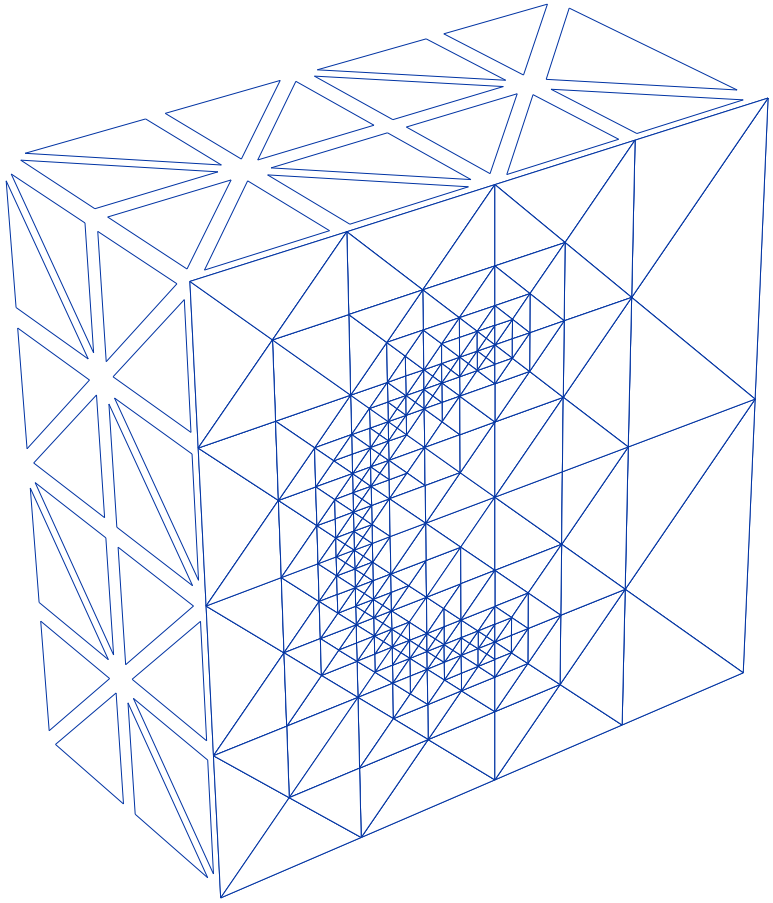
In 2D meshes are usually made of triangles and/or quadrilaterals (sometimes allowing curvilinear elements along boundaries and interfaces).

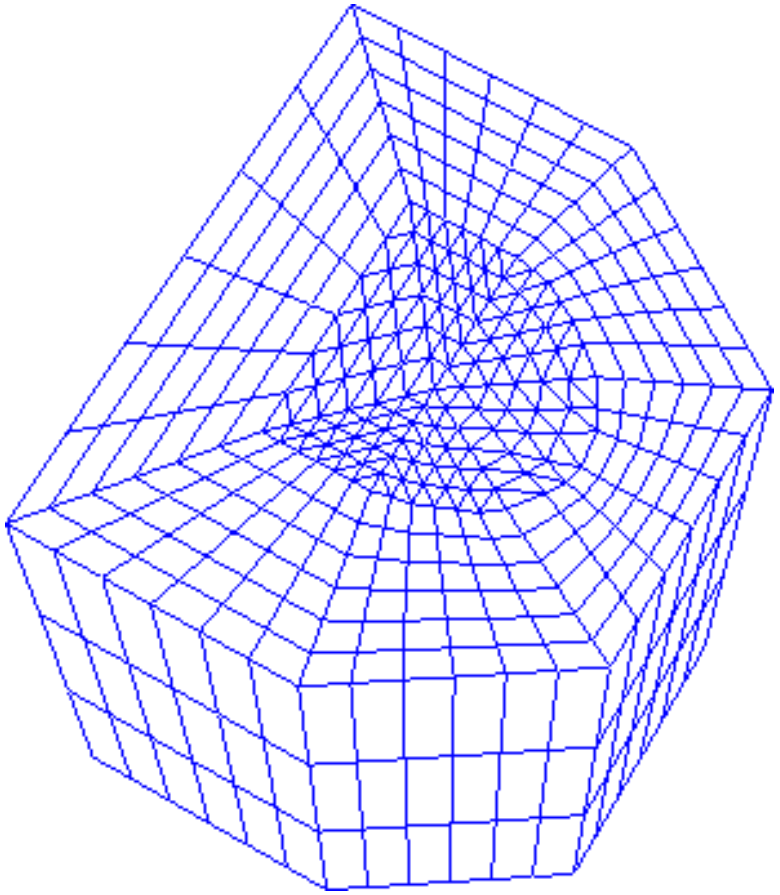


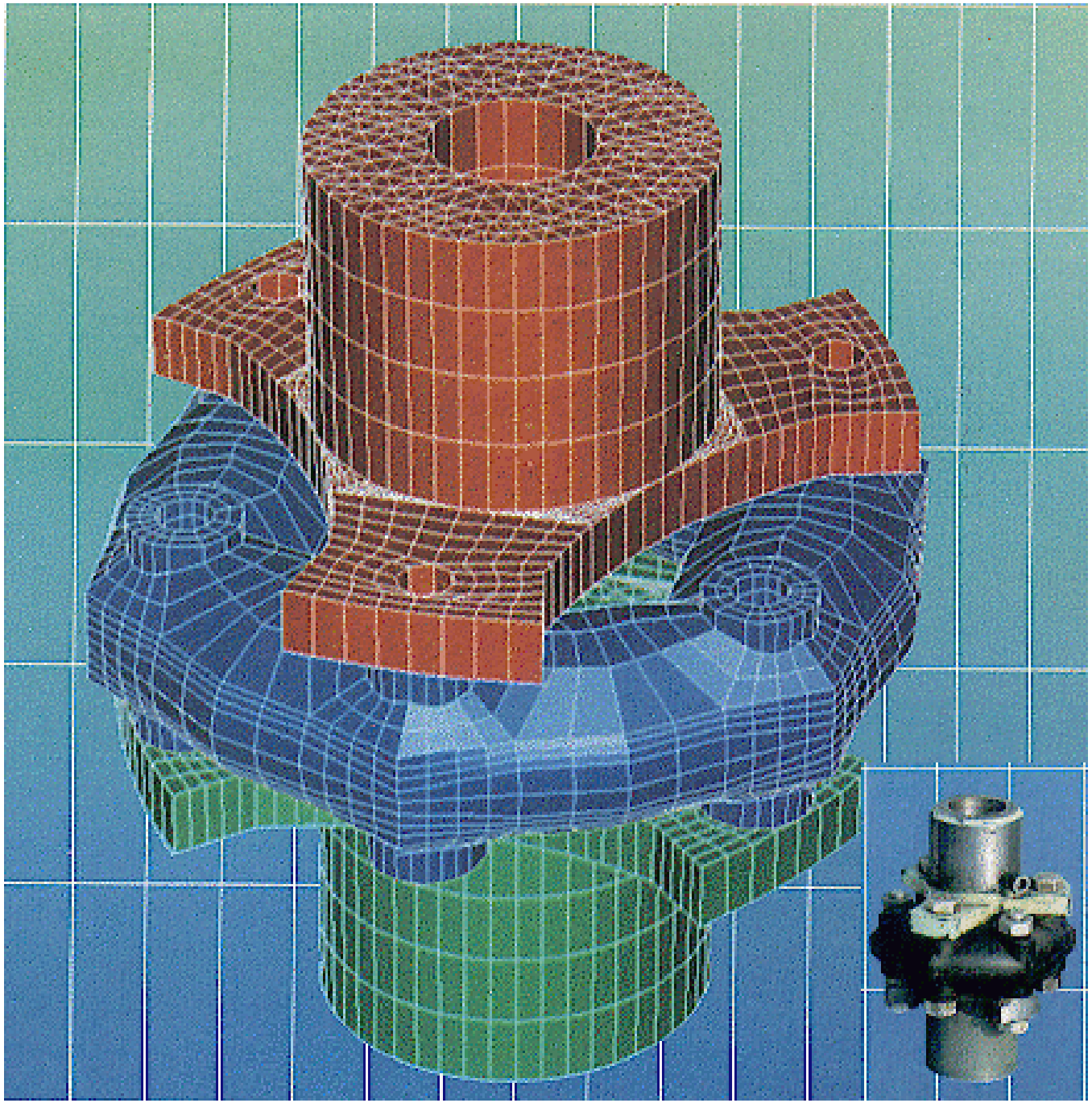


Finite element meshes in 3D

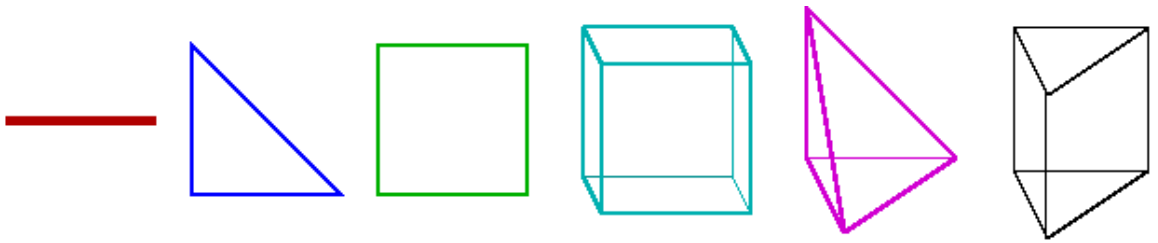
In 3D, tetrahedra, bricks, and prisms are most common.

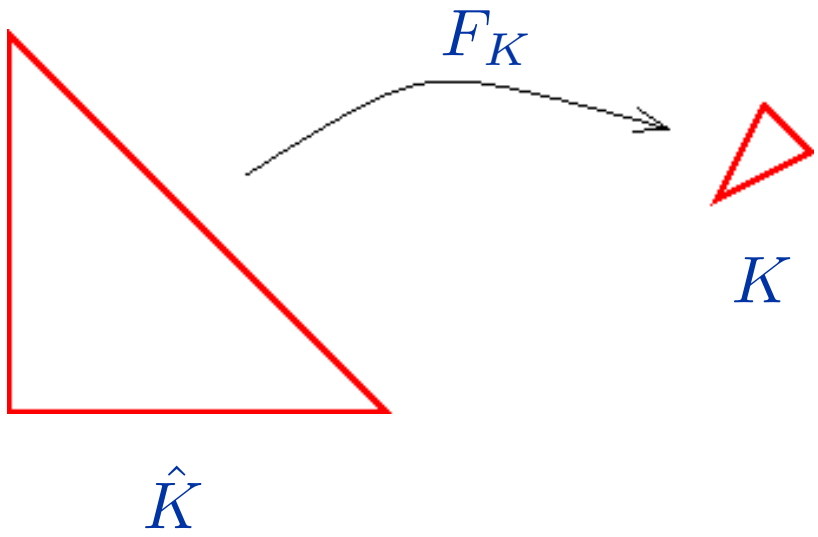


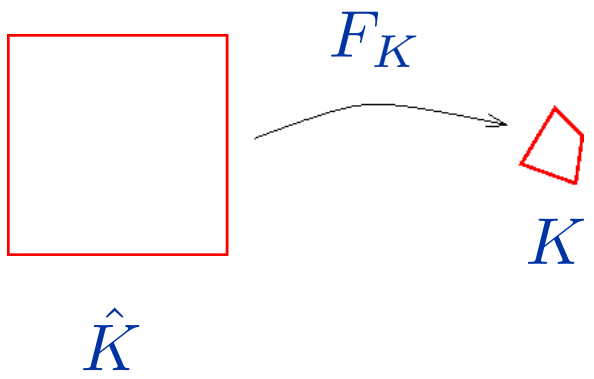




All the elements are images of simple reference elements (unit simplex, unit cube, . . .) under low degree polynomial mappings.







Construction of finite elements spaces

1. Fix a function space $\hat{S}$ on the reference element $\hat{K}$, usually consisting of polynomials.

2. Specify degrees of freedom: a set of linear functionals $C^\infty(\hat{K}) \to \mathbb{R}$ which are unisolvent on $\hat{S}$.

3. Fix a map F_K from the reference element to the actual element and use it to transfer the reference element functions to the actual element:
   $S(K) = \{ \hat{u} \circ F_K^{-1} \mid \hat{u} \in \hat{S} \}$

4. Define the global finite element space S_h as functions which restrict to $S(K)$ on each K, and for which corresponding degrees of freedoms agree.

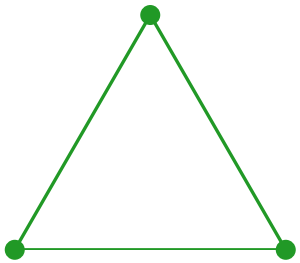
The canonical projection operator

In view of the unisolvence of the degrees of freedom, we can define a projection operator $\hat{\Pi} : C^\infty(\hat{K}) \to \hat{S}$ on the reference element, and then transfer it to the actual element $\Pi_K : C^\infty(K) \to S$:

$$(\Pi_K u) \circ F_K = \hat{\Pi}(u \circ F_K),$$

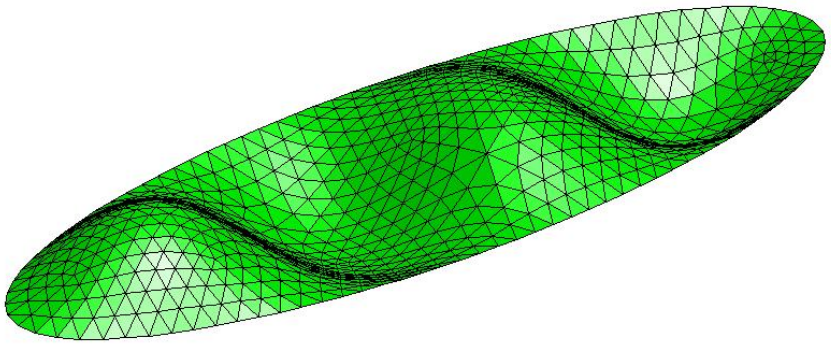
and finally piece these maps together to get $\Pi_h : C^\infty(\Omega) \to S_h$.

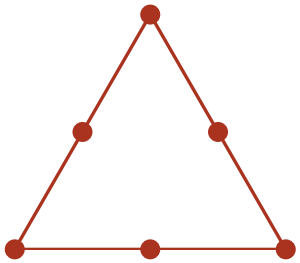
Example: Lagrange elements of degree 1 and 2



$\mathcal{P}_1$

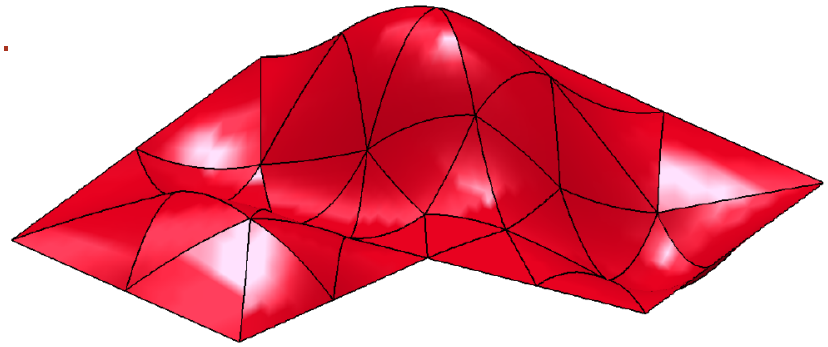
Reference maps are affine. Assembled finite element spaces consist of all cont. p.w. linear functions.



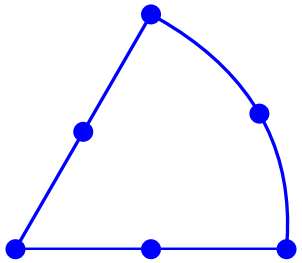


$\mathcal{P}_2$

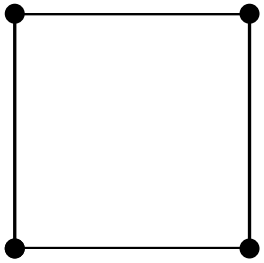
Reference maps again affine. Assembled finite element spaces consist of all cont. p.w. quadratics.



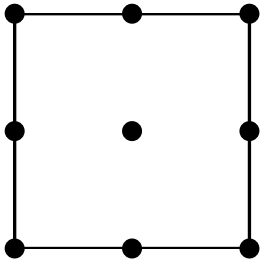
With $\mathcal{P}_2$ we may also use quadratic maps from the reference element to match curved boundaries. In this case the function space on the curved element contains non-polynomial (rational) functions.



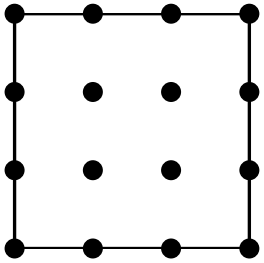
Common finite elements on the reference square



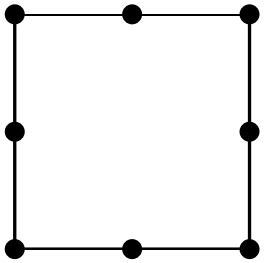
$\mathcal{Q}_1$



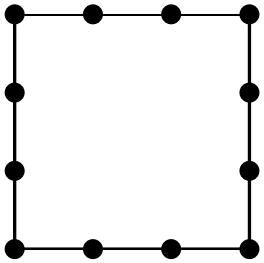
$\mathcal{Q}_2$



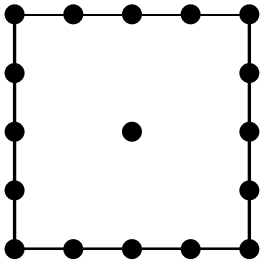
$\mathcal{Q}_3$



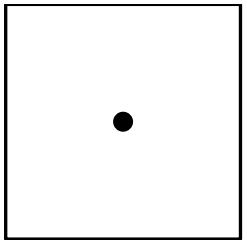
$\mathcal{Q}'_2$



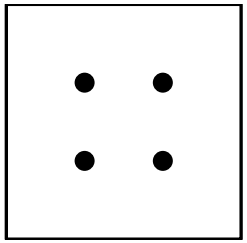
$\mathcal{Q}'_3$



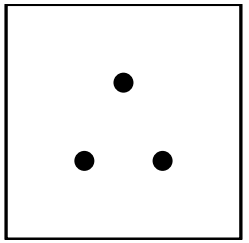
$\mathcal{Q}'_4$



$\mathcal{Q}_0$

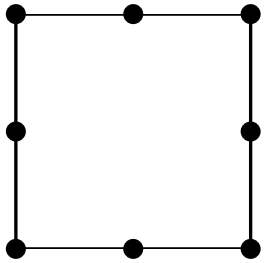


$\mathcal{Q}_1$

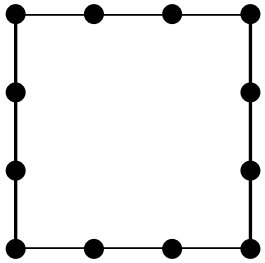


$\mathcal{P}_1$

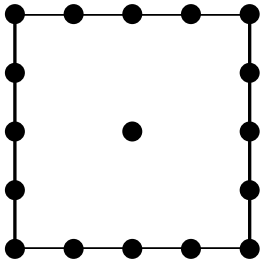
The serendipity space $\mathcal{Q}'_p$ is the span of $\mathcal{P}_p$ together with the two additional monomials $x^p y$ and $x y^p$. It is a strict subspace of $\mathcal{Q}_p$ for all $p > 1$.



$$1$$
$$x \quad y$$
$$x^2 \quad xy \quad y^2$$
$$x^2y \quad xy^2$$



$$1$$
$$x \quad y$$
$$x^2 \quad xy \quad y^2$$
$$x^3 \quad x^2y \quad xy^2 \quad y^3$$
$$x^3y \qquad xy^3$$

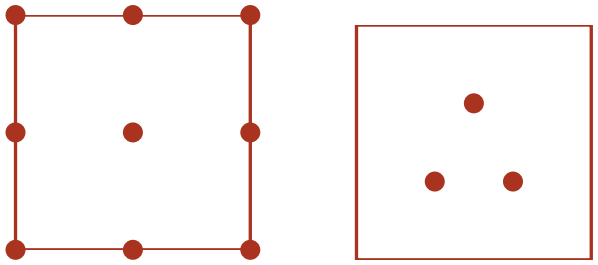


$$1$$
$$x \quad y$$
$$x^2 \quad xy \quad y^2$$
$$x^3 \quad x^2y \quad xy^2 \quad y^3$$
$$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$$
$$x^4y \qquad\qquad xy^4$$

Serendipity elements spaces have been popular for thirty years.

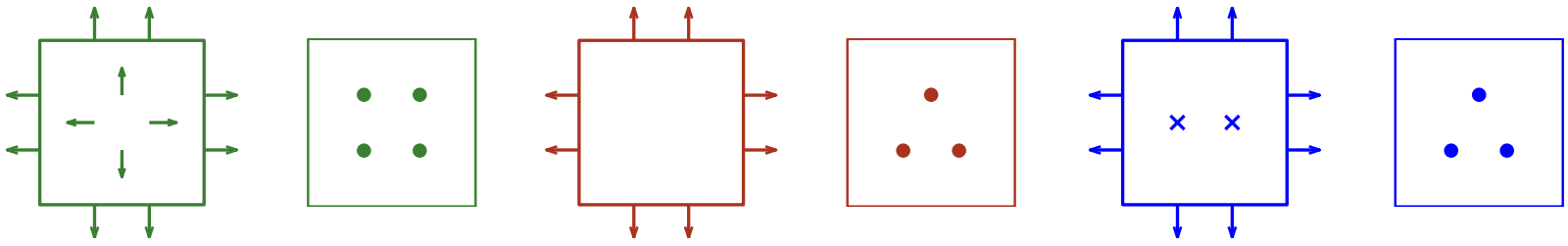
Popular quadrilateral mixed finite elements

For the Stokes equations:



$\mathcal{Q}_2$ $\mathcal{P}_1$

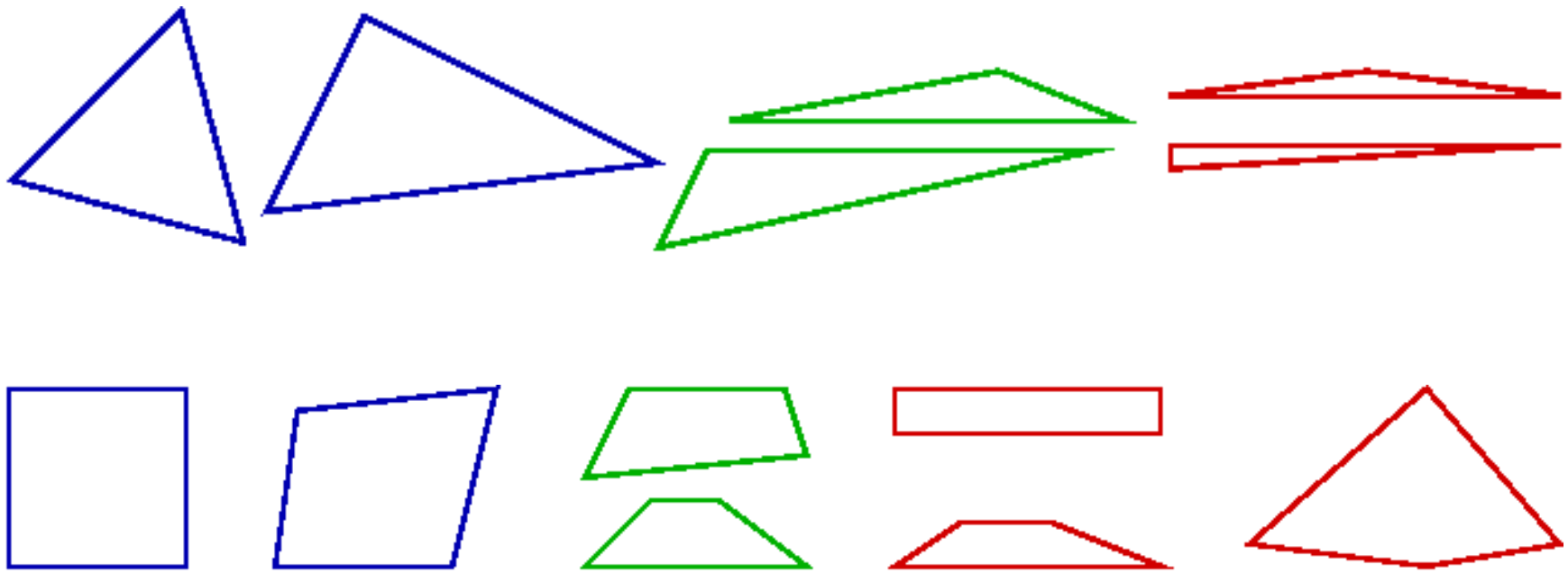
For the mixed Laplacian:



RT_1 $\mathcal{Q}_1$ BDM_2 $\mathcal{P}_1$ $BDFM_2$ $\mathcal{P}_1$

Shape regularity

The condition number of the Jacobian of F_K gives a measure of the shape regularity of the element K: how far it deviates from a dilation of $\hat{K}$. Almost all finite element approximation theorems require a uniform bound on the shape regularity of elements.



Approximation by affine finite elements

In the case when the F_K are *affine*, the basic approximation theory, based on the Bramble–Hilbert lemma, is a pillar of finite element analysis.

Theorem. *Suppose that $S \supset \mathcal{P}_r(\hat{K})$ and that $\hat{\Pi}$ is bounded on $H^{r+1}(\hat{K})$. Then*

$$\|u - \Pi_h u\|_{L^2(\Omega)} \leq C h^{r+1} |u|_{H^{r+1}(\Omega)},$$

where C only depends on the shape regularity of the elements in the mesh.

Similarly $|u - \Pi_h u|_{H^1(\Omega)} \leq C h^r |u|_{H^{r+1}(\Omega)}$; also L_p and W_p^1

Q. Is it necessary that $\hat{S} \supset \mathcal{P}_r(\hat{K})$?

A. Obviously yes, for a bound in terms of $|u|_{H^{r+1}}$.

Q. What if we just ask that $\|u - \Pi_h u\| = O(h^{r+1})$ for smooth u?

Theorem. *Suppose*

$$\inf_{\chi \in S_h} \|u - \chi\| = o(h^r) \quad \forall u \in \mathcal{P}_r(\Omega).$$

Then $\hat{S} \supset \mathcal{P}_r(\hat{K})$.

There are analogous results for H^1, L_p, W_p^1.

The above theory applies for any reference element: triangle, square, whatever. But when we apply it with $\hat{K}$ square, the restriction to affine reference mappings restricts us to K parallelogram. In order to allow arbitrary convex quadrilaterals, we must allow *bilinear* mappings. In this case, the spaces $S(K)$ will almost always contain non-polynomials. There is a positive result for quadrilaterals involving $\mathcal{Q}_r$ in place of $\mathcal{P}_r$.

Theorem. *Suppose that* $S \supset \mathcal{Q}_r(\hat{K})$ *and that* $\hat{\Pi}$ *is bounded on* $H^{r+1}(\hat{K})$. *Then*

$$\|u - \Pi_h u\|_{L^2(\Omega)} \leq C h^{r+1} |u|_{H^{r+1}(\Omega)},$$

where C *only depends on the shape regularity of the elements.*

Note that this theorem does *not* imply $O(h^{r+1})$ approximation for Q'_r on quadrilateral meshes—although we know it holds for parallelogram meshes.

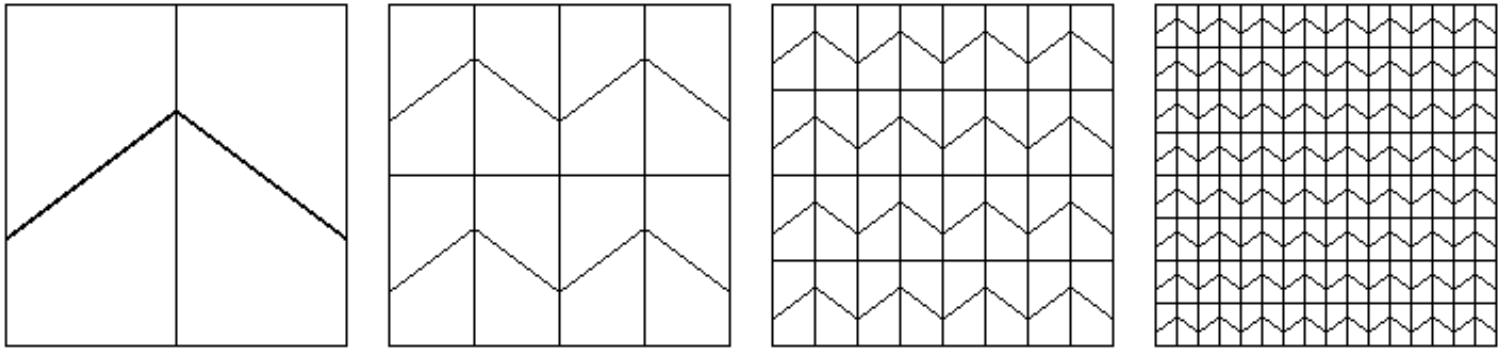
Q. Is it really necessary that $\hat{S}$ contain all of $\mathcal{Q}_r(\hat{K})$ for optimal approximation?

A. Yes indeed!

Theorem. *Suppose that*

$$\inf_{\chi \in S_h} \|u - \chi\| = o(h^r) \quad \forall u \in \mathcal{P}_r(\Omega)$$

and for (e.g.) the following sequence of meshes.
Then $\hat{S} \supset \mathcal{Q}_r(\hat{K})$.



Consequences: serendipity elements

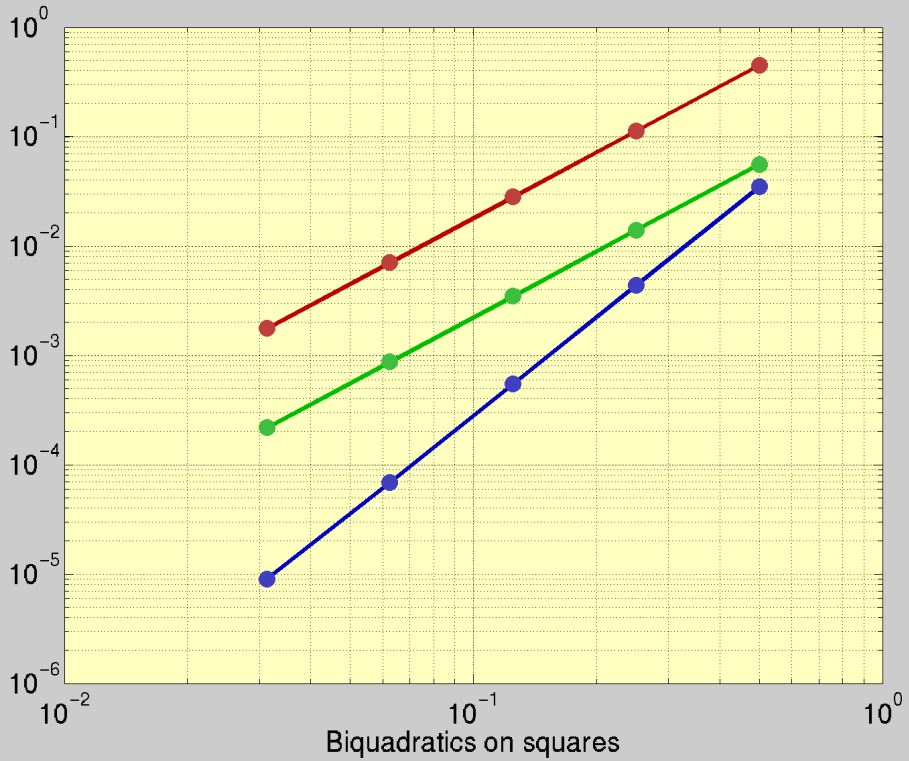
As a consequence, serendipity elements are suboptimal on quadrilateral meshes. On the unit square Q_2' contains Q_1 but not Q_2. Thus it only gives $O(h^2)$ approximation in L^2

Serendipity cubics don't achieve a better rate, since they don't contain Q_2 either. In general Q_k' only contains $Q_{\lfloor k/2 \rfloor}$.

$$\begin{array}{ccc} & 1 & \\ & x \quad y & \\ x^2 & xy & y^2 \\ x^2y & & xy^2 \end{array}$$

$$\begin{array}{ccccc} & & 1 & & \\ & & x \quad y & & \\ & x^2 & xy & y^2 & \\ x^3 & x^2y & & xy^2 & y^3 \\ & x^3y & & xy^3 & \end{array}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & x \quad y & & & \\ & & x^2 & xy & y^2 & & \\ & x^3 & x^2y & & xy^2 & y^3 & \\ x^4 & x^3y & x^2y^2 & & xy^3 & y^4 & \\ & x^4y & & & xy^4 & & \end{array}$$


10^{0}
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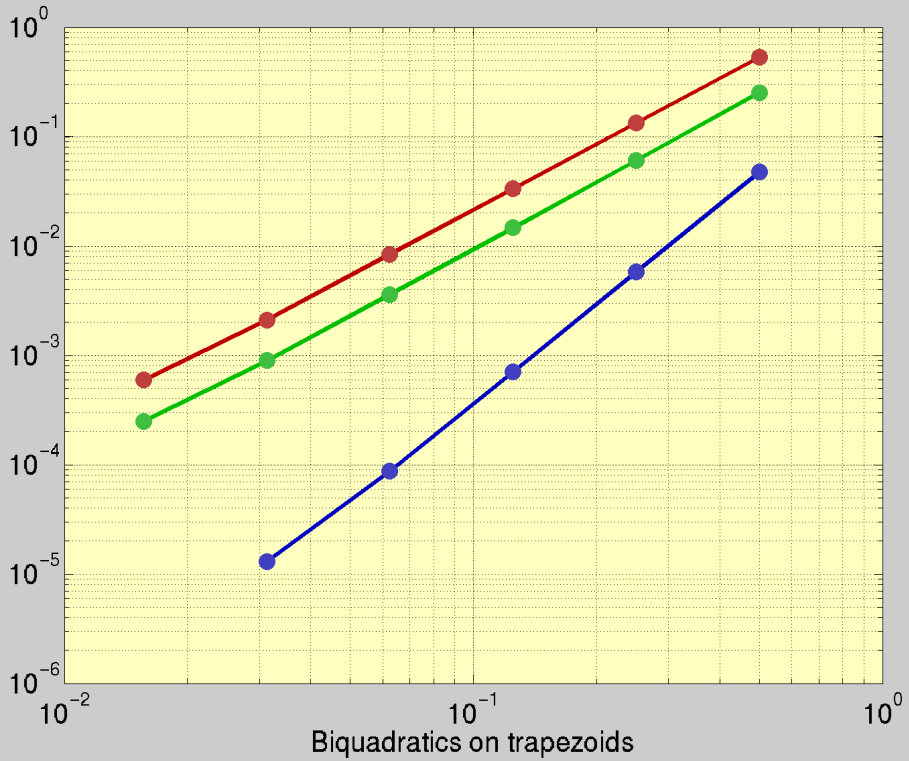
Biquadratics on squares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Biquadratics on trapezoids

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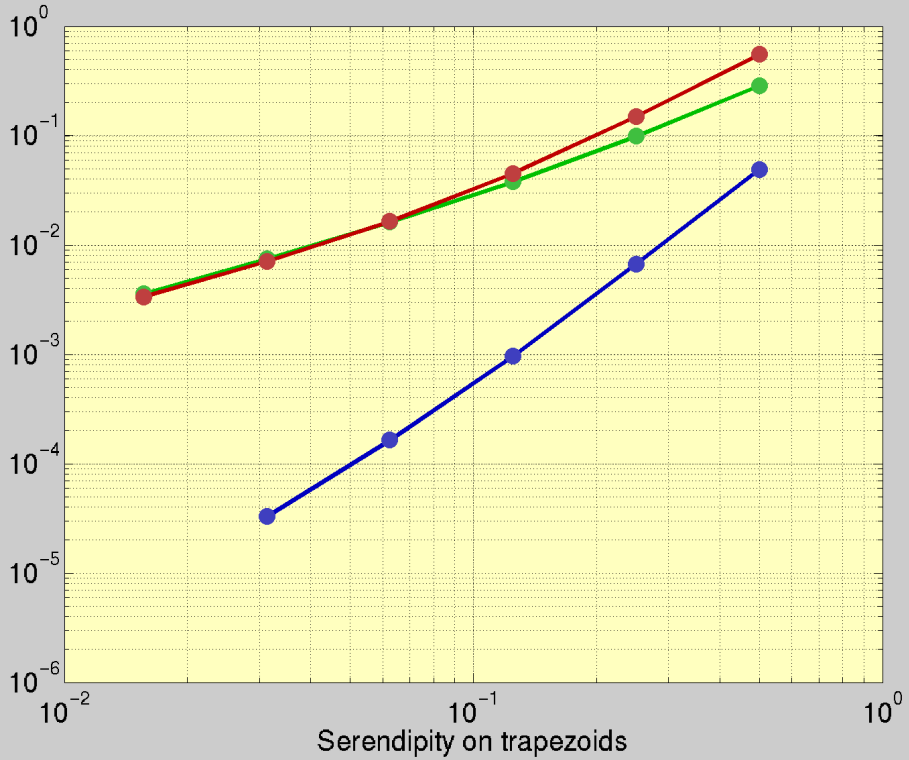
Serendipity on squares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10^{-6}
10^{-2}
10^{-1}
10^{0}
Serendipity on trapezoids

Errors on trapezoidal meshes

Errors in $|\nabla u|_{L^\infty}$ for Poisson's problem when

$$u(x,y) = 5y^2 + x^3 - 10y^3 + y^4$$

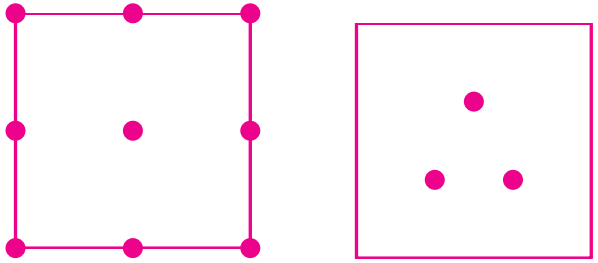
Q_2 on trapezoidal meshes

n	error	%err.	rate
2	5.9E-01	48.58	
4	1.5E-01	12.08	2.0
8	3.7E-02	3.02	2.0
16	9.2E-03	0.75	2.0
32	2.3E-03	0.19	2.0
64	5.7E-04	0.05	2.0

Q_2' on trapezoidal meshes

n	error	%err.	rate
2	6.2E-01	51.21	
4	1.8E-01	14.72	1.8
8	5.9E-02	4.84	1.6
16	2.3E-02	1.89	1.4
32	1.0E-03	0.84	1.2
64	4.9E-03	0.40	1.1

The $\mathcal{Q}_2$–$\mathcal{P}_1$ Stokes element

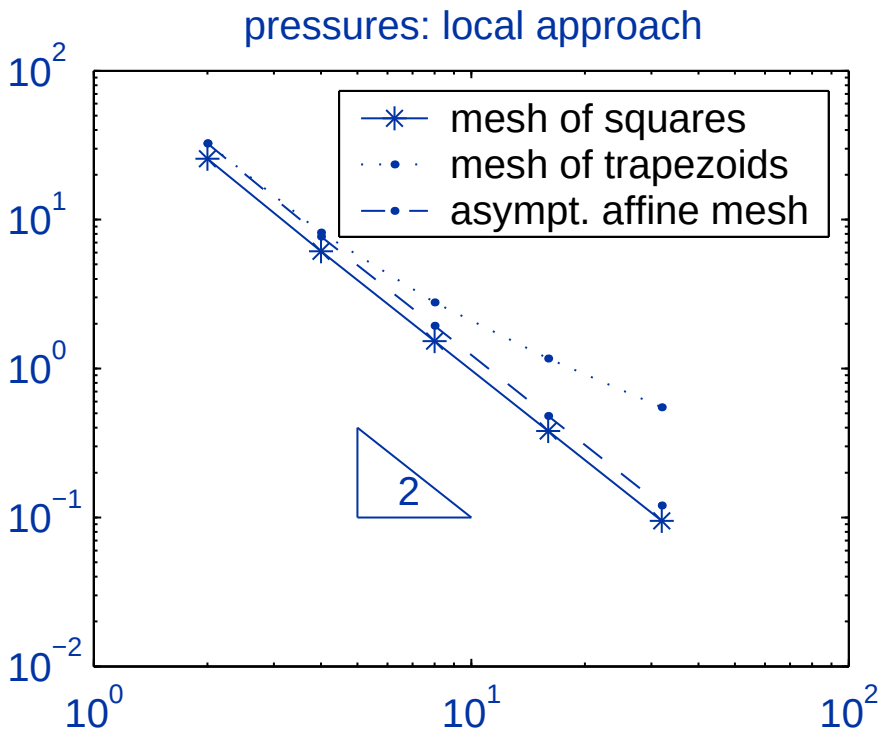


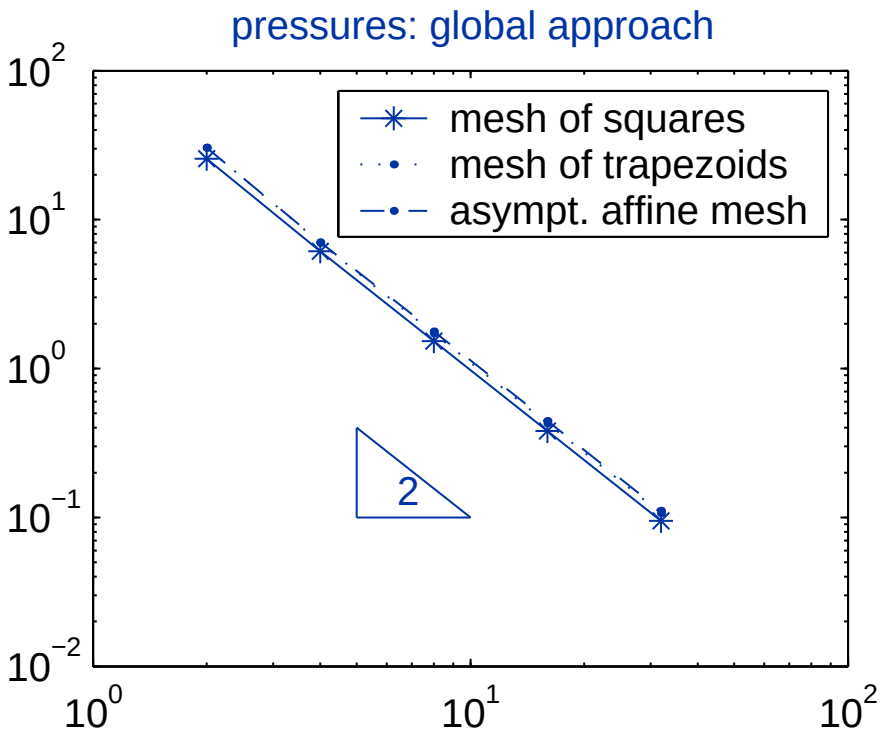
$\mathcal{Q}_2$ $\mathcal{P}_1$

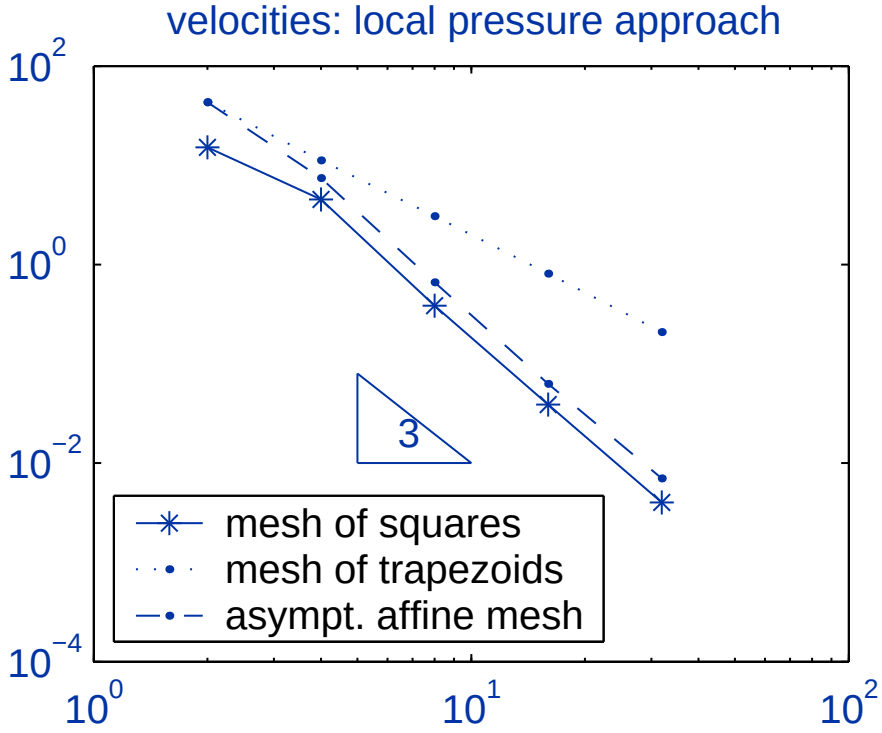
For the $\mathcal{Q}_2$–$\mathcal{P}_1$ Stokes element, we *must* map the velocity space to the reference element. But the pressures are discontinuous, so we can take pressures which are linear on the reference element (i.e., in the local coordinate system) or linear in global coordinates.

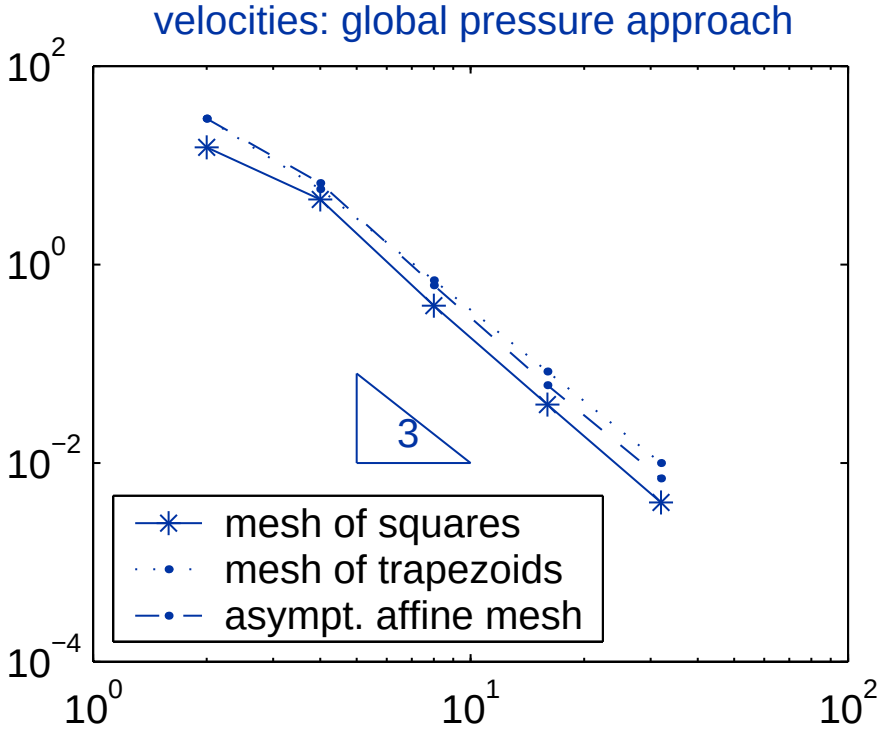
Q. Which is right?

A. If we want to have good approximation on quadrilaterals, we *must not* map the pressure space. Fortunately, the most natural proof of stability holds in that case as well.


pressures: local approach
mesh of squares
mesh of trapezoids
asympt. affine mesh
2


pressures: global approach
mesh of squares
mesh of trapezoids
asympt. affine mesh
2


velocities: local pressure approach
mesh of squares
mesh of trapezoids
asympt. affine mesh
3


velocities: global pressure approach
mesh of squares
mesh of trapezoids
asympt. affine mesh
3

Mixed finite elements for scalar elliptic problems.

$$\mathcal{L}(\boldsymbol{u}, p) = \frac{1}{2}\int_\Omega |\boldsymbol{u}|^2\, dx + \int_\Omega p \operatorname{div} \boldsymbol{u}\, dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\, dx$$

The Raviart–Thomas elements give optimal order approximation on quadrilateral meshes, but BDM and BDFM must be suboptimal because of bad approximation of the scalar variable.

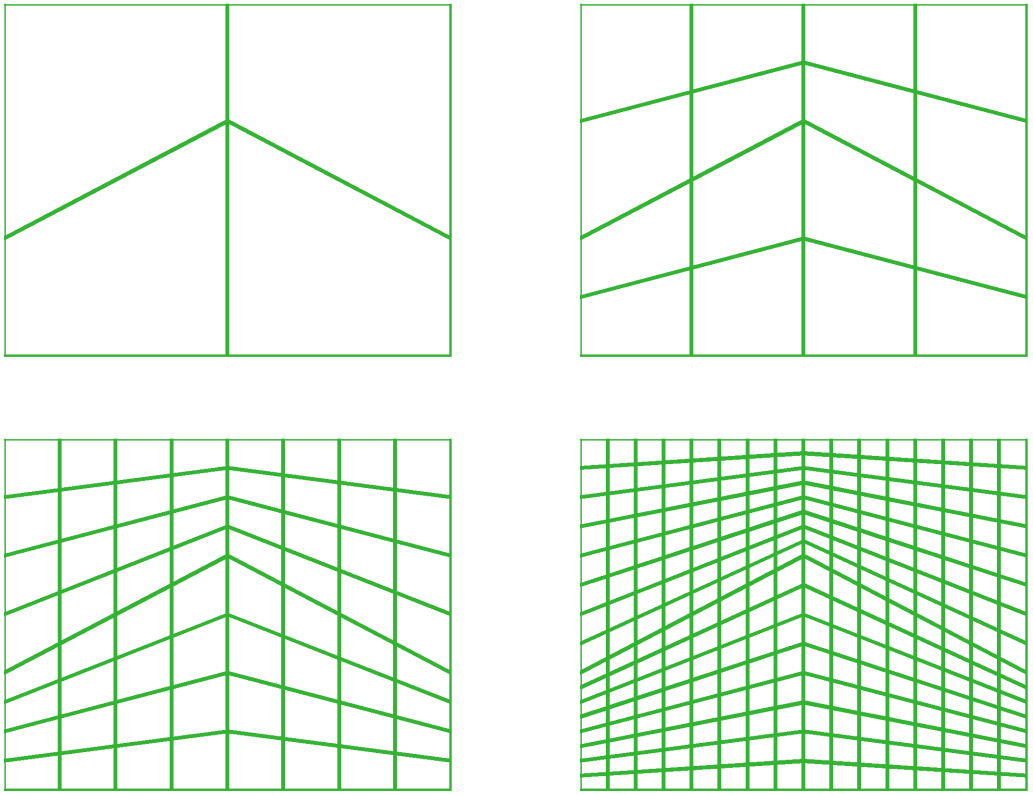
Reissner–Mindlin plate elements. Most of the locking free quadrilateral elements proposed for give suboptimal approximation on general quadrilaterals.

Asymptotically parallelogram meshes

The deviation of a quadrilateral K from being a parallelogram may be quantified by the maximum of the angles between opposite sides. We call a sequence of meshes *asymptotically parallelogram* if this quantity is bounded on each element by a multiple of the element diameter.

Theorem. *For an asymptotically affine sequence of shape regular quadrilateral meshes, $\hat{S} \supset \mathcal{P}_r(\hat{K})$ is a necessary and sufficient condition for optimal order approximation.*

This theorem applies, in particular, to any sequence of meshes coming from regular refinement of an initial quadrilateral mesh.



Conclusions

- The condition that $\hat{S} \supset \mathcal{Q}_r(\hat{K})$ is necessary and sufficient for optimal order approximation on general quadrilateral meshes.
- On affine (parallelogram) meshes, or, more generally, asymptotically affine meshes, the weaker condition $\hat{S} \supset \mathcal{P}_r(\hat{K})$ is necessary and sufficient.
- As a result, lots of standard elements do not attain the same optimal rate on general quadrilateral meshes as they do on affine meshes (contrary to folklore).
- This result applies to serendipity elements and lots of mixed finite element methods.