# On Homotopy Continuation Method for Computing Multiple Solutions to the Henon Equation

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Motivated by numerical examples in solving semilinear elliptic PDEs for multiple solutions, some properties of Newton homotopy continuation method, such as its continuation on symmetries, the Morse index, and certain functional structures, are established. Those results provide useful information on selecting initial points for the method to find desired solutions. As an application, a bifurcation diagram, showing the symmetry/peak breaking phenomena of the Henon equation, is constructed. © 2007 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 24: 728–748, 2008

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## I. INTRODUCTION

The Henon equation modeling spherical stellar systems and its energy function are given by

$$-\Delta u(x) = |x|^r |u(x)|^{q-1} u(x) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$
$$J(u) = \int_\Omega \left[ \frac{1}{2} |\nabla u(x)|^2 - \frac{|x|^r}{q+1} |u(x)|^{q+1} \right] dx, \tag{1.1}$$

where $u \in H = H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is an open bounded domain, $r \geq 0$ and $1 < q \leq \frac{N+2}{N-2}$ [1, 2]. Hence $J \in C^2(H, \mathbb{R})$ and weak solutions of (1.1) coincide with critical points of $J$. Note that $J$ is bounded neither from below nor from above, and has a unique local minimum at 0. In this paper, we aim to find multiple solutions to (1.1) by a Newton homotopy continuation method (NHCM). Let us first introduce some notions.

Let $H$ be a Hilbert space, $J \in C^2(H, \mathbb{R})$, $J'$ and $J''$ be the first and second Frechet derivatives. A point $u^* \in H$ is a critical point of $J$ if $J'(u^*) = 0$. $u^*$ is called nondegenerate if $J''(u^*)$ has a bounded inverse. The Morse index (MI) of $u^*$ is the dimension of the maximum subspace where $J''(u^*)$ is negative definite. The most well-studied critical points are the local extrema. Classical

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calculus of variations and traditional numerical (variational) methods focus on finding such stable solutions. Critical points that are not local extrema are called saddle points, which appear as unstable equilibria or transient excited states in physical systems.

Many nonlinear boundary value problems can be reduced to solving $F(u) = 0$ for an operator $F$ on $H$. A problem is variational if there is $J \in C^1(H, \mathbb{R})$ s.t. $F = J'$. To solve $F(u) = 0$, a Newton's method of the form [3]

$$u_{k+1} = u_k - s_k v_k \tag{1.2}$$

can be used where $s_k > 0$ is a stepsize, e.g., in Armijo's rule, $s_k > 0$ is chosen s.t.

$$\|F(u_{k+1})\| - \|F(u_k)\| < -\frac{1}{2}s_k\|F(u_k)\| \tag{1.3}$$

and $v_k$, the Newton direction, is the least-norm solution to the linear system

$$F'(u_k)v_k = F(u_k), \quad \text{i.e.,} \quad J''(u_k)v_k = J'(u_k), \tag{1.4}$$

or the minimization problem $\min_{v\in H} \|F'(u_k)v - F(u_k)\|$. In this paper, we assume that $J''(u)$ is a self-adjoint Fredholm operator with index zero. Then it has a finite dimensional kernel, $\ker(J''(u))$, and closed range. It is known that (1.4) may have none, unique or infinitely many solution(s), and (1.4) has a solution if and only if $F(u) = J'(u) \perp \ker(J''(u))$. In this case, the Newton direction is the least-norm solution to (1.4).

A standard Newton's method usually requires the invertibility or nondegeneracy of $J''$ along the trajectory generated by the method. But degeneracy exists in every multiple saddle point problem due to a sign change of the eigenvalues of $J''$ between two saddle points with different Morse indices. Additionally, in order for a Newton's method to converge to a desired unknown critical point $u^*$, its initial guess $u_0$ must be chosen sufficiently close to $u^*$. Otherwise it can be extremely slow or divergent, or lead to an unwanted or known critical point. This is really challenging when multiple critical points are to be found.

A primary motivation to introduce a NHCM is to relax the restriction on the initial guess/point in Newton's method. To be precise, let $u^* \in H$ be a desired solution of the equation $F(u) = 0$, then one can choose a smooth function $Q$ as well as a point $u_0 \in H$ with $Q(u_0) = 0$ and define a homotopy $\mathcal{H} : [0, 1] \times H \to H$ s.t. $\mathcal{H}(0, u) = Q(u), \mathcal{H}(1, u) = F(u)$. In particular, one may choose a convex homotopy of the form [4]

$$\mathcal{H}(\lambda, u) = \lambda F(u) + (1 - \lambda)Q(u), \tag{1.5}$$

and trace a curve $c(s) = (\lambda(s), u(s))(0 \le s \le \bar{s})$ implicitly defined by the equation

$$\mathcal{H}(\lambda(s), u(s)) = 0 \tag{1.6}$$

from an initial point $(\lambda(0), u(0)) = (0, u_0)$ to a final point $(\lambda(\bar{s}), u(\bar{s})) = (1, u^*)$. Typically, one may set $\lambda = \lambda(s) \equiv s$ with $s \in [0, 1]$ and use a partition on $[0, 1]$: $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_m = 1$ for some $m > 1$. For each $k = 1, \ldots, m$, one applies Newton's method (1.2) with the initial guess $u_{k-1}$ to obtain a solution $u_k$ of $\mathcal{H}(\lambda_k, u) = 0$. If it is successful, when $k = m$, one can expect that $u_m$ will be a good approximation of $u^*$.

In practice, several techniques on selecting the function $Q$ have been proposed [4,5] for NHCM. For example, if choosing some $u_0 \in H$ with $J'(u_0) \neq 0$ and setting $Q(u) = J'(u) - J'(u_0)$, then (1.5) becomes a global homotopy [4], i.e.,

$$\mathcal{H}(\lambda, u) = J'(u) - (1 - \lambda)J'(u_0), \tag{1.7}$$

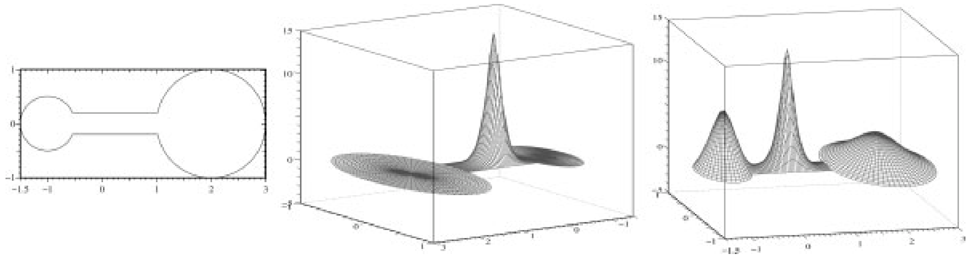

FIG. 1. Equation (1.1) with $r = 0, q = 3$. The domain (left). A ground state (center) and a 3-peak positive solution (right) with MI $\geq$3.

and if selecting some $u_0 \in H$ s.t. $J''(u_0)$ is invertible and setting $Q(u) = J''(u_0)(u - u_0)$, it leads to a mixed homotopy, i.e.,

$$\mathcal{H}(\lambda, u) = \lambda J'(u) + (1 - \lambda)J''(u_0)(u - u_0). \tag{1.8}$$

But the success of those techniques depends on both the selected function $Q$ and the generic properties (such as dimension, definiteness, invariance) of the problem to be solved. Thus, as long as multiple critical point problems are concerned, NHCM is still not well-understood. In particular, without knowing some basic properties (like nodal structures, symmetries, or invariance) of a desired solution $u^*$ and the functional $J$, finding multiple solutions to (1.1) by NHCM is no more than an initial point guessing and/or a function $Q$ selecting problem.

Instability analysis is an important issue when multiple unstable solutions are concerned. It is known that [6, 7] the Morse index can be used to measure local instabilities of nondegenerate critical points. Recently several researchers [8–10] suggested to use the eigenfunctions of $-\Delta$, i.e., the linear part of $J'$, as or to construct initial points for NHCM to find multiple solutions to (1.1) and then estimated their Morse indices and other solution structures as well. However, our numerical experiments raised some doubts about their approach. Let us look at two numerical cases on (1.1) which motivate this work. Note that for (1.1), positive solutions are more interesting in physical applications.

**Case 1.** $\Omega \subset \mathbb{R}^2$ is a dumbbell-shaped domain as in Fig. 1 (left), $q = 3$ and $r = 0$. We found five positive solutions by the minimax-Newton method [11]. Two of them have a peak located on the central corridor, see Fig. 1 (center) and (right). We also computed the first 52 eigenfunctions of $-\Delta$ on $\Omega$ by Matlab subroutine PDEEIG and found that their values on the central corridor are almost zero. Hence those 52 eigenfunctions are not enough to construct an initial point for NHCM to approximate those two positive solutions.

**Case 2.** $\Omega$ is the unit disk in $\mathbb{R}^2$, $q = 3$ and $r = 9$. Then (1.1) is radially symmetric and depends explicitly on $x$. Thus, symmetry-breaking [12] may occur depending on the value of $r (> 0)$. Indeed, in addition to the radial positive solution, we have found several nonradial positive solutions including the ground states, see Figs. 2 and 3. Since, a rotation of a nonradial solution by any angle still yields a solution, any nonradial solution is not isolated and hence is degenerate. We call a solution $u$ $n$-symmetric if there exists an integer $n \geq 1$ s.t. $u(r, \theta) \equiv u(r, \theta + k\frac{2\pi}{n}), \forall k = 1, \ldots, n$. We have the following two inspiring findings.

**(a)** Let $v_0$ be the first eigenfunction of $-\Delta$ or any other 1-peak radial positive function. Starting with $u_0 = tv_0$, NHCM (1.7) obtains either the trivial solution 0 when $t$ is small or the radial

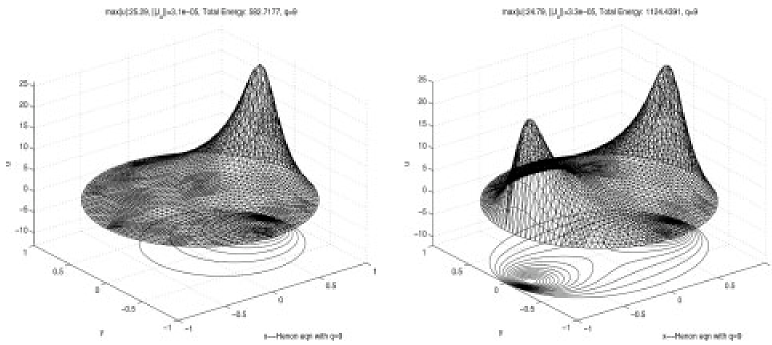

FIG. 2. Equation (1.1) with $r = 9, q = 3$. A nonradial ground state and its contours, $J = 582.72$ (left). A 2-symmetric 2-peak solution of MI = 2 and its contours, $J = 1124.44$ (right).

positive solution $u^*$ as in Fig. 3 (right) when $t$ is large. We miss all other nonradial positive solutions including the ground states, see Fig. 2 (left). One sees that $u_0$, 0 and $u^*$ are in the same invariant subspace consisting of all radial function, and the Morse index of $u^*$ restricted to this invariant subspace is 1.

**(b)** Construct a normalized 4-peak positive function $v_0$ in $H$ by

$$v_0 = 0.78413 \exp(-0.45((x^2 - 1.44)^2 + y^2)(x^2 + (y^2 - 1.44)^2)) \cos((x^2 + y^2)\pi/2).$$

We first find $t_0 \approx 54.437$, $t_1 \approx 94.2877$ s.t. $\int_\Omega [J''(t_0 v_0)v_0] \cdot v_0 dx = 0$, $\int_\Omega J'(t_1 v_0) \cdot v_0 dx = 0$, i.e., $t_0 v_0$ is an inflection point and $t_1 v_0$ is a peak point of $J$ along the direction of $v_0$. Using NHCM (1.7) with the initial point $u_0 = t v_0$, we obtain

i. the trivial solution 0 when $t = 44.437 < t_0$,
ii. the 4-peak positive solution as in Fig. 3 (center) when $t_0 < t = 64.437 < t_1$,
iii. the radial positive solution as in Fig. 3 (right) when $t = 94.437 > t_1$.

The purpose of this work is not to propose any NEW algorithm on NHCM, but to explore some of its properties, such as its continuation on symmetries, the Morse index and other functional structures, and to provide some useful information on selecting initial points.

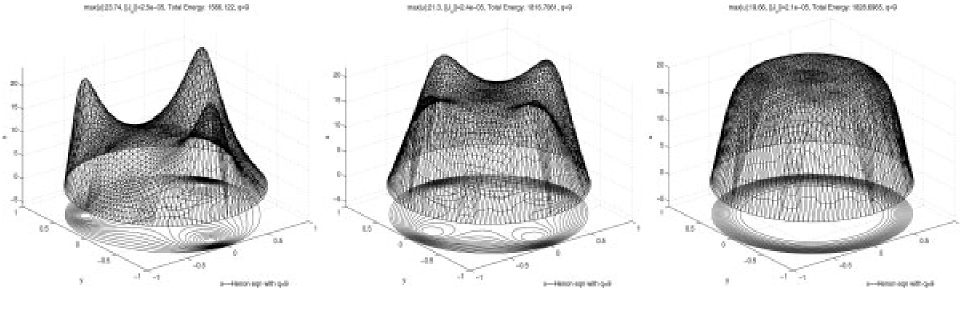

FIG. 3. Equation (1.1) with $r = 9, q = 3$. A 3-symmetric 3-peak solution of MI $\geq 3$ and its contours, $J = 1566.12$ (left). A 4-symmetric 4-peak solution of MI $\geq 4$ and its contours, $J = 1816.71$ (center). A radial solution of MI $\geq 5$ and its contours, $J = 1828.70$ (right).

## II. SOME PROPERTIES OF NHCM

Note that a numerical NHCM always assumes that

**Assumption 2.1.** There is $\delta > 0$ s.t. if a solution $u(s_0)$ of $\mathcal{H}(\lambda(s_0), u) = 0$ is known for some $0 \le s_0 < \bar{s}$, then any solution $u(s)$ of $\mathcal{H}(\lambda(s), u) = 0$ with $s \in [0, \bar{s}]$ and $|s - s_0| < \delta$ can be approximated by Newton's method (1.2) starting from $u(s_0)$, where the Newton direction $v_k$ at $u_k$ is the least-norm solution to

$$\mathcal{H}_2'(\lambda(s), u_k)v_k = \mathcal{H}(\lambda(s), u_k). \tag{2.1}$$

### A. Continuation on Symmetries

Symmetries exist in many nature phenomena. They can lead to the existence of multiple saddle-type solutions to a nonlinear variational problem [13] and may also cause (symmetric) degeneracy. Newton's method is invariant to symmetries and this invariance is insensitive to computational errors [11]. If one knows a symmetry of a desired solution $u^*$, then it is advantageous to use it, i.e., one can choose an initial point $u_0$ with the same symmetry. Otherwise it will become a trap, i.e., if $u_0$ has a symmetry different from that of $u^*$, then the whole sequence generated by Newton's method will be trapped in the invariant subspace induced by the symmetry of $u_0$ and fail to reach $u^*$. Thus this invariance is double-edged.

We need some preliminaries on transformation groups and invariant functionals. Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, $\mathcal{G}$ be a compact group that acts isometrically on $H$ and $J \in C^2(H, \mathbb{R})$ be $\mathcal{G}$-invariant, i.e., $J(gu) = J(u)$, $\forall g \in \mathcal{G}, u \in H$, and $J''(u)$ has a closed range for each $u \in H$. For a subgroup $G$ of $\mathcal{G}$, let $H_G = \{u \in H : gu = u, \forall g \in G\}$ be the invariant subspace of $H$ under the group actions of $G$. For $u \in H$, the $\mathcal{G}$-orbit of $u$ is the set $\mathcal{G}u = \{gu : g \in \mathcal{G}\}$ and the isotropy subgroup of $u$ is $\mathcal{G}_u = \{g \in \mathcal{G} : gu = u\}$. Denote $e$ the identity element in $\mathcal{G}$. When $Gu$ is differentiable at $u$, we denote $T_u(Gu)$ the tangent space of $Gu$ at $u$. The following lemma will be used in this paper.

**Lemma 2.1.** ([11])

a. *$J'$ is $\mathcal{G}$-equivariant, i.e., $J'(gu) = gJ'(u)$, $\forall u \in H, g \in \mathcal{G}$;*
b. *$J'(u) \in H_G$ for any subgroup $G \subset \mathcal{G}$ and $u \in H_G$;*
c. *$\langle J''(u)w, v\rangle = \langle J''(gu)gw, gv\rangle$ $\forall u, v, w \in H, g \in \mathcal{G}$, and in particular, we have $J''(u)(H_G) \subset H_G$ for any subgroup $G \subset \mathcal{G}$ and $u \in H_G$.*

Lemma 2.1(c) also implies $J''(u)w = g^{-1}J''(gu)gw, \forall g \in G, u, w \in H$. If $u \in H_G$, then

$$J''(u)w = g^{-1}J''(u)gw \quad \text{or} \quad gJ''(u)w = J''(u)gw \quad \forall g \in G, w \in H.$$

**Theorem 2.1.** *Under Assumption 2.1, if the homotopy $\mathcal{H}(\lambda(s), \cdot)$ is $\mathcal{G}(G)$-equivariant and $u_0 \in H_G$ for a subgroup $G$ of $\mathcal{G}$, then $u(s) \in H_G$, $\forall s \in [0, \bar{s}]$.*

**Proof.** By Assumption 2.1, $u(s)(0 < s \le \bar{s})$ can be obtained by Newton's method (1.2). Since $u_0 \in H_G$ and Newton's method is invariant in $H_G$, we have $u(s) \in H_G$, $\forall s \in [0, \bar{s}]$. ■

**Corollary 2.1.** *If $J \in C^2(H,\mathbb{R})$ is $\mathcal{G}$-invariant and $u_0 \in H$, then both the global homotopy* (1.7) *and the mixed homotopy* (1.8) *are $\mathcal{G}_{u_0}$-equivariant. Consequently, if starting with $u(0) = u_0$, then $u(s) \in H_{\mathcal{G}_{u_0}}$ $(0 \le s \le \bar{s})$.*

**Proof.** $J$ is $\mathcal{G}$-invariant implies $J'$ is $\mathcal{G}$-equivariant. Thus we only need to verify that for each $g \in \mathcal{G}_{u_0}$, $J'(u_0) = gJ'(u_0)$ for the global homotopy (1.7) and $J''(u_0)(gu - u_0) = gJ''(u_0)(u-u_0)$ for the mixed homotopy (1.8). Since $\mathcal{G}_{u_0}$ is an isotropy subgroup of $u_0$ w.r.t. $\mathcal{G}$, then $J'(u_0) = J'(gu_0) = gJ'(u_0)$ and $J''(u_0)(gu - u_0) = J''(u_0)(gu - gu_0) = J''(u_0)g(u - u_0) = gJ''(u_0)(u - u_0)$ by Lemma 2.1(c). Thus in either case $\mathcal{H}(\lambda(s), u(s))$ is $\mathcal{G}_{u_0}$-equivariant, and the last conclusion follows from Theorem 2.1. ∎

Thus Corollary 2.1 actually verifies Case 2(a) in Section I. More precisely, a radial function (including the first eigenfunction of $-\Delta$ on a radial domain) cannot be used as an initial solution for either NHCM (1.7) or NHCM (1.8) to approximate a nonradial solution.

### B. Continuation on the Morse Index in the Subspace $H_G$

Let $G$ be a subgroup of $\mathcal{G}$, $\mathcal{H} : \mathbb{R}\times H \to H$ be a $C^1$ homotopy and $c(s) = (\lambda(s), u(s))$ $(0 \le s \le \bar{s})$ be a smooth curve (solution trajectory) as defined in (1.6) s.t. $\lambda(s)$ is monotone increasing in $s$ and $\lambda(0) = 0, \lambda(\bar{s}) = 1$. By differentiating equation (1.6) with respect to $s$, it follows that the tangent $c'(s) = (\lambda'(s), u'(s))$ satisfies

$$\mathcal{H}'(c(s))c'(s) = \mathcal{H}_1'(\lambda(s), u(s))\lambda'(s) + \mathcal{H}_2'(\lambda(s), u(s))u'(s) = 0, \quad \forall s \in [0, \bar{s}], \tag{2.2}$$

where $\mathcal{H}'(c(s)) = (\mathcal{H}_1'(\lambda(s), u(s)), \mathcal{H}_2'(\lambda(s), u(s)))$.

Before establishing the continuation of NHCM on the Morse index, we need two lemmas.

**Lemma 2.2.** *If the homotopy $\mathcal{H}(\lambda, u)$ is $G$-equivariant in $u$, then so is $\mathcal{H}_1'(\lambda, u)$.*

**Proof.**

$$\begin{aligned}\mathcal{H}_1'(\lambda, gu) &= \lim_{\Delta\lambda\to 0} \frac{1}{\Delta\lambda}\,(\mathcal{H}(\lambda + \Delta\lambda, gu) - \mathcal{H}(\lambda, gu))\\ &= g\lim_{\Delta\lambda\to 0} \frac{1}{\Delta\lambda}\,(\mathcal{H}(\lambda + \Delta\lambda, u) - \mathcal{H}(\lambda, u)) = g\mathcal{H}_1'(\lambda, u).\end{aligned}$$ ∎

**Lemma 2.3.** *If $u(s) \in H_G$ and $\mathcal{H}(\lambda(s), u(s))$ is $\mathcal{G}(G)$-equivariant in $u$ for every $s \in [0, \bar{s}]$, then $u'(s) \in H_G$ for any $s \in [0, \bar{s}]$ where $u'(s)$ is the least-norm solution to* (2.2).

**Proof.** Follow a similar proof of Lemma 2.4 in [11]. ∎

In view of (2.2), a smooth curve $c(\cdot)$ is said to be nondegenerate if the operator

$$T(s) \equiv \begin{bmatrix} c'(s) \\ \mathcal{H}'(c(s)) \end{bmatrix} = \begin{bmatrix} \lambda'(s) & u'(s) \\ \mathcal{H}_1'(c(s)) & \mathcal{H}_2'(c(s)) \end{bmatrix} \tag{2.3}$$

is invertible from $\mathbb{R} \times H \to \mathbb{R} \times H$ for all $s \in [0, \bar{s}]$. In the next theorem, we will weaken this condition by assuming that

(h0) $T(s)(r, w)^T = (\alpha, v)^T$ has a unique solution$(r, w) \in \mathbb{R} \times H, \forall(\alpha, v) \in \mathbb{R} \times H_G$.

**Theorem 2.2.** *Let $u_0 \in H_G$. Assume $c(s) = (\lambda(s), u(s))$ $(s \in [0, \bar{s}])$ is a smooth solution trajectory implicitly defined by a differentiable and G-equivariant homotopy $\mathcal{H}(\lambda(s), u(s)) = 0$ s.t. $\lambda(s)$ is monotone increasing in s, $c(0) = (0, u_0)$ and $c(\bar{s}) = (1, u^*)$. For the operator $T(s)$ defined in* (2.3), *if (h0) holds, then*

a. *the operator $\mathcal{H}_2'(c(s))$ is invertible from $H_G \to H_G, \forall s \in [0, \bar{s}]$;*
b. *the operator $T(s)$ is invertible from $\mathbb{R} \times H_G \to \mathbb{R} \times H_G, \forall s \in [0, \bar{s}]$, i.e., the implicit solution trajectory $c(\cdot)$ is nondegenerate in $\mathbb{R} \times H_G$;*
c. *consequently, $u_0$ and $u^*$ are nondegenerate and have the same Morse index in $H_G$.*

**Proof.** First, by Theorem 2.1, $u(s) \in H_G, \forall s \in [0, \bar{s}]$. To prove (a) is equivalent to show $\mathcal{H}_2'(c(s)) : H_G \to H_G$ is 1-1 and onto. By Lemma 2.1(c), $\mathcal{H}_2'(c(s))(H_G) \subset H_G$. Then, assumption (h0) implies that $\mathcal{H}_2'(c(s))$ is 1-1. Next, we show it is also onto. For any $v \in H_G$, under assumption (h0), let $w \in H$ be the unique solution to $\mathcal{H}_2'(c(s))w = v$. With Lemma 2.1(c), we have

$$g\mathcal{H}_2'(c(s))w = \mathcal{H}_2'(c(s))gw, \forall g \in G.$$

Denote $w_G = \int_G g(w)dg \in H_G$ the Haar projection of $w$ onto $H_G$. Since $v \in H_G$ and the Haar integral is linear and normalized, we have

$$\mathcal{H}_2'(c(s))w_G = \int_G \mathcal{H}_2'(c(s))gw\, dg = \int_G g\mathcal{H}_2'(c(s))w\, dg = \int_G gv dg = v.$$

By the uniqueness, $w_G = w \in H_G$. Thus it is onto.

Part (b). By Theorem 2.1 and Lemma 2.3, $c(s) = (\lambda(s), u(s)), c'(s) = (\lambda'(s), u'(s)) \in \mathbb{R} \times H_G$ for each $s \in [0, \bar{s}]$. Since $\lambda(s)$ is monotone increasing in $s$, it follows that $\lambda'(s) > 0$ and $\|c'(s)\|^2 = (\lambda'(s))^2 + \|u'(s)\|^2 > 0$. Then, with $u'(s) \in H_G$, conclusion (a) and equation (2.2), we see that the operators

$$\begin{bmatrix} \lambda'(s) & 0 \\ u'(s) & \mathrm{Id_H} \end{bmatrix} \text{ and } \begin{bmatrix} \|c'(s)\|^2 & u'(s) \\ 0 & \mathcal{H}_2'(c(s)) \end{bmatrix} = T(s) \begin{bmatrix} \lambda'(s) & 0 \\ u'(s) & \mathrm{Id_H} \end{bmatrix}$$

are invertible from $\mathbb{R} \times H_G \to \mathbb{R} \times H_G$. Thus $T(s)$ is also invertible from $\mathbb{R} \times H_G$ to $\mathbb{R} \times H_G$.

Part (c). Only need to prove the second conclusion since Part (a) implies that $u_0$ and $u^*$ are nondegenerate in $H_G$. In this paper we assume $\mathcal{H}_2'(c(s))$ is a self-adjoint Fredholm operator of index zero, then all its eigenvalues are isolated with finite multiplicities. By Chapter IV §3.5 in [14], if $A(s)$ is a bounded linear operator on a Banach space $X$ which depends on a parameter $s$ continuously, then any finite family of isolated eigenvalues of $A(s)$ with finite multiplicities varies continuously with $A(s)$.

Suppose $\mathrm{MI}(u_0) \neq \mathrm{MI}(u^*)$, say $\mathrm{MI}(u_0) < \mathrm{MI}(u^*)$. Then one positive eigenvalue $\mu(0)$ of $\mathcal{H}_2'(c(0)) = \mathcal{H}_2'(0, u_0)$ will change to a negative eigenvalue $\mu(\bar{s})$ of $\mathcal{H}_2'(c(\bar{s})) = \mathcal{H}_2'(1, u^*)$ when $s$ goes from 0 to $\bar{s}$. By the continuous dependence of a finite family of eigenvalues on the operator $\mathcal{H}_2'(c(s))$, there must exist $0 < s' < \bar{s}$ s.t. the eigenvalue $\mu(s')$ of $\mathcal{H}_2'(c(s'))$ is 0. Then $\mathcal{H}_2'(c(s'))$ is not invertible in $H_G$, which contradicts (a). The proof is complete. ∎

**Remark 2.1.** (a) Nonradial solutions to (1.1) on a radial domain (including Case 2 of Section I) are degenerate in $H$ but can be nondegenerate in certain invariant subspace $H_G$, see Section II.C. Thus Theorem 2.2 can be applied to $\mathbb{R} \times H_G$ but not to $\mathbb{R} \times H$; (b) We use the assumption that $\lambda(s)$ is monotone increasing in $s$ to avoid possible turning points in the first coordinate of $c(s)$ which can lead to a bifurcation [4].

### C. Removal of Symmetric Degeneracy in Computation

It is known that degeneracy can result from various factors. In this section, we consider degeneracy caused by symmetries only, or more precisely by a differentiable group of rotation actions, and discuss how to avoid it in numerical computations.

Let $\Omega \subset R^N (N \geq 1)$ be a bounded open set and $\mathcal{O}(N)$ be the set of all orthogonal matrices in $R^{N\times N}$. Denote $\mathcal{G} = \{g \in \mathcal{O}(N) : g(\Omega) = \Omega\}$.

**Definition 2.1.** Let $G$ be a differentiable subgroup of $\mathcal{G}$. A point $u$ of $J$ is called $G$-degenerate if $\ker(J''(u)) \cap T_u(Gu) \neq \{0\}$, where $T_u(Gu)$ is the tangent space of the orbit $Gu$ at $u$. For a degenerate critical point $u$ of $J$, we called the degeneracy of $u$ caused only by the subgroup $G$ if $0 \neq \ker(J''(u)) \subseteq T_u(Gu)$.

Then either $Gu = \{u\}$ if $u \in H_G$ or $Gu$ is differentiable if $u \notin H_G$. In particular, for a radial domain $\Omega \subset \mathbb{R}^2$, a differentiable Abelian group $G$ representing rotations in $\mathbb{R}^2$ is defined by

$$G = \left\{ g(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} : t \in [-\pi, \pi] \right\}. \tag{2.4}$$

For $u \in H$ and $g(t) \in G$, let $\gamma(t)(x) = g(t)u(x) = u(g(t)x)$. Then $Gu = \{\gamma(t) : t \in [\pi, \pi]\}$.

**Lemma 2.4.** $\gamma'(0) = 0 \Leftrightarrow T_u(Gu) = \{0\} \Leftrightarrow u \in H_G$.

**Proof.** Let $x = (x_1, x_2)$ be a point on the circle of radius $r$ centered at 0. Differentiating $\gamma(t)$, we have $\gamma'(t)(x) = \nabla u(g(t)x) \cdot g'(t)x$. Then $\gamma'(0)(x) = \nabla u(x) \cdot \vec{\tau} = \frac{\partial u}{\partial \vec{\tau}}$ where $\vec{\tau} = (-x_2, x_1)$ is a tangent vector of the circle at $(x_1, x_2)$. Then $\gamma'(0) = 0 \Leftrightarrow \frac{\partial u}{\partial \vec{\tau}} = 0 \Leftrightarrow T_u(Gu) = \{0\}$ iff $u(x_1, x_2)$ is constant along the circle, i.e., $u$ is radial or $u \in H_G$. ∎

Consequently $u \in H_G$ implies $T_u(Gu) = \{0\}$ and then $u$ is not $G$-degenerate. Conversely, $u \in H \setminus H_G$ implies that $Gu$ is a differentiable path in $H$.

**Lemma 2.5.** *Assume $J \in C^2(H, \mathbb{R})$ is $\mathcal{G}$-invariant and $G$ is a differentiable Abelian subgroup of $\mathcal{G}$. For any $u \notin H_G$ with $J'(u) \in H_G$, then $\{0\} \neq T_u(Gu) \subseteq \ker(J''(u))$, i.e., $u$ is $G$-degenerate.*

**Proof.** For each $v (\neq 0) \in T_u(Gu)$, there is a scalar $\alpha \neq 0$ s.t. $\gamma'(0) = \alpha v$ and $\gamma(t) = g(t)u$, where $g(t) \in G$. $J$ is $\mathcal{G}(G)$-invariant implies that $J'$ is $\mathcal{G}(G)$-equivariant. Since $J'(u) \in H_G$, $J'(\gamma(t)) = J'(g(t)u) = g(t)J'(u) = J'(u)$. Differentiating $J'(\gamma(t))$ at $t = 0$ yields $J''(u)\gamma'(0) = \frac{d}{dt}|_{t=0} J'(\gamma(t)) = J''(u)(\alpha v) = 0$, i.e., $v \in \ker(J''(u))$. ∎

Lemma 2.5 not only provides us a sufficient condition for checking $G$-degeneracy but also suggests that any nonradial critical point of a $G$-invariant functional $J$ on a radial domain is $G$-degenerate. Next, we define subgroups $G_1 = \{e, h_1\}$ and $G_2 = \{e, h_2\}$ of $\mathcal{G}$, where $h_1(u)(r, \theta) = u(r, -\theta)$ and $h_2(u)(r, \theta) = -u(r, -\theta)$ for each $u \in H$ and each $x = (r, \theta) \in \Omega \subset \mathbb{R}^2$. Then $h_1$ ($h_2$) represents an even (odd) reflection, and hence $H_{G_1}$ ($H_{G_2}$) represents the invariant subspace of $H$ consisting of even (odd) functions. Clearly, $H_{G_1} \perp H_{G_2}$.

**Theorem 2.3.** *Let $G$ be defined as in* (2.4) *and $J \in C^2(H, \mathbb{R})$ be $G$-invariant. Then $T_u(Gu) \subseteq H_{G_2}$, $\forall u \in H_{G_1}$. Moreover, if $u \in H_{G_1}$ s.t.* $\ker(J''(u)) \subseteq T_u(Gu)$, *then $J''(u) : H_{G_1} \to H_{G_1}$ is invertible.*

**Proof.** For each $v(\neq 0) \in T_u(Gu)$, the one-parameter differentiable curve $\gamma$ defined by $Gu$ satisfies $\gamma(t)(r,\theta) = (g(t)u)(r,\theta) = u(r,\theta + t)$, $\gamma(0) = u$ and $\gamma'(0) = \alpha v$ for some $\alpha \neq 0$. Since $u \in H_{G_1}$,

$$\alpha v(r,-\theta) = \gamma'(0)(r,-\theta) = \frac{d}{dt}\bigg|_{t=0} u(r,-\theta + t) = \frac{d}{dt}\bigg|_{t=0} u(r,\theta - t)$$
$$= -\gamma'(0)(r,\theta) = -\alpha v(r,\theta),$$

i.e., $v \in H_{G_2}$. Thus $T_u(Gu) \subseteq H_{G_2}, \quad \forall u \in H_{G_1}$.

To prove the second part, let $u \in H_{G_1}$ s.t. $\ker(J''(u)) \subseteq T_u(Gu)$. By the first part, we obtain $\ker(J''(u)) \subseteq T_u(Gu) \subseteq H_{G_2}$. Since $H_{G_1} \perp H_{G_2}$, we have $\ker(J''(u)) \perp H_{G_1}$. Thus $J''(u) : H_{G_1} \to H_{G_1}$ is 1-1 and the equation $J''(u)v = w$ is always solvable in $H$ for each $w \in H_{G_1}$. Applying the Haar integral both sides over $G_1$ yields $J''(u)v_{G_1} = w$, where $v_{G_1} \in H_{G_1}$ represents the Haar projection of $v$ onto $G_1$. Hence, $J''(u) : H_{G_1} \to H_{G_1}$ is also onto. ∎

**Corollary 2.2.** *Under assumptions in Theorem 2.3, if, in addition, $u \in H_{G_1} \backslash H_G$ s.t. $J'(u) \in H_G$, then (i)* $\ker(J''(u)) = T_u(Gu) \neq \{0\}$; *(ii) $J''(u) : H_{G_1} \to H_{G_1}$ is invertible.*

**Proof.** By Lemma 2.5, $\{0\} \neq T_u(Gu) \subseteq \ker(J''(u))$. Then, with $\ker(J''(u)) \subseteq T_u(Gu)$, one obtains (i). For (ii), see Theorem 2.3. ∎

In Theorem 2.3 and Corollary 2.2, $G_1$ and $G_2$ are interchangeable. For any nonradial critical point $u$ of $J$, by Lemma 2.5, $u$ is $G$-degenerate and hence $\ker(J''(u)) \neq \{0\}$, i.e., $J''(u)$ is not invertible in $H$. However, when such a degeneracy is caused only by the subgroup $G$, then by Theorem 2.3 or Corollary 2.2, $J''(u)$ becomes invertible in the subspace $H_{G_1}$ or $H_{G_2}$. Thus $G$-degeneracy can be avoided by using an even or odd reflection symmetry in numerical computations.

## III. APPLICATION TO SEMILINEAR ELLIPTIC PDE

The Henon equation and its energy functional in (1.1) can be generalized as below

$$-\Delta u(x) = f(x, u(x)) \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0,$$
$$J(u) = \int_\Omega \left[\frac{1}{2}|\nabla u(x)|^2 - F(x, u(x))\right] dx \tag{3.1}$$

where $\Omega \subset \mathbf{R}^N (N \geq 2)$ is an open bounded domain, $F(x,t) = \int_0^t f(x,\tau)d\tau$ and $f$ is a $C^1$ function satisfying

**(h1)** $\forall \varepsilon > 0$ there is $c_1 = c_1(\Omega)$ s.t.

$$\left|f'_u(x,u)\right| \leq \varepsilon + c_1|u|^{s-1}, \ \forall x \in \Omega, u \in \mathbb{R}, \text{ where } 1 < s < 2^* = \frac{N+2}{N-2} \text{ (see [2]);}$$

**(h2)** $\frac{f(x,t\xi)}{t\xi} \nearrow +\infty$ monotonically in $t$ or $f'_u(x,t\xi) > \frac{f(x,t\xi)}{t\xi}$, $\forall t > 0$, for a.e. $x \in \Omega$.

We seek multiple weak solutions to (3.1) in $H = H_0^1(\Omega) = W_0^{1,2}(\Omega)$, which correspond to multiple critical points of $J$. Let $\|\cdot\|$ be the norm induced by the inner product $\langle\cdot,\cdot\rangle$ in $H$.

### A. The Gradient and Newton Direction

Since the dual of $H$ is $W^{-1,2}(\Omega)$, the Frechet derivative of $J$ as in (3.1) at each $u \in H$ is $J'(u) = -\Delta u - f(x,u) \in W^{-1,2}(\Omega)$. To increase the smoothness for numerical computations, we find the gradient $\nabla J(u) = \Delta^{-1}(-J'(u)) = u + \Delta^{-1}(f(x,u)) \in H$, the canonical identity of $J'(u)$ in $H$. By taking the second derivative, the Newton direction $\nu$ of $J$ at $u$ can be obtained from solving the equation

$$\begin{cases} -\Delta \nu(x) - f'_u(x,u(x))\nu(x) = -\Delta u(x) - f(x,u(x)), & x \in \Omega, \\ \nu(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

It is known that a Newton direction is independent of the norm designated for the space.

In this paper, we will discuss three types of homotopy methods, i.e.,

$$\mathcal{H}(\lambda,u) = \lambda\nabla J(u) + (1-\lambda)\nabla Q(u) \quad \text{(the convex homotopy)}, \tag{3.3}$$

$$\mathcal{H}(\lambda,u) = \nabla J(u) - (1-\lambda)\nabla J(u_0) \quad \text{(the global homotopy)}, \tag{3.4}$$

$$\mathcal{H}(\lambda,u) = \lambda\nabla J(u) + (1-\lambda)J''(u_0)(u-u_0) \quad \text{(the mixed homotopy)}, \tag{3.5}$$

corresponding respectively to (1.5), (1.7), (1.8). Here and henceforth, we choose $\lambda = \lambda(s) \equiv s$ with $s \in [0,1]$ and treat $J''(\cdot) \in L(H,H)$ because of the canonical identity relationship between $H$ and $W^{-1,2}(\Omega)$.

### B. Continuation on Mountain Pass Structure

Motivated by the mountain pass lemma [2], a functional $J \in C^1(H,\mathbb{R})$ is said to have a mountain pass structure (MPS) if there is $t_0 > 0$ s.t. for each $v \in H$ with $\|v\| = 1$, it holds

$$+\infty > t_v = \arg\max_{t>0} J(tv) > t_0.$$

Consider a nonlinear functional of the form

$$J(u) = \frac{1}{2}\|u\|^2 - h(u) = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx - h(u), \quad u \in H, \tag{3.6}$$

where $h \in C^2(H,\mathbb{R})$ satisfies the following conditions
**(h1')** $h(u) = o(\|u\|^2)$ as $\|u\| \to 0$ or $\|u\| \to +\infty$;
**(h2')** $\frac{\langle\nabla h(tu),u\rangle}{tu^2}$ or $\frac{\langle\nabla h(tu),u\rangle}{t} \to +\infty$ monotonically in $t > 0$, $\quad \forall u \in H, u \neq 0$.

In this case, (h2') $\Longleftrightarrow$ (h2) and we have

$$\langle\nabla J(u),v\rangle = \int_\Omega (-\Delta u - \nabla h(u))v\,dx = 0, \forall v \in H \Longleftrightarrow \Delta u + \nabla h(u) = 0,$$

$$\frac{d}{dt}J(tu) = \int_\Omega t|\nabla u|^2 dx - \langle\nabla h(tu),u\rangle = t - \langle\nabla h(tu),u\rangle,\ \forall u \in H, \|u\| = 1.$$

By (h1') and (h2'), there is a unique $t_u > 0$ s.t. $\frac{d}{dt}\big|_{t=t_u} J(tu) = 0$. Then (h2') implies

$$\frac{1}{t^2}(t\langle h''(tu)u, u\rangle - \langle \nabla h(tu), u\rangle) > 0 \quad \text{or} \quad \langle h''(tu)u, u\rangle > \frac{\langle \nabla h(tu), u\rangle}{t}.$$

Thus

$$\frac{d^2}{dt^2}\bigg|_{t=t_u} J(tu) = 1 - \langle h''(t_u u)u, u\rangle < 1 - \frac{\langle \nabla h(t_u u), u\rangle}{t_u} = 0.$$

By the implicit function theorem, the Nehari (solution) manifold

$$\mathcal{N} = \left\{ t_u u : u \in H, \|u\| = 1, \frac{d}{dt}\bigg|_{t=t_u>0} J(tu) = \langle \nabla J(t_u u), u\rangle = 0 \right\}$$

is a continuously differentiable manifold, which is the boundary of the set

$$\mathcal{N}^I = \{tu : u \in H, \|u\| = 1, 0 \le t \le t_u, t_u u \in \mathcal{N}\}.$$

It is known that any nontrivial critical point of $J$ must be on $\mathcal{N}$. Since zero is a minimum point of $J$ and $t_u u$ is the only local maximum point of $J$ along the direction of $u$, there is a unique $t_u^0 : 0 < t_u^0 < t_u$ s.t. $t_u^0 u$ is an inflection point of $J$ along the direction of $u$, i.e., $\frac{d^2}{dt^2}\big|_{t=t_u^0} J(tu) = 0$. Denote the manifold of inflection points of $J$ by

$$\mathcal{S} = \left\{ t_u^0 u : \|u\| = 1, \frac{d^2}{dt^2}\bigg|_{t=t_u^0} J(tu) = \left\langle J''\left(t_u^0 u\right)u, u\right\rangle = 0, 0 \le t_u^0 < t_u, t_u u \in \mathcal{N} \right\}, \tag{3.7}$$

then we have:

**Lemma 3.1.** *$\mathcal{S}$ is a continuous and closed manifold, where $J$ is given as in* (3.6).

**Proof.** $\mathcal{S}$ is closed since $J \in C^2$. Let $u_0, u_n \in H$ with $\|u_0\| = \|u_n\| = 1$ and $t^0 u_0, t_n^0 u_n \in \mathcal{S}$. We need to show that if $u_n \to u_0$ then $t_n^0 u_n \to t^0 u_0$, or equivalently, $\{t_n^0 u_n\}$ has a subsequence converging to $t^0 u_0$. To see this, one notes that the Nehari manifold $\mathcal{N}$ is continuously differentiable, thus it is locally bounded near $u_0$, i.e., there exist some $\eta > 0, M > 0$ s.t. for any $u \in H$ with $\|u\| = 1$ and $\|u - u_0\| < \eta$, $\|t_u u\| = t_u < M$ where $t_u u \in \mathcal{N}$. Since $u_n \to u_0$, there exists some $N$ s.t. $\|u_n - u_0\| < \eta, \forall n > N$. By the definitions of $\mathcal{N}$ and $\mathcal{S}$, we have $0 < t_n^0 < t_n < M, \forall n > N$, i.e., $t_n^0$ is bounded. Then there is a subsequence of $\{t_n^0\}$, still denoted by $\{t_n^0\}$, s.t. $t_n^0 \to t' \ge 0$. Since $J$ is $C^2$ and $t_n^0 u_n \in \mathcal{S}$ (i.e., $\langle J''(t_n^0 u_n)u_n, u_n\rangle = 0$), then $u_n \to u_0$ and $t_n^0 \to t'$ lead to $\langle J''(t' u_0)u_0, u_0\rangle = 0$, i.e., $t' u_0 \in \mathcal{S}$. By uniqueness, $t' = t^0$. Thus $t_n^0 u_n \to t^0 u_0$. ∎

Note that $\mathcal{S}$ is also the boundary of the following set

$$\mathcal{S}^I = \left\{ tu : u \in H, \|u\| = 1, 0 \le t \le t_u^0, t_u^0 u \in \mathcal{S} \right\}.$$

Since $0 \le t_u^0 < t_u$, we have $\mathcal{S}^I \subsetneq \mathcal{N}^I$. If, in addition, $0 \notin \mathcal{S}$, we see that $\mathcal{N}$ and $\mathcal{S}$ exhibit a MPS. In this case, by Lemma 3.1, $\mathcal{S}$ actually divides the whole space $H$ into two separate parts, i.e., the interior (containing the origin) and the exterior. For a general functional $J$, MPS may not exist even $0 \notin \mathcal{S}$.

Next, we introduce a homogeneous and stronger condition
**(h3')** $\exists 2 < k < 2^* + 1$ s.t. $h(tu) = |t|^k h(u), \forall u \in H, \forall t \in \mathbb{R}$, and $h(u) > 0, \forall u \neq 0$.
It is easy to check that (h3') implies both (h1')-(h2') and (h1)-(h2).

**Lemma 3.2.** *For a functional $J$ of the form* (3.6)*, if $h(\cdot)$ satisfies* (h3') *and $t^0 \equiv \inf\{t_u^0 : \|u\| = 1, t_u^0 u \in \mathcal{S}\} > 0$, then $\inf\{\|\nabla J(t_u^0 u)\| : \|u\| = 1, t_u^0 u \in \mathcal{S}\} > 0$.*

**Proof.** By assumption, for each $u \in H$ with $\|u\| = 1$, $\exists\, t_u^0 > 0$ s.t. $t_u^0 u \in \mathcal{S}$. From (h3'), $J(t_u^0 u) = \frac{(t_u^0)^2}{2} - (t_u^0)^k h(u)$ and $\langle \nabla J(t_u^0 u), u\rangle = t_u^0 - k(t_u^0)^{k-1} h(u) = t_u^0 - \frac{k}{t_u^0} h(t_u^0 u)$. Then

$$
\begin{aligned}
0 &= \left\langle J''\left(t_u^0 u\right)u, u\right\rangle = 1 - k(k-1)\left(t_u^0\right)^{k-2} h(u) \\
&= \frac{1}{t_u^0}(k-1)\left(t_u^0 - \frac{k}{t_u^0} h\left(t_u^0 u\right)\right) - (k-1) + 1 = \frac{k-1}{t_u^0}\left\langle \nabla J\left(t_u^0 u\right), u\right\rangle - k + 2,
\end{aligned}
$$

i.e., $\langle \nabla J(t_u^0 u), u\rangle = \frac{k-2}{k-1} t_u^0$. Thus $\|\nabla J(t_u^0 u)\| \geq \langle \nabla J(t_u^0 u), u\rangle = \frac{k-2}{k-1}\, t_u^0 \geq \frac{k-2}{k-1}\, t^0 > 0$. ∎

If replacing $h(u)$ in (3.6) by

$$h_1(u) = \int_\Omega F(x, u(x))dx$$

where $F(x,t) = \int_0^t f(x,\tau)d\tau > 0$ $(\forall t > 0)$ and $f(x,\cdot)$ satisfies $f(x,tu) = |t|^{p-1} t f(x,u)$ for some $1 < p < 2^*$, then $h_1(\cdot)$ satisfies (h3') with $k = p + 1 > 2$. Denote the target functional

$$J_1(u) = \frac{1}{2}\|u\|^2 - h_1(u). \tag{3.8}$$

For the convex homotopy as in (1.5) or (3.3), there are numerous ways to choose an initial functional $J_0(\cdot)$ with a point $u_0$ being its critical point. For instance, one can replace the term $h(u)$ in (3.6) with a nonlinear term $h_0(u) = \frac{1}{q}(\int_\Omega \frac{1}{2} u^2(x)dx)^q$ and set

$$J_0(u) = \frac{1}{2}\|u\|^2 - h_0(u), \tag{3.9}$$

where $q = (p+1)/2 > 1$ and $p$ is defined as in $h_1(u)$. Clearly, $h_0(\cdot)$ also satisfies (h3') with $k = 2q = p + 1 > 1$.

**Fact 3.1.** *$(\sigma, u) \in \mathbb{R} \times H$ is an eigenpair of $-\Delta$ if and only if there is $\tau_u > 0$ s.t. $\sigma = \left[\int_\Omega \frac{1}{2}\tau_u^2 u^2(x)dx\right]^{q-1}$ and $\tau_u u \in H$ is a critical point of $J_0(\cdot)$ as in* (3.9).

**Proof.** Since $u \neq 0$, there is $\tau_u > 0$ s.t. $\sigma = \left[\int_\Omega \frac{1}{2}\tau_u^2 u^2(x)dx\right]^{q-1}$ and for any $v \in H$,

$$
\begin{aligned}
&\int_\Omega (-\Delta \tau_u u(x) - \sigma \tau_u u(x))v(x)dx, \\
&= \int_\Omega \left(-\tau_u \Delta u(x) - \left[\int_\Omega \frac{1}{2}\tau_u^2 u^2(x)dx\right]^{q-1} \tau_u u(x)\right) v(x)dx = \langle \nabla J_0(\tau_u u), v\rangle.
\end{aligned}
$$
∎

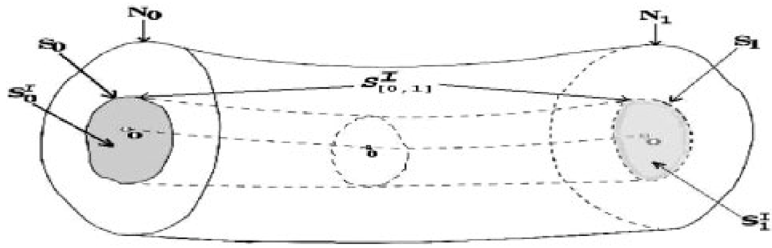


FIG. 4. Continuation on a mountain pass structure by a convex homotopy.

Now we can define a convex homotopy

$$\mathcal{H}(\lambda, u) = \nabla J_\lambda(u) = (1-\lambda)\nabla J_0(u) + \lambda \nabla J_1(u), \lambda \in [0,1], \tag{3.10}$$

where $J_1(u)$, $J_0(u)$ are given respectively as in (3.8), (3.9) and

$$J_\lambda(u) = (1-\lambda)J_0(u) + \lambda J_1(u).$$

If denoting $h_\lambda(u) = (1-\lambda)h_0(u) + \lambda h_1(u)$, then $J_\lambda(u) = \frac{1}{2}\|u\|^2 - h_\lambda(u)$ and $h_\lambda$ satisfies (h3').

Next, let $c(\lambda) = (\lambda, u(\lambda))\,(0 \le \lambda \le 1)$ be a solution trajectory implicitly defined by $\mathcal{H}(\lambda, u(\lambda)) = 0$ where $\mathcal{H}(\cdot,\cdot)$ is given as in (3.10). For each $\lambda \in [0,1]$, denote the Nehari manifold $\mathcal{N}_\lambda$ and the set $\mathcal{S}_\lambda$ of inflection points of $J_\lambda$ by

$$\mathcal{N}_\lambda = \left\{ t_u u : \|u\| = 1, \left.\frac{d}{dt}\right|_{t=t_u>0} J_\lambda(tu) = \langle \mathcal{H}(\lambda, t_u u), u\rangle = 0 \right\},$$

$$\mathcal{S}_\lambda = \left\{ t_u^0 u : \|u\| = 1, \left.\frac{d^2}{dt^2}\right|_{t=t_u^0} J_\lambda(tu) = \left.\frac{d}{dt}\right|_{t=t_u^0} \langle \mathcal{H}(\lambda, tu), u\rangle = 0, 0 \le t_u^0 < t_u, t_u u \in \mathcal{N}_\lambda \right\}.$$

Then $\mathcal{N}_\lambda$ and $\mathcal{S}_\lambda$ are respectively the boundaries of the sets

$$\mathcal{N}_\lambda^I = \{tu : 0 \le t \le t_u, t_u u \in \mathcal{N}_\lambda\} \quad \text{and} \quad \mathcal{S}_\lambda^I = \left\{tu : 0 \le t \le t_u^0, t_u^0 u \in \mathcal{S}_\lambda\right\}.$$

If define a "cylinder" $\mathcal{S}_{[0,1]}^I$ (refer to Fig. 4) by

$$\mathcal{S}_{[0,1]}^I = \left\{ \{\lambda\} \times \mathcal{S}_\lambda^I : \lambda \in [0,1] \right\}, \tag{3.11}$$

then we have the following theorem:

**Theorem 3.1.** $t_\lambda^0 = \inf\{t_u^0 : \|u\| = 1, t_u^0 u \in \mathcal{S}_\lambda\} > 0,\ \forall \lambda \in [0,1]$. *Consequently, any implicit continuous solution trajectory* $\{c(\lambda) : \lambda \in [0,1]\} \subset \mathcal{H}^{-1}(0)$ *will stay outside the "cylinder"* $\mathcal{S}_{[0,1]}^I$ *for each* $u_0 \in H \backslash \mathcal{S}_0^I$ *s.t.* $\mathcal{H}(0, u_0) = 0$, *where* $\mathcal{H}$ *is given as in* (3.10).

**Proof.** Only need to verify the first conclusion. The second conclusion follows directly from the first conclusion and Lemmas 3.1-3.2. With (h3'), for each $0 \le \lambda \le 1$, we have

$$J_\lambda(tu) = \frac{t^2}{2} - h_\lambda(tu) = \frac{t^2}{2} - t^k[(1-\lambda)h_0(u) + \lambda h_1(u)], \forall t \ge 0, \quad \text{and}$$

$$0 = \frac{d^2}{dt^2}\bigg|_{t=t_u^0 \ge 0} J_\lambda(tu) = 1 - k(k-1)\left(t_u^0\right)^{k-2}[(1-\lambda)h_0(u) + \lambda h_1(u)]$$

(By integrating the inequality in (h1) twice)

$$\ge 1 - k(k-1)\left(t_u^0\right)^{k-2}\left[\frac{1-\lambda}{q}\left(\int_\Omega \frac{1}{2}u^2 dx\right)^q + \lambda \int_\Omega [\epsilon|u|^2 + c_1|u|^{s+1}]dx\right]$$

(By the Poincare and Sobolev inequalities, there exist positive constants $c_2, c_3$ that depend only on $\Omega$)

$$\ge 1 - k(k-1)\left(t_u^0\right)^{k-2}\left[\frac{(1-\lambda)c_2}{q2^q}\left(\int_\Omega |\nabla u(x)|^2 dx\right)^q\right.$$

$$\left. + \lambda\left(\varepsilon c_2 \int_\Omega |\nabla u(x)|^2 dx + c_1 c_3 \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{((s+1)/2)}\right)\right]$$

$$= 1 - k(k-1)\left(t_u^0\right)^{k-2}\left[\frac{(1-\lambda)c_2}{q2^q} + \lambda(\varepsilon c_2 + c_1 c_3)\right] \text{ for some } k > 2$$

$$\ge 1 - k(k-1)\left(t_u^0\right)^{k-2} \max\left[\frac{c_2}{q2^q}, \varepsilon c_2 + c_1 c_3\right] \text{ for some } k > 2.$$

Hence, $t_\lambda^0 > 0, \forall \lambda \in [0,1]$. ∎

The above theorem suggests that the convex homotopy (3.10) gives a continuation on a MPS, see Fig. 4, and that the manifold of inflection points $\mathcal{S}_\lambda$ serves as a "barrier" which a continuous solution trajectory can never pass through. Note that for the convex homotopy (3.10), it involves solving a differential-integral equation to find the Newton direction, therefore it is very expensive and difficult for numerical realization.

### C. Continuation between Manifolds of Inflection Points

In this section, we discuss a global homotopy method. Let us first prove the following lemma.

**Lemma 3.3.** *For the functional $J$ as in (3.1),*

$$t^0 = \inf\left\{t_u^0 : u \in H, \|u\| = 1, t_u^0 u \in \mathcal{S}\right\} > 0,$$

*where the inflection manifold $\mathcal{S}$ is defined as in (3.7). Consequently, $J$ has a MPS.*

**Proof.** By (h1), for any small $\varepsilon > 0$, there is $c_1 = c_1(\Omega)$ s.t. $|f_u'(x,u)u^2| \le \varepsilon|u|^2 + c_1|u|^{s+1}$ $(1 < s < 2^*)$. Then for each $u \in H$ with $\|u\| = 1$, $0 = \frac{d^2}{dt^2}|_{t=t_u^0} J(tu) =$

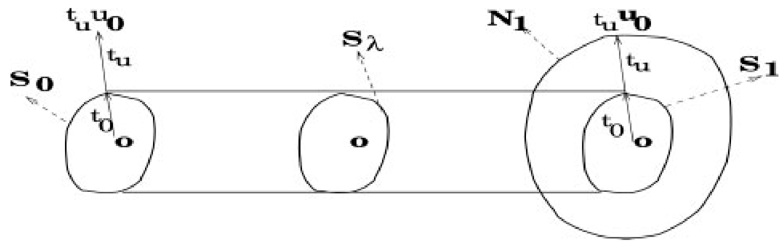


FIG. 5. Homotopy continuation between the (identical) manifolds of inflection points $\mathcal{S}_0$ and $\mathcal{S}_1$.

$1 - \int_\Omega f_u'(x, t_u^0 u(x)) u^2(x) dx$ leads to

$$1 \le \int_\Omega \left| f_u'\big(x, t_u^0 u(x)\big) u^2(x) \right| dx \le \int_\Omega \left( \varepsilon |u(x)|^2 + c_1 \big(t_u^0\big)^{s-1} |u(x)|^{s+1} \right) dx$$

(By the Poincare and Sobolev inequalities, $\exists c_0 = c_0(\Omega), c_2 = c_2(\Omega)$)

$$\le \varepsilon c_0 \int_\Omega |\nabla u(x)|^2 dx + c_1 c_2 \big(t_u^0\big)^{s-1} \left( \int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{s+1}{2}} = \varepsilon c_0 + c_1 c_2 \big(t_u^0\big)^{s-1}.$$

Since $\varepsilon$ is arbitrary, we have $t_u^0 \ge \left[\frac{1-\varepsilon c_0}{c_1 c_2}\right]^{1/s-1} \ge \left[\frac{1}{2c_1 c_2}\right]^{1/s-1} > 0$. The proof is complete. ■

Now, let our target functional $J_1(u) = J(u)$ where $J(u)$ is defined in (3.1). For a preferred non-critical point $u_0 \in H$ of $J_1$, define an initial functional $J_0(u) = J_1(u) - \langle \nabla J_1(u_0), u \rangle$. Then the global homotopy reads as

$$\mathcal{H}(\lambda, u) = \nabla J_\lambda(u) = (1-\lambda)\nabla J_0(u_0) + \lambda \nabla J_1(u) = \nabla J_1(u) - (1-\lambda)\nabla J_1(u_0), \qquad (3.12)$$

where $J_\lambda(u) = (1-\lambda) J_0(u_0) + \lambda J_1(u)$. Clearly, $u_0$ is a critical point of $J_0$ and

$$\mathcal{H}_2'(\lambda, u) = J_1''(u), \quad \forall \lambda \in [0, 1]. \qquad (3.13)$$

Although, by Lemma 3.3, $J_1$ has a MPS, the initial functional $J_0$ which depends on the point $u_0$ may fail to have a MPS. Thus, we approach this case differently. Thanks to (3.13), for each $\lambda \in [0, 1]$, we have $\mathcal{S}_\lambda = \mathcal{S}_1$ where, as before, $\mathcal{S}_\lambda$ is the set of inflection points of $J_\lambda$. Hence, the global homotopy (3.12) gives an identical continuation between $\mathcal{S}_0$ and $\mathcal{S}_1$ as shown in Fig. 5.

For a moment, let $J_1$ be a more general target functional (not necessarily taking the form as in (3.1)) s.t. $\mathcal{S}_1$ is a continuous and closed manifold and divides the whole space $H$ into two disjoint parts, i.e., the interior $H_S^I$ and the exterior $H_S^E$. Let $\{c(\lambda) = (\lambda, u(\lambda)) : \lambda \in [0, 1]\} \subset \mathcal{H}^{-1}(0)$ be a continuous solution trajectory implicitly defined by the homotopy (3.12), then we have the following theorem.

**Theorem 3.2.** *Assume $r_1 = \inf_{u \in \mathcal{S}_1} \|\nabla J_1(u)\| > 0$. If the initial point $u_0 \in H$ such that $\|\nabla J_1(u_0)\| < r_1$, then $u(\lambda) \notin \mathcal{S}_\lambda, \forall \lambda \in [0, 1]$; consequently, the solution path $\{u(\lambda) : \lambda \in [0, 1]\}$ will never cross the manifold $\mathcal{S}_1 = \mathcal{S}_0$, i.e., it belongs to either $H_S^I$ or $H_S^E$.*

**Proof.** For each $\lambda \in [0, 1]$, we have

$$\begin{aligned} r_\lambda &= \inf_{u\in\mathcal{S}_1} \|\mathcal{H}(\lambda, u)\| = \inf_{u\in\mathcal{S}_1} \|\nabla J_1(u) - (1-\lambda)\nabla J_1(u_0)\| \\ &\geq \inf_{u\in\mathcal{S}_1} |\|\nabla J_1(u)\| - \|(1-\lambda)\nabla J_1(u_0)\|| = \inf_{u\in\mathcal{S}_1} \|\nabla J_1(u)\| - \|(1-\lambda)\nabla J_1(u_0)\| \\ &= r_1 - (1-\lambda)\|\nabla J_1(u_0)\| \geq r_1 - \|\nabla J_1(u_0)\| > 0, \end{aligned}$$

i.e., $\|\mathcal{H}(\lambda, u)\| > 0, \forall u \in \mathcal{S}_\lambda$. Since $\mathcal{H}(\lambda, u(\lambda)) = 0, \forall\lambda \in [0, 1]$, the conclusion follows. ∎

In the above theorem, one requires $r_1 > 0$, a condition which can be verified by the following corollary.

**Corollary 3.1.** *For the function $f(x, u)$ as in* (3.1)*, if, in addition, $f(x, tu) = |t|^{p-1}tf(x, u)$ for some $p > 1$, then the global homotopy* (3.12) *has the following properties*

a. $t^0 = \inf\{t_u^0 : u \in H, \|u\| = 1, t_u^0 u \in \mathcal{S}_1\} > 0$*, i.e., $J_1$ has a MPS;*
b. $r_1 = \inf\{\|\nabla J_1(u)\| : u \in \mathcal{S}_1\} > 0$*;*
c. *The conclusions of Theorem* 3.2 *are applied.*

**Proof.** (a) See Lemma 3.3. (b) Denote $h(u) = \int_\Omega F(x, u)dx$, then $h$ satisfies (h3') and (b) follows from Lemma 3.2. (c) By Lemma 3.1, $\mathcal{S}_1$ is a continuous closed manifold and hence divides $H$ into two disjoint parts. With (b), conclusions of Theorem 3.2 follow. ∎

Corollary 3.1 states that, for the target functional $J$ as in (1.1) which possesses a MPS, if a point $u_0 \in \mathcal{S}_0^I$ with $\|\nabla J(u_0)\| < r_1$ is chosen as an initial solution for NHCM (1.7) or (3.4), then any continuous solution trajectory $\{(\lambda, u(\lambda)) : \lambda \in [0, 1], u(0) = u_0\}$ will be trapped in the "cyclinder" $\mathcal{S}_{[0,1]}^I$, and hence the final solution $u(1) = 0$ since 0 is the only critical point of $J$ in $\mathcal{S}_1^I$. Therefore, to find a nontrivial critical point of $J$, an initial solution $u_0$ had better be chosen outside $\mathcal{S}_0^I$, which can be easily fulfilled by choosing $u_0 \in \mathcal{N}_1$ due to the fact $\mathcal{S}_0^I = \mathcal{S}_1^I \subset \mathring{\mathcal{N}}_1^I$, the interior of $\mathcal{N}_1^I$. This explains why we obtained those numerical results in Case 2(b) in Section I, since wherein $tv_0 \in \mathcal{S}_1^I$ for (i) and $tv_0 \notin \mathcal{S}_1^I$ for (ii)–(iii).

## D. On the Mixed Homotopy Continuation Method

For a preferred point $u_0 \in H$, choose an initial functional $J_0(u) = \frac{1}{2}\langle J_1''(u_0)(u - u_0), u - u_0\rangle$ and define a mixed homotopy

$$\mathcal{H}(\lambda, u) = \lambda\nabla J_1(u) + (1-\lambda)J_1''(u_0)(u - u_0) \tag{3.14}$$

where the target functional $J_1$ takes the form as in (3.1). Then $u_0$ is a critical point of $J_0$ with $J_0''(u_0) = J_1''(u_0)$. Note $J_0(u)$ is the second order term of the Taylor expansion of $J_1(u)$ at $u_0$. If $J_1''(u_0)$ is nondegenerate, $u_0$ is the only critical point of $J_0$. Since, for this homotopy method, MPS can be completely altered/lost as $\lambda$ goes from 0 to 1, the "cylinder" $\mathcal{S}_{[0,1]}^I$ as in (3.11) may not be well-defined and hence the solution trajectory $\{(\lambda, u(\lambda)) : \lambda \in [0, 1], u(0) = u_0\}$ is not necessarily trapped. Also, due to high dependence of the nature of $J_1''(u_0)$, it is impossible to give a general analysis for (3.14).

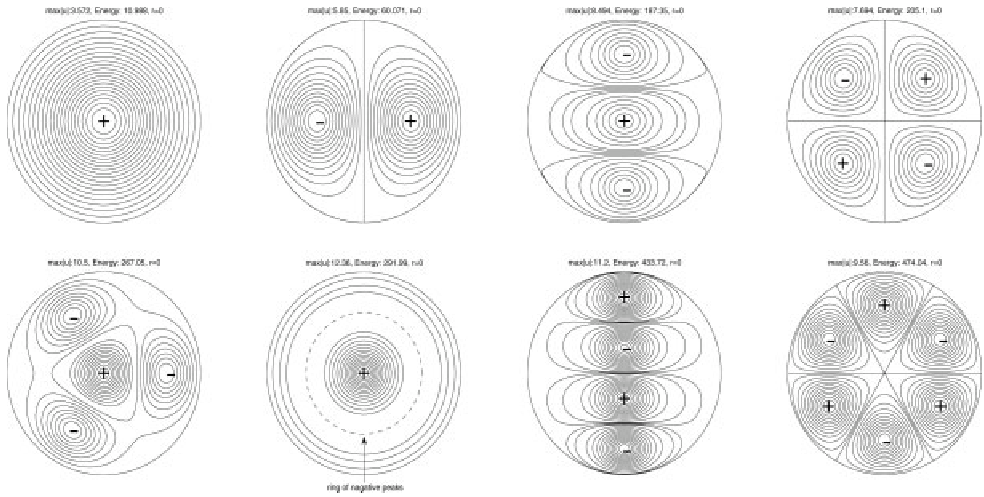


FIG. 6. The contours of the first 8 solutions to (1.1) with $r = 0, q = 3$. The sign "+/–" represents a positive/negative peak. The 6th contour plot is a radial sign-changing solution, where the dashed circle denotes a ring of negative peaks (a one-dimensional peak set).

## IV. MORE NUMERICAL RESULTS

In this section, based on the theoretical results obtained in the previous sections, we compute multiple solutions to the Henon equation (1.1) along the $r$-axis by using NHCM (1.7). The domain $\Omega$ used here is the unit disk in $\mathbb{R}^2$. Thus the problem (1.1) possesses many symmetries and all nonradial solutions are degenerate due to the differentiable group of rotation actions. We use $\varepsilon = 1 \times 10^{-5}$ to terminate iterations in our algorithm and employ a uniform partitioning over [0,1] with $\Delta\lambda \leq 0.04$ in NHCM (1.7).

i. By our analysis in Section III.C, the inflection manifold $\mathcal{S}_1$ acts like a "barrier". To avoid the trivial solution of (1.1), we choose an initial point $u_0 \notin \mathcal{S}_0^I = \mathcal{S}_1^I$. Here, we do not employ any perturbation technique as suggested in [4] to "jump over" a bifurcation point or to switch a branch. The main reason is that after "jumping over" or "switching", essential information of the initial point $u_0$ such as symmetry, the Morse index, is lost; secondly, even a solution is found through "jumping over" or "switching", it may not be a desirable one. For example, "jumping over" a bifurcation point may prevent one from finding the nonradial ground state as in Fig. 2 (left) due to the change of the Morse index.

ii. A nonradial $G$-degenerate solution, where the differential group $G$ is defined as in (2.4), can be successfully approximated by using its odd (even) symmetry w.r.t. certain axis, due to the invariance of NHCM (Section II.A) and the solvability of the Newton direction on $H_{G_1}$ ($H_{G_2}$) (Theorem 2.3). Thus a more complicated Haar projection can be avoided in numerical computations.

iii. To obtain a bifurcation diagram (Fig. 9) of multiple solutions to (1.1) in $r$, we first compute the first 8 solutions (Fig. 6) for $r = 0$ following a sequential order (the Morse index). With $r$ increasing gradually, a branch is found by using initial points with the same or similar (nodal) structures of one of those eight solutions. Meanwhile, for a fixed $r$, we use an initial point whose symmetries are exactly the same as those of one of the 8 solutions to capture

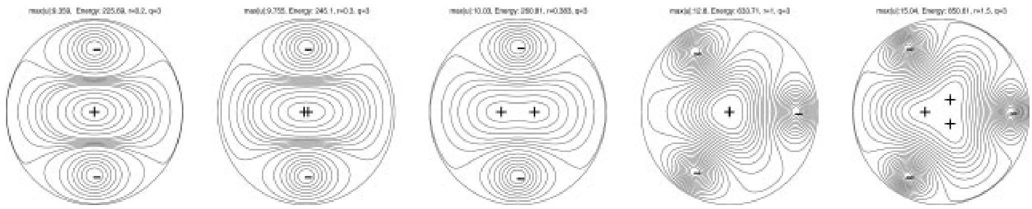

FIG. 7. Peak-breaking on sign-changing solutions. A peak breaks from one to two (1–3) and a peak breaks from one to three (4–5).

a solution created by peak-breaking and use initial points with less and less symmetries to locate solutions caused by symmetry-breaking.

In our numerical experiments, both symmetry-breaking and peak-breaking phenomena have been observed. Figure 6 lists the first eight solutions of (1.1) for $r = 0$, $q = 3$. Except the 1st and 6th radial solutions, all others (nonradial solutions) are representatives of their group orbits. Figure 7 shows peak-breaking phenomena, where the first three contours display a process that the central peak breaks from one to two and the last two contours illustrate a picture where the central peak breaks from one to three. Before and after the peak-breaking process, the symmetries remain the same. Figure 8 shows three solutions which bifurcate from the sign-changing radial solution as in Fig. 6 (6th) through symmetry-breaking.

Figure 9 is a bifurcation diagram of multiple solutions of (1.1) along the r-axis for $q = 3$. For convenience, we label each solution. The reader can also see Fig. 6 (Fig. 8) for the contour plots of solutions 1–8 (solutions 10–11), and see Figs. 2 and 3 for the profiles and contour plots of solutions 1'-4'. From the bifurcation diagram, one can see that whenever a symmetry-breaking occurs, new branches of solutions with fewer symmetries and lower energy levels are produced by certain main branch. For example, at the bifurcation point $a$ in the figure, a new branch S1 consisting of all nonradial ground states emerges from the main branch P1. For the case of a radial domain, once a symmetry-breaking takes place, infinitely many new branches are created simultaneously. Meanwhile, when a peak-breaking happens, a solution changes its own structure in number of peaks but not in symmetries, see also Fig. 7.

In Fig. 9, the branches denoted by the solid lines (see P1–P5) are called the *primary branches* due to two reasons. First, for each solution in those branches, there is a corresponding eigenfunction of $-\Delta$ in $H$ with a similar (nodal) structure. Secondly, the existence of those branches is independent of the parameter $r$. The dashed lines denote the *secondary branches* (see S1–S4 and B1–B5), which either bifurcate from the primary branches or exist only when the parameter $r$ is

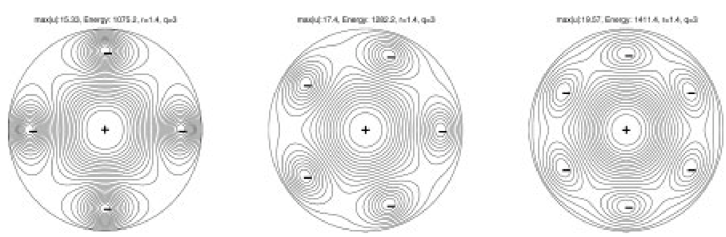

FIG. 8. Contour plots of three nonradial sign-changing solutions created by symmetry-breaking for $r = 1.4$ and $q = 3$.

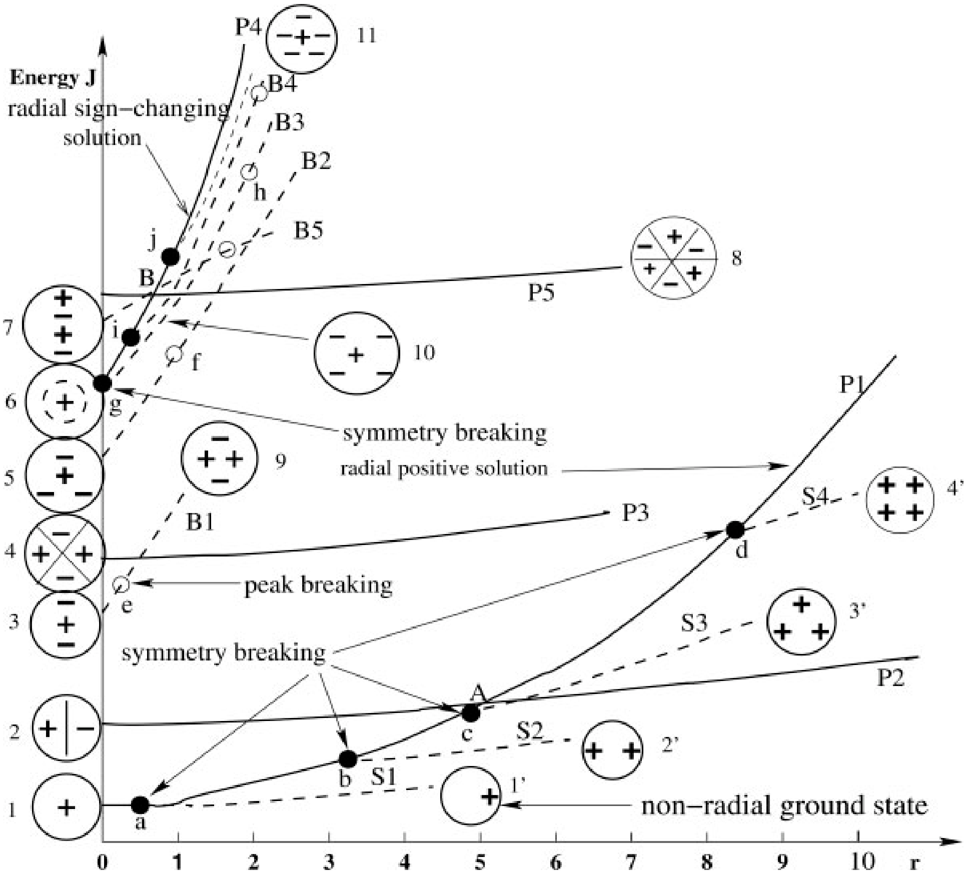


FIG. 9. A bifurcation diagram of solutions to (1.1) along the $r$-axis for $q = 3$. The sign "+/−" represents a positive/negative peak. The dots or little circles indicate where symmetry-breaking or peak-breaking takes place, respectively. For convenience, each solution is labelled. Letters A and B show where the primary branches P1, P2 and P4, P5 switch their order in energy levels. S1–S4 are the branches created by symmetry-breaking.

in certain range. For solutions on those secondary branches, there is no eigenfunction of $-\Delta$ with similar nodal structures. Besides, all solutions on the secondary branches are nonradial. Of secondary branches, B1–B5 are the peak-breaking branches, i.e., a peak-breaking happens on those branches. One interesting phenomenon is that, while all symmetry-breaking is observed on radial primary branches, see a–d, g, i, j, a peak-breaking is observed only on the secondary branches, see B1–B5. For the primary branches with radial symmetry, see P1 and P4, both their energy levels and Morse indices grow and thus more unstable, as $r$ increases. But for the nonradial primary branches, see P2, P3, and P5, although their energy levels increase as well, their Morse indices remain the same as $r$ increases. This suggests that solutions with more symmetries become more unstable when $r$ increases. Thus symmetry-breaking constantly takes place on radial solutions. In this situation, the solution with the least symmetry is the nonradial ground state, the most stable solution, refer also to solution 1' in Figs. 9 or 2 (left).

Another interesting phenomenon is that, peak-breaking branches are "dead" branches, i.e., numerically we can no longer locate those branches/solutions when $r$ becomes large enough (for instance, $r > 3$). They vanish. Why? We may explain as follows. Take the peak-breaking branch B1, for example, the central positive peak of solution 3 first splits into twin peaks (see Figs. 7, 1–3) as $r \geq 0$ increases. Then these twin peaks become further apart as $r$ increases further. Meanwhile, when the distance of such twin peaks exceeds certain value, they become too unstable to sustain their positions due to the existence of the primary branch P3, which is right below the branch B1 for $r \geq 1$. As a compromise, the branch B1 vanishes when $r$ becomes large enough. Similar compromise happens for other peak-breaking branches.

## V. CONCLUSION

From the bifurcation diagram in Fig. 9, one can see that there are more solution patterns of the Henon equation (1.1) than the patterns of eigenfunctions of $-\Delta$. If the approach in [8–10] is used to solve (1.1), very likely, only the solutions on those primary branches can be approximated and accordingly all the solutions including the ground states on the secondary branches will be missed. The properties (such as continuation on symmetry, the Morse index, the mountain pass structure) of NHCM established in this paper will help us finding multiple desired solutions to semilinear elliptic PDEs and carrying out more intrinsic investigations for exploring new phenomena on those solutions (either nondegenerate or $G$-degenerate) and examining their qualitative behaviors, such as locating the points of symmetry/peak-breaking numerically.

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