Mean Field Convergence of a rate model of multiple TCP connections through a buffer implementing RED

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Summary

- N TCP/IP connections
- implementing TCP Reno
- with fast recovery and fast retransmit
- routed through a bottleneck queue in the departmental router.
- We assume the router implements RED.

Mean Field Limit of the window histogram

- $p(t,w)$ is the density of window sizes at time t

- $R(t)$ is the average round trip time.

- $$\frac{\partial p(t,w)}{\partial t} = -\frac{1}{R(t)}\frac{\partial p(t,w)}{\partial w} + \Big(4wp(t,2w) - wp(t,w)\Big)\frac{K(t-R(t))}{R(t-R(t))} \qquad (1)$$

Mean Field Limit of the queue

- $Q(t)$ is the relative queue size at time t

- $$\frac{dQ(t)}{dt} = \int_w \frac{w}{R(t)} p(t,w)\,dw\,(1 - K(t)) - C \qquad (2)$$

Steady state

- When the loss probability $K(t) = F(Q(t))$ i.e. RED then (1) and (2) may stabilize;

- The loss probability $K(t)$ tends to a k

- The window distribution stabilizes to a fixed distribution f_k,

- $Q(t)$ tends to a constant q

- The RTT, $R(t)$, tends to a constant r.

- For stable systems (1) and (2) become:

$$\frac{df_k(w)}{dw} = k\left(2(2w)f_k(2w) - wf_k(w)\right) \quad (3)$$

$$C = (1-k)\frac{1}{r}\int_w wf_k(w)dw. \quad (4)$$

Steady state density

- The unique density $f_k(w)$ solving (3) is given by

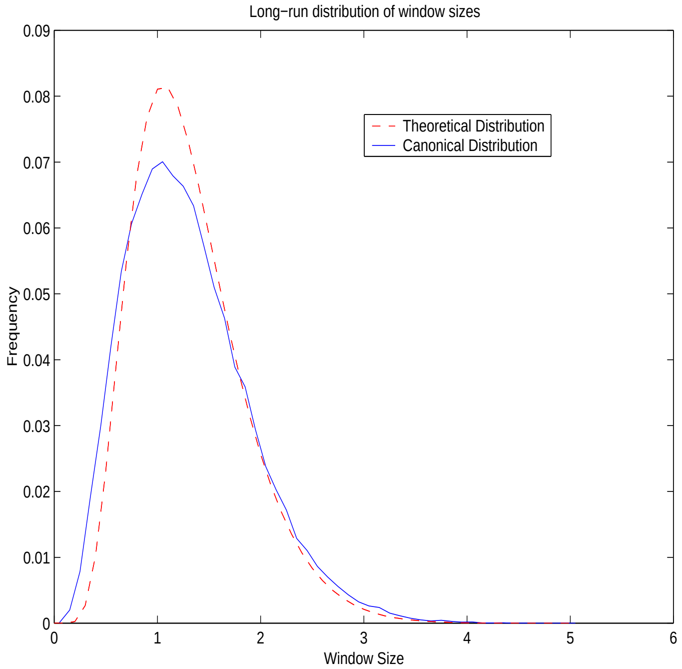
$$f_k(w) = \sum_{i=0}^{\infty} a_i \exp(-\frac{k}{1-k} 4^i \frac{w^2}{2}) \qquad (5)$$

- where

$$\begin{aligned} a_0 &= \sqrt{\frac{2}{\pi}} \frac{1}{\Psi} \sqrt{\frac{k}{1-k}}; \\ a_i &= a_{i-1} \frac{4}{1-4^i} = a_0 \frac{4^i}{\prod_{j=1}^{i}(1-4^j)}. (6) \end{aligned}$$

- and where $\Psi = \sum_{i=0}^{\infty} \frac{2^i}{\prod_{j=1}^{i}(1-4^j)}$ ($\Psi \approx 0.4194$).

The canonical window distribution



Model of the N window system

- The N window sizes $\mathbf{W}^N := (W_1^N, \ldots, W_N^N)$

- are N dynamical systems which evolve independently

- except through a shared resource Q^N.

- The randomness comes from the RED mechanism.

Model of the N window system II

To first order, the rate of a window reductions between time t and $t+h$ is

$$\frac{1}{h}\int_{t-R_n^N(t)}^{t+h-R_n^N(t+h)} \frac{W_n^N(s)}{R_n^N(s)} F(Q^N(s))ds$$
$$\sim \ [1-\frac{d}{dt}R_n^N(t)]\frac{W_n^N(t-R_n^N(t))}{R_n^N(t-R_n^N(t))}F(Q^N(t-R_n^N(t)))$$

- since the rate packets are dropped is proportional to $W_n^N(t-R_n^N(t))/R_n^N(t-R_n^N(t))$, the transmission rate one time in the past, times $F(Q^N(t-R_n^N(t)))$, the drop probability one round trip in the past.

- We omit the Doppler term $[1-\frac{d}{dt}R_n^N(t)]$

Model of the N window system II

- Therefore model the process of window reductions by a Poisson point process with stochastic intensity

$$\lambda_n^N(t) := \frac{W_n^N(t - R_n^N(t))}{R_n^N(t - R_n^N(t))} F(Q^N(t - R_n^N(t)))$$

- The Poisson assumption models the fact that under RED packets are dropped randomly with probability $p = F(Q^N(t))$ at time t and not, for instance, deterministically one every $1/p$ packets.

- We can construct the simple point process of window reductions:

$$N_n^N(t) = \int_0^t \int_0^\infty \chi_{[0,\lambda_n^N(v)]}(u) \Upsilon_n(du, dv)$$

- where the $\Upsilon_n(u, v)$ are independent two dimensional Poisson processes with intensities 1 on $[0, T] \times [0, \infty)$.

- Hence, the window equations are

$$dW_n^N(t) \;=\; \frac{1}{R_n^N(t)}dt - \frac{W_n^N(t^-)}{2}dN_n^N(t),$$

with $W_n^N(0) = w_n(0)$, $n = 1, \ldots, N$ specified.

Evolution of the fluid queue

- We assume that the instantaneous window and throughtput are linked by a Little type formula.

- Hence, $K^N(t) = F(Q^N(t))$, the rate of change of the fluid buffer is given by

$$N\frac{dQ^N(t)}{dt} = \sum_{n=1}^{N} \frac{W_n^N(t)}{R_n^N(t)}(1 - K^N(t)) - NL$$

- since the proportion $K^N(t) := F(Q^N(t))$ of the fluid is lost.

- Dividing by N gives

$$\frac{dQ^N(t)}{dt} = \frac{1}{N}\sum_{n=1}^{N} \frac{W_n^N(t)}{R_n^N(t)}(1 - K^N(t)) - L$$

Existence of a mean field limit (bring back the particles)

- Create a modified system $(\mathcal{W}^N, \mathcal{Q}^M)$ where $\mathcal{W}^N := (\mathcal{W}_1^N, \ldots \mathcal{W}_N^N))$ with

- $$\frac{dQ^N(t)}{dt} = \frac{1}{N}\sum_{n=1}^{N} \frac{E\mathcal{W}_n^N(t)}{\mathcal{R}_n^N(t)}(1 - K^N(t)) - L$$

- Since the modified shared resource is deterministic the modified dynamical systems are independent.

- Moreover it is easy to pick a convergent subsequence $\mathcal{Q}^N \to \mathcal{Q}$.

- It is then easy to prove $\mathcal{W}_n^N$ converges to a limit $\mathcal{W}_n$ along the subsequence for each component n.

- This gives the existence of an infinite modified system $(\mathcal{W}, \mathcal{Q})$.

- Next, the key remark is that by the law of large numbers (and boundedness)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\mathcal{W}_n^N(t)}{\mathcal{R}_n^N(t)} = E \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\mathcal{W}_n^N(t)}{\mathcal{R}_n^N(t)}$$

  i.e. the infinite modified system is in fact a limit of the original system!

- Here this means we show the existence of an infinite limiting system $(\mathbf{W}, Q^\infty)$.

Convergence to the mean field limit (bring back the particles)

- We must define the distance between the the marginal process $\mathbf{W}^N(t) \equiv (W_1^N(t), \ldots W_N^N(t)$ and $Q^N(t)$ and the limit processes $\mathbf{W}(t) \equiv (W_1(t), \ldots W_N(t), \ldots)$ and $Q(t)$

- Define $||\mathbf{W}^N(t) - \mathbf{W}(t)||$ as

$$\frac{1}{N} \sum_{n=1}^{N} \sup_{0 \le \tau \le t \wedge \rho^N} |W_n^N(\tau) - W_n(\tau)|$$

  where τ is a stopping time with respect to $\mathcal{F}_t$.

- ρ to the (deterministic) time when $Q(t)$ first hits the boundary and ρ^N to the stopping time when $Q^N(t)$ first hits the boundary.

- Define $D_N(t)$ as

$$E \sup_{\tau \le t \wedge \rho^N} |Q^N(\tau) - Q(\tau)| + E||\mathbf{W}^N(t) - \mathbf{W}(t)||.$$

- We will establish a Gronwall inequality:

$$D_N(t) \le B_N + C \int_0^t D_N(s)ds$$

for $t < \rho$ where $B_N < \epsilon$ for N sufficently large where ϵ is arbitrarily small and where C will be a canonical constant throughout this calculation.

- It will then immediately follow that

$$\lim_{N \to \infty} D_N(t) = 0$$

on $[0, T]$.

Evolution equation for the mean field

- As $N \to \infty$, the random measure of the window sizes $M^N(t, dw)$ converges in probability to a deterministic measure $M(t, dw)$; i.e. $||M^N(t, dw) - M(t, dw)||_w \to 0$ in probability as $N \to \infty$.

- $M(t, dw)$ is the marginal of

$$M(s - R(s), dv; s, dw)$$

the deterministic joint distribution of the window sizes at time t and at time $t - R(t)$.

Evolution equation for the mean field II

For $g \in C_b^1(R^+), g(0) = 0$

$$
\begin{aligned}
&\langle g, M(t)\rangle - \langle g, M(0)\rangle \\
= \;& \int_0^t \Big[\frac{1}{R(s)}\langle \frac{dg(w)}{dw}, M(s, dw)\rangle \\
+ \;& \langle (g(w/2) - g(w))v, M(s - R(s), dv; s, dw)\rangle \\
& \cdot \frac{1}{R(s - R(s))} K(s - R(s))\Big] ds.
\end{aligned}
$$

Evolution equation for the mean field III

- Moreover the queue size converges in probability to a deterministic limit $Q(t)$ satisfying

$$\frac{dQ(t)}{dt} = \int_w wM(t,dw)\frac{(1-K(t))}{R(t)} - L$$