Mathematical Challenges of the Energy Industry

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Financial Engineering for the Power Markets

Traditional Role of the Financial Engineer (in academia)

- New instrument valuation

  (take-or-pay, swing options, weather derivatives, gas storage, ...)

- Design and analysis of risk measures

  (VaR, Expected Shortfall, ... coherent/convex/... measures of risk)

- Search for optimal hedging strategies

  (VaR hedging, Delta hedging, Delta-Gamma hedging, ...)

FE for the New Power Markets

Degradation of credit exacerbates liquidity problems

- Credit Risk
  - Understanding the statistics of credit migration
  - Including counter-party risk in valuation
  - Credit derivatives
- Could clearing be a solution?
  - Exchange traded instruments pretty much standardized, but OTC ......
  - Design of a minimal set of instruments for standardization
- Collateral requirements / margin calls
  - Objective valuation algorithms widely accepted for frequent Mark-to-Market
  - Netting
    - Challenge of the Dependencies (Correlations, copulas, ...)
    - Integrated approach to risk control

First Challenge: Constructing Forward Curves

- Mathematician: How can it be a challenge?
  - Just do a PCA !
    - Almost OK for Crude Oil (backwardation/contango –¿ 3 factors)
    - Not settled for Gas
    - Does not work for Electricity
  - Extreme complexity & size of the data (location, grade, peak/off peak, firm/non firm)
  - Incomplete and inconsistent sources of information
  - Wide Bid-ask spreads (smoothing)
  - Length of the curve (extrapolation)
- Dynamic models: Seasonality? Mean reversion? HJM? Factor Models? Consistency? Historical? Risk Neutral Models?

Only you – financial mathematician – can prevent forest fires!

Another Statistical Issue
Modelling Demand

- Depending on end user, demand for energy highly correlated with Temperature
  - Heating Season (winter) HDD
  - Cooling Season (summer) CDD
- Stylized Facts
  - Electricity demand = β * weather + α
    - Not true all the time
    - Time dependent β by filtering !
  - Correlation (Gas,Power) = f(weather)
- Weather dynamics need to be included in models
  - Incompleteness

Second Challenge: Marking-to-Market

- Crucial for Clearing
- Major differences
  - Great variety of risk sources
  - Highly complex financial contracts
  - Physical commodities (storage, demand elasticity,....)
  - Extreme volatility
  - Negative prices
- Mark cannot be liquidation value
- Netting requires understanding complex dependencies

Weather Derivatives:
A Lot has Already Been Done

- Temperature Options: Actuarial/Statistical Approach
- Temperature Options: Diffusion Models
- Precipitation Options: Markov Models
  - *Problem:* Pricing in an Incomplete Market
  - *Solution:* Indifference Pricing à la Davis

Weather Derivatives: What is Really Going On?

- Extremely Illiquid Markets
- Weather Derivative = Insurance-like Products
  - No secondary market
  - Mark-to-Market (or Model) does not change
- Until Meteorology kicks in
  - Mark-to-Market (or Model) changes every day
  - Contracts change hands
  - That's when major losses occur and money is made
- This *hot period* is not considered in academic studies
  - Need for updates: new information coming in (temperatures, forecasts, ...)
  - Filtering is (again) the solution

Social & Political Issues

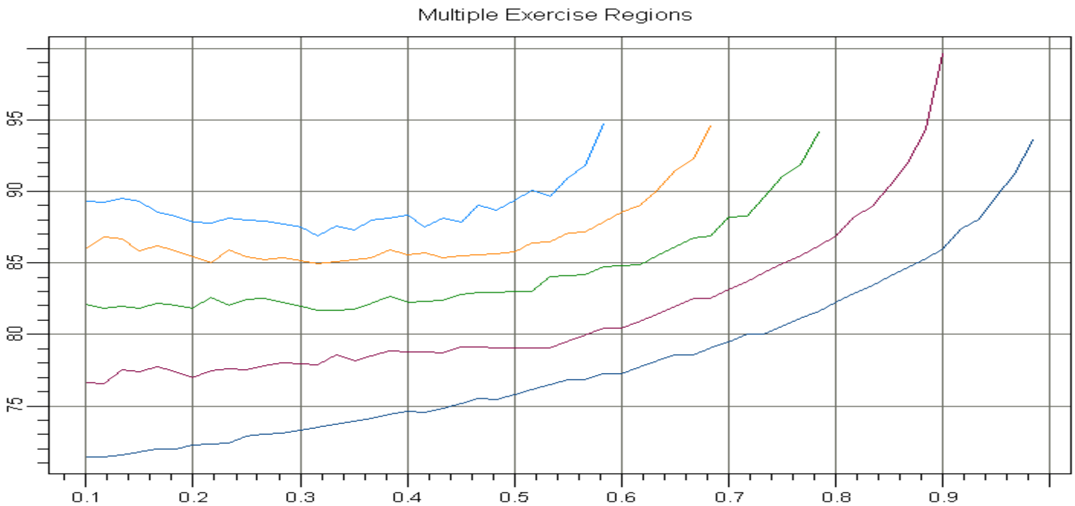
- Social function of the weather market
  - Existence of a Market of Professionals (for weather risk transfer)
- Under attack from
  - (Re-)Insurance industry
  - Utilities (trying to pass weather risk to end-customer)
    - EDF program in France
    - Weather Normalization Agreements in US
- Emission Trading
  - Does it create Pollution Hot Spots?

Stochastic Control Problems

- Gas Storage
  - Dimension: TOO HIGH
- Simpler Problems: Hedging using forward contracts
  - Exposure to demand (weather) fluctuations
    - ∗ Dynamic hedging: still too hard
    - ∗ Static hedging: OK
    - ∗ Several variations in between: still OK

Instruments with Multiple American Exercises

- Ubiquitous in Energy Sector
  - Take-or-Pay and/or Swing contracts
  - End user contracts (EDF)
- Present in other contexts
  - Fixed Income (Chooser Flexible Caps)
  - Executive option programs
  - Purchase of airplanes
- Challenges
  - Valuation
  - Optimal exercise policies



Exercise regions for $N = 5$ rights and finite maturity computed by Malliavin-Monte-Carlo.

Temperature Options: Mathematical Model

$$\begin{aligned} d\theta_t &= p(t,\theta)dt + q(t,\theta)dW_t^{(\theta)} \\ dS_t &= \mu(t,\theta)S_t dt + \sigma(t,\theta)S_t dW_t^{(S)} \end{aligned}$$

- θ_t weather underlying (temperature, rainfall, ...)
- S_t (strongly) correlated tradable (electricity future, gas)

Problem

Pricing an option with pay-off $\Phi = f(\theta_T)$ at time T

Assumptions

- $d[W^{(\theta)}, W^{(S)}]_t = \rho dt$
- Traded market price of risk

$$m(\theta,t) = \frac{\mu(\theta,t) - r}{\sigma(\theta,t)}, \qquad \text{Lipschitz \& bounded}$$

Simple Remarks

- Abstract **BS** Theory
  - Epitomy of an **Incomplete Model**
  - **CMG** Theory gives all the Equivalent Martingale Measures
  - But which one should one choose?
- Practical **Risk Neutrality**
  - Pricing correctly the options quoted by the market
  - Finitely many (linear in EMM)) constraints

First Approach

Choose

- among all the EMM's pricing correctly the quoted options
- the one which minimizes a given cost functional

(for example, the information distance from the historical distribution.)

Rubinstein, Jacwerth, Follmer-Schweizer, Avellaneda et al., C-Xu

Implementation

Strategy

- Handle constraints with Lagrange multipliers
- Stochastic Dynamical Formulation
- Hamilton Jacobi Bellman PDE

Shortcomings

- Numerically very difficult, NOT stable

Possible Fix

- Discretize Space & Time
- Generate (Monte Carlo) Scenarios
- Standard Constrained Optimization Problem

Problems to Overcome

- Works well when $\rho \sim \pm 1$!
- Feasibility: constraints may be incompatible with scenarios
- Lost the natural notion of history (the filtration is trivial)

Possible Solution: add constraints to enforce "martingale pty"

- Nonparametric approach (Longstaff-Schwartz)
- Malliavin Calculus (*Lions et al.*)

Too many constraints !

Second Approach: Utility Maximization

Albina Danilova

Example of a European call

- Choose a utility function $U(x) = -e^{-\gamma x}$
- Use Merton-Davis indifference pricing approach
- HJB nonlinear PDE
- Probabilistic representation of the solution

$$\tilde{p}_t = \frac{\mathbb{E}\{\tilde{\phi}(Y_T) e^{-\int_t^T V(s,Y_s)ds}\}}{\mathbb{E}\{e^{-\int_t^T V(s,Y_s)ds}\}}$$

Second Approach: Utility Maximization

$$\tilde{p}_t = \frac{\mathbb{E}\{\tilde{\phi}(Y_T)e^{-\int_t^T V(s,Y_s)ds}\}}{\mathbb{E}\{e^{-\int_t^T V(s,Y_s)ds}\}}$$

where

- $\tilde{\phi} = e^{-\gamma(1-\rho^2)f}$
  where $f(\theta_T)$ is the pay-off function of the European call on the temperature
- $\tilde{p}_t = e^{-\gamma(1-\rho^2)p_t}$
  where p_t is price of the option at time t
- Y_t is the diffusion:

$$dY_t = [p(t,Y_t) - \frac{\mu(t,Y_t) - r}{\sigma(t,Y_t)} q(t,Y_t)]dt + q(t,Y_t)d\tilde{W}_t$$

  starting from $Y_0 = y$
- V is the time dependent potential function:

$$V(t,y) = -\frac{1-\rho^2}{2}\frac{(\mu(t,y)-r)^2}{\sigma(t,y)^2}$$

Set-Up

$$\begin{aligned} d\theta_t &= p(t,\theta)dt + q(t,\theta)dW_t^{(\theta)} \\ dS_t &= S_t[\mu(t,\theta)dt + \sigma(t,\theta)dW_t^{(S)}] \end{aligned}$$

where $W^{(S)}$ and $W^{(\theta)}$ are Wiener processes with correlation ρ

- For any self-financing portfolio, π_t amount in S_t
- X_t^π wealth at time t given by:

$$dX_t^\pi = rX_t^\pi dt + (\mu(t,\theta_t) - r)\pi_t dt + \sigma(t,\theta_t)\pi_t dW_t^{(S)}$$

with $\pi_t = \overline{\pi}_t X_t$

- Buyer's indifference price p solves

$$\max_\pi \mathbb{E}\{U(X_T^\pi)|X_0 = x, S_0, \theta_0\} = \max_\pi \mathbb{E}\{U(X_T^\pi + f(\theta_T))|X_0 = x - p, S_0, \theta_0\}$$

HJB Equations

Value function

$$u^{(f)}(t,x,\theta) = \max_{\pi} \mathbb{E}\{U(X_T^{\pi} + f(\theta_T))|X_0 = x, \theta_0 = \theta\}$$

Solution of the HJB equation:

$$\begin{aligned} u_t &+ \frac{1}{2}q(t,\theta)u_{\theta,\theta} + p(t,\theta)u_\theta \\ &\qquad + \max_{\pi}[\pi(\mu(t,\theta) - r)u_x + \frac{1}{2}\pi^2\sigma(t,\theta)^2 + \rho\pi\sigma(t,\theta)q(t,\theta)u_{x,\theta}] = 0 \end{aligned}$$

- solve explicitly the optimization (its a quadratic)& plug in the result
- nonlinear PDE

$$u_t + \frac{1}{2}q(t,\theta)^2 u_{\theta,\theta} + p(t,\theta)u_\theta - \frac{[(\mu(t,\theta) - r)u_x + \rho q(t,\theta)\sigma(t,\theta)u_{x,\theta}]^2}{2\sigma(t,\theta)u_{\theta,\theta}} = 0$$

and setting $u(t,x,\theta) = e^{\gamma x + \delta \log v(t,\theta)}$ we linearize HJB into

$$v_t + \frac{1}{2}q(t,\theta)^2 v_{\theta,\theta} + [p(t,\theta) - \frac{\rho q(t,\theta)(\mu(t,\theta) - r)}{\sigma(t,\theta)}]v_\theta - \frac{1-\rho^2)}{2}\frac{[(\mu(t,\theta) - r)]^2}{\sigma(t,\theta)^2}v = 0$$

with the terminal condition

$$v(T,\theta) = e^{-\gamma(1-\rho^2)f(\theta)}$$

HJB Equations

So Cameron-Martin and Feynman-Kac give

$$v(t,y) = \mathbb{E}\{e^{-\gamma(1-\rho^2)f(Y_T)}e^{-\int_t^T V(s,Y_s)ds}|Y_t = y\}$$

where the potential V is given by

$$V(t,y) = \frac{1-\rho^2}{2}\frac{[(\mu(t,y)-r)]^2}{\sigma(t,y)^2}$$

and the stochastic process $\{Y_t\}_t$ is solution of the SDE

$$dY_t = [p(t,Y_t) - \frac{\mu(t,Y_t)-r}{\sigma(t,Y_t)}q(t,Y_t)]dt + d\tilde{W}_t$$

Indifference p_t at time t is solution of:

$$u^{(f)}(t,x-p,y) = u^{(f=0)}(t,x,y) \quad \text{QED}$$

Let's be Realistic !

Unfortunately

Exercise of temperature options is of the Asian type

- Replace temperature process $\{\theta_t\}_t$ by $\{Z_t\}_t$ with

$$Z_t = \int_{t_1}^{t} g(s, \theta_s) ds$$

- Replace SDE driving $\{\theta_t\}_t$ by dynamics of $\{Z_t\}_t$
- Same exact computations give similar formula

Static Hedge with Liquid Options

Assume that at time $t = 0$ I can buy/sell options with

- dates of maturity $t = 0 < T_1 < T_2 < \cdots < T_N < T$
- pay-off's $f_1(\theta_{T_1}), f_2(\theta_{T_2}), \cdots, f_N(\theta_{T_N})$
- prices $c_1, c_2, \cdots, c_N$

Can I compute the value function (i.e. solve the (MAXUTIL) problem);

$$u^{(f)}(t, x, \theta) = \max_{\alpha, \pi} \mathbb{E}\left\{ U\left(X_T^{\pi,\alpha} + f(\theta_T) + \sum_{j=1}^{N} \alpha_j f_j(\theta_{T_j}) \right) |X_t^{\pi,\alpha} = x - \sum_{j=1}^{N} \alpha_j c_j, \theta_t = \theta \right\}$$

and can I price the option with maturity T and pay-off $f(\theta_T)$ by indifference as before?

Connection with MAXENT

- If the pay-off functions f_j are bounded
- If the (MAXENT) optimization problem

$$\min_{\mathbb{E}^{\mathbb{Q}}\{e^{-rT_j} f(\theta_j)\}=c_j,\ j=1,\cdots,N} H(\mathbb{Q}|\mathbb{P}^*)$$

where $\mathbb{P}^*$ is the indifference measure, is feasible

then

- (MAXENT) has a solution given by N Lagrange multipliers λ_j
- the original (MAXUTIL) has a solution, the optimal α_j's being given by the Lagrange multipliers λ_j

THANK YOU