A New Friendly Method of Computing Prolate Spheroidal Wave Functions and Wavelets

G. Walter T. Soleski*

April 12, 2005

Abstract

Prolate spheroidal wave functions, because of their many remarkable properties leading to new applications, have recently experienced an upsurge of interest. They may be defined as eigenfunctions of either a differential operator or an integral operator (as observed by Slepian in the 1960's). There are various ways of calculating their values based on both approaches. The standard one uses an approximation based on Legendre polynomials, which, however, is valid only on a finite interval. An alternative, valid in a neighborhood of infinity, uses a Bessel function approximation. In this paper we present a new method based on an eigenvalue problem for a matrix operator equivalent to that of the integral operator. Its solution gives the values of these functions on the entire real line and is computationally more efficient.

Key words: sampling theory, prolate spheroidal wave functions.

1 Introduction

In this work we shall be concerned with the construction of prolate spheroidal wave functions (PSWFs) and their associated prolate spheroidal wavelets (PS wavelets). The former were introduced in a classic paper [6] by David Slepian and his collaborators in Bell Labs as solutions of an energy concentration problem. They had previously been known as solutions of a Sturm-Liouville problem, from which many of their properties could be derived. The scaling function of the PS wavelets introduced in [12] was based on the first PSWF.

Both sets of functions have many interesting, even unique properties, that make them desirable as bases [11, 12]. Some of these properties are listed in the next section, and will be used to convert the energy concentration problem from one involoving integrals to one involving sequences. This in turn will enable us to constuct the PSWFs and the PS scaling function from a discrete eigenvalue problem. It should be pointed out that this discrete problem is not the one arising from some standard numerical methods, but rather

*Department of Mathematical Sciences, University of Wisconsin–Milwaukee, P.O. Box 413, Milwaukee, WI 53201-0413, USA.

is unique to this setting and gives *exactly* the same eigenvalues as the continuous integral equation. These eigenvalues have the surprising "step function property": they are very close to 1 for $n < \pi\tau$, and very close to 0 for larger values of n [6].

After Slepian, Pollak and Landau discovered the connection between PSWFs and the energy concentration problem during the 1960's, the PSWFs were shown to be an important tool for analyzing some problems raised in signal processing and telecommunications [4]. But they did not have a standard representation in terms of trigonometric functions and were, and still are, regarded as somewhat mysterious. They are seldom used in practice because of this and because the computation of the PSWF function values themselves is a complex numerical problem [3]. Most of the standard methods of computing PSWFs involve an expansion in Legendre polynomials for small values of t and expansion in Bessel functions for large values [10]. In practice, it is often more convenient to use published tabulated values [1, 2] to construct the PSWFs, but then one is restricted to the values of the parameters in the tables. Although some computer programs for evaluating the PSWFs are available [9, 13], many are not portable, or not have been tested thoroughly. Our method can be easily programmed in MAPLE, it holds for all values of t simultaneously, it is easily extended to higher dimensions, and does not involve calculating integrals. It should make this useful tool more widely accessible to both researches and students.

2 Background

The prolate spheroidal wave functions, (PSWFs) $\{\varphi_{n,\sigma,\tau}(t)\}$, constitute an orthonormal basis of the space of $\sigma-$bandlimited functions on the real line, i.e., functions whose Fourier transforms have support on the interval $[-\sigma, \sigma]$. The PSWFs are maximally concentrated on an interval $[-\tau, \tau]$ in a sense described below and depend on parameters σ and τ. There are several ways of characterizing them:

- as the eigenfunctions of a differential operator arising from a Helmholz equation on a prolate spheroid:
$$(\tau^2 - t^2)\frac{d^2\varphi_{n,\sigma,\tau}}{dt^2} - 2t\frac{d\varphi_{n,\sigma,\tau}}{dt} - \sigma^2 t^2 \varphi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau}\varphi_{n,\sigma,\tau};$$
- as the maximum energy concentration of a $\sigma-$ bandlimited function on the interval $[-\tau, \tau]$; that is, $\varphi_{0,\sigma,\tau}$ is the function of total energy 1 ($= ||\varphi_{0,\sigma,\tau}||^2$) such that
$$\int_{-\tau}^{\tau} |f(t)|^2 dt$$
is maximized, $\varphi_{1,\sigma,\tau}$ is the function with the maximum energy concentration among those functions orthogonal to $\varphi_{0,\sigma,\tau}$, etc.;

or

- as the eigenfunctions of an integral operator with kernel arising from the sinc function $S(t) = \sin(\pi t)/\pi t$:

$$\frac{\sigma}{\pi}\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(x) S\left(\frac{\sigma}{\pi}(t-x)\right) dx = \lambda_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}(t). \tag{1}$$

These rather surprising connections were labelled a "lucky accident" by Slepian [7], and enable one to use properties of both the differential operator and integral operator in studying properties of the PSWFs. In addition to the equation (1), the $\{\varphi_{n,\sigma,\tau}\}$ satisfy an integral equation over $(-\infty, \infty)$

$$\frac{\sigma}{\pi}\int_{-\infty}^{\infty} \varphi_{n,\sigma,\tau}(x) S\left(\frac{\sigma}{\pi}(t-x)\right) dx = (\varphi_{n,\sigma,\tau} * S_\sigma)(t) = \varphi_{n,\sigma,\tau}(t) \tag{2}$$

with the same kernel.

As one might expect, PSWFs are closely related to their Fourier transforms. Indeed, the Fourier transform of $\varphi_{n,\sigma,\tau}$ is given by

$$\widehat{\varphi}_{n,\sigma,\tau}(\omega) = (-1)^n \sqrt{\frac{2\pi\tau}{\sigma\lambda_{n,\sigma,\tau}}}\, \varphi_{n,\sigma,\tau}\left(\frac{\tau\omega}{\sigma}\right) \chi_\sigma(\omega) \tag{3}$$

where $\chi_\sigma(\omega)$ is the characteristic function of $[-\sigma, \sigma]$.

It is possible to find the relation between these functions at different scales by using the above definitions and formulas. By a straightforward change of scale in the integral equation (1), we find that both $\varphi_{n,\sigma\tau,1}(x)$ and $\sqrt{\tau}\,\varphi_{n,\sigma,\tau}(\tau x)$ are solutions of the same eigenvalue problem. Since each of the eigenvalues has multiplicity one it follows that each is a multiple of the other, and after normalization, we get

$$\varphi_{n,\sigma\tau,1}(x) = \sqrt{\tau}\,\varphi_{n,\sigma,\tau}(\tau x). \tag{4}$$

Then (4) leads to the following relation between scales for $n = 0$ in particular:

$$\varphi_{0,\sigma,\tau}(2x) = (1/\sqrt{2})\varphi_{0,2\sigma,\tau/2}(x). \tag{5}$$

Some of these properties were used to define the PS wavelets [12] with the restriction to $n = 0$, i.e., the $\sigma-$bandlimited function $\varphi_{0,\sigma,\tau}(x)$ whose concentration on the interval $[-\tau, \tau]$ is maximum. While these functions are entire functions and therefore cannot vanish on any interval, they can be made uniformly small outside of $[-\tau, \tau]$ for τ sufficiently large, so that computationally they behave like functions with compact support. The total energy of such a function outside of this interval is just $1 - \lambda_0$, which is quite small even for moderate values of τ. For example, for $\tau = 2$, and $\sigma = \pi$, this value is about 0.00006 (see Table 1).

In order to construct these PS wavelets [12], the scaling function $\phi = \varphi_{0,\pi,\tau}$, where τ is any positive number, was first introduced. The integer translates of ϕ form a Riesz basis of a space $V_0 \subset L^2(\mathbf{R})$ which turns out to be the Paley-Wiener space B_π of $\pi-$bandlimited functions, no matter what the choice of τ. The PSWFs $\{\varphi_{n,\pi,\tau}\}$ also constitute an orthonormal basis of the same space B_π.

This space then becomes part of the family of nested subspaces of a multiresolution analysis (MRA). The other spaces are obtained, as usual, by dilations by factors of two and consist of the Paley-Wiener spaces $V_m = B_{2^m\pi}$. This MRA has been widely studied and has as its standard scaling function the sinc function $S(t) = \sin \pi t/\pi t$ mentioned above. This function has very good frequency localization, but not very good time localization. This has limited its use as a wavelet basis in comparison to the Daubechies wavelets which have compact support in the time domain. However, this MRA has properties which make it possible to carry out analysis in the spaces V_m, something which cannot be done when other MRA are used. In particular, all derivatives and translations of functions in V_m are again in V_m and have explicit formulas in terms of the scaling function approximations (see [11] for details). The following results may be found there:

Proposition 1 *Let $\phi(t)$ be the PS scaling function, then $\phi^{(k)} \in V_0$ and has expansion coefficients given by*

$$a_n^{(k)} = \begin{cases} \frac{k!(-1)^{n+k+1}}{n^k}, & n \neq 0 \\ \frac{(i\pi)^k(1-(-1)^k)}{2(k+1)}, & n = 0; \end{cases}$$

Let the translation operator by an amount $\beta \in \mathbf{R}$ be denoted by T_β, $T_\beta f(t) = f(t-\beta)$, then $T_\beta \phi \in V_0$ and has expansion coefficients given by

$$a_n(\beta) = \frac{\sin \pi(n-\beta)}{\pi(n-\beta)}, n \in \mathbf{Z}, \beta \notin \mathbf{Z}.$$

Thus, by using these results, the derivative and translation of any function in V_0 can be found in terms of its scaling function expansion.

3 Discrete Maximimization

The maximization problem mentioned above consists of maximizing the ratio

$$\rho = \int_{-\tau}^{\tau} |f(t)|^2 dt \Big/ \int_{-\infty}^{\infty} |f(t)|^2 dt \tag{6}$$

for $\pi-$bandlimited functions f. Such functions may be represented by the Shannon sampling theorem as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)S(t-n),$$

where $S(t)$ is again the sinc function. This sequence of functions $\{S(t-n)\}$ is also an orthonormal basis of B_π and hence the coefficients $\{f(n)\} \in l^2$. We then can substitute this series into both integrals in (6) to get, after an interchange of integals and summations,

$$\sum_{n=-\infty}^{\infty} f(n) \sum_{k=-\infty}^{\infty} \overline{f(k)} \int_{-\tau}^{\tau} S(t-n)S(t-k)dt$$

in the numerator and

$$\sum_{n=-\infty}^{\infty} f(n) \sum_{k=-\infty}^{\infty} \overline{f(k)} \int_{-\infty}^{\infty} S(t-n)S(t-k)dt = \sum_{n=-\infty}^{\infty} |f(n)|^2$$

in the denominator. The former is valid because of the dominated convergence theorem, while the latter is a result of Parseval's equality for the orthonormal basis $\{S(\cdot - n)\}$ of B_π with an inner product in the sense of $L^2(\mathbf{R})$.

We now denote by A_τ the doubly infinite matrix

$$A_\tau := [a_\tau(n,k)] = \left[\int_{-\tau}^{\tau} S(t-n)S(t-k)dt\right]. \tag{7}$$

Thus the ratio in (6) can be expressed as

$$\rho = \frac{< \mathbf{f}, A_\tau \mathbf{f} >}{< \mathbf{f}, \mathbf{f} >}, \tag{8}$$

where $\mathbf{f}$ now denotes the sequence $\{f(n)\}$ and the inner product is just the l^2 inner product. This doubly infinite matrix is clearly real and symmetric. We denote by the same symbol A_τ the operator on l^2 arising from this matrix which then is self-adjoint. It is also positive definite since the inner product in the numerator of (8) is

$$< \mathbf{f}, A_\tau \mathbf{f} >= \int_{-\tau}^{\tau} |f(t)|^2 dt.$$

This value must be positive for non-zero f, since if it were not the entire function f would be zero on the interval $[-\tau, \tau]$, a distinct impossibilty. The operator is also a Hilbert-Schmidt operator since by Schwarz' inequality $[\int_{-\tau}^{\tau} S(t-n)S(t-k)dt]^2 \leq \int_{-\tau}^{\tau} S^2(t-n)dt \int_{-\tau}^{\tau} S^2(t-k)dt$ and hence

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |a_\tau(n,k)|^2 \leq \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_\tau(n,n)a_\tau(k,k) < \infty.$$

This follows from the fact that

$$a_\tau(n,n) = \int_{-\tau}^{\tau} S^2(t-n)dt = \int_{-\tau}^{\tau} \frac{\sin^2 \pi(t-n)}{\pi^2(t-n)^2} dt < \frac{2\tau}{\pi^2(n^2-\tau^2)} \tag{9}$$

for $n^2 > \tau^2$.

Proposition 2 *Let A_τ be the operator on l^2 given by (7); it is a self-ajoint, positive definite, and compact; its eigenvalues are simple and positive and satisfy* $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots > 0$.

The conclusion about the eigenvalues holds for all such operators except for the statement that they are always less than 1. But this is easy to show directly.

We now turn to the problem of maximizing the quotient in (8). This is a standard problem in optimization and is solved by finding the the maximum eigenvalue of the operator A_τ and its associated eigenvector, i.e., by finding the sequence $\boldsymbol{\phi} = \{\phi_n\}$ and λ such that $A_\tau \boldsymbol{\phi} = \lambda \boldsymbol{\phi}$ where λ is maximum value of the ratio and $||\boldsymbol{\phi}|| = 1$ in the sense of l^2.

We then turn to the original maximizartion problem and use this solution of (8) to solve it. Indeed we find the function maximizing (6) to be

$$\phi(t) = \sum_{n=-\infty}^{\infty} \phi_n S(t-n), \tag{10}$$

and the ratio for which (6) is maximum is just $\rho = \lambda$. The function $\phi(t)$ is the scaling function for the PS wavelets and is the first prolate spheroidal wave function.

We may extend this result to other eigenvalues and eigenvectors of A_τ in the usual way ([8], p. 392). We let $\boldsymbol{\phi}_1 = \{\phi_{1,n}\}$ be the sequence othogonal to $\boldsymbol{\phi}$ such that (8) is maximized amoung all such sequences, let $\boldsymbol{\phi}_2 = \{\phi_{2,n}\}$ be the sequence othogonal to $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}$ such that (8) is maximized, etc... Then the sequence of eigenvectors $\boldsymbol{\phi}, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2, ...$ is an orthonormal basis of l^2. The sequence of functions $\{\phi_k(t)\}$ given by

$$\phi_k(t) = \sum_{n=-\infty}^{\infty} \phi_{k,n} S(t-n) \tag{11}$$

will also be orthogonal with respect to the $L^2(\mathbf{R})$ inner product since the correspondence $f_k \to f(t)$ is an isometry from l^2 into $L^2(\mathbf{R})$. In fact, the $\{\phi_k(t)\}$ constitute an orthonormal basis of B_π (they are normalized and the kernel of this correspondence is the zero sequence).

The original maximization problem of (6) is solved by these functions. Hence they must be the prolate spheroidal wave functions which are the unique (up to a unimodular constant) solution of this problem by a repetition of the same arguments. We summarize this in

Proposition 3 *The prolate spheroidal wave functions $\phi_k(t)$ are given by (11) where the $\{\phi_{k,n}\}$ are the eigenfunctions of the discrete operator A_τ on l^2 given by (7) and the eigenvalues $\{\lambda_n\}$ are given by $\lambda_n = \int_{-\tau}^{\tau} |\phi_n(t)|^2 dt$.*

It should be observed that the set $\{\phi_k(t)\}$ also constitutes an orthogonal basis of $L^2[-\tau,\tau]$, but need a different normalization than in B_π, i.e., normalization by the eigenvalues.

4 Finite Approximations

Any practical computations involving these matrices and eigenvectors must be based on finite approximations to both. The most natural way to do this is to truncate the matrix A_τ; we denote the truncated matrix as A_τ^m which has $2m+1$ rows and columns and is given by

$$A_\tau^m := [a_\tau(n,k)], |n| \le m, |k| \le m.$$

We also use the same notation for the infinite matrix consisting of A_τ^m extended by zeros. By (9) and the inequality preceding it, we obtain a bound for the elements of A_τ :

$$|a_\tau(n,k)|^2 \le a_\tau(n,n)a_\tau(k,k) \le \frac{2\tau}{\pi^2(n^2-\tau^2)}\frac{2\tau}{\pi^2(k^2-\tau^2)}$$

provided $|k| > \tau, |n| > \tau$. Furthermore, if we asume that both are bounded below by multiples of τ, $|k| \ge p\tau, |n| \ge p\tau$, then we get

$$|a_\tau(n,k)| \le \frac{2}{\pi^2(p^2-1)\tau}, \qquad |p| > 1. \tag{12}$$

Thus the maximum error between A_τ^m and A_τ is just that given in (12) for $p\tau > m$. If, in addition, the parameter p is chosen to satisfy

$$|p| > \sqrt{1 + 2/(\pi\tau\varepsilon)}$$

for any $\varepsilon > 0$, then it immediately follows from (12) that this same error can be made arbitrary small. The eigenvalues and eigenvectors of A_τ^m are then used to approximate those of A_τ.

Unfortunately, the discrete eigenvalue-eigenvector calculations fail as τ becomes progressively larger. While they work well for the concentration parameter up to about $\tau = 4$, for higher values of τ the eigenvectors are hard to find. Data from Table 1 explain why. This table contains five largest eigenvalues of our operator A_τ corresponding to various values of the parameter τ ranging from 0.5 to 6 (to obtain these values, the series in (10) was truncated to $N = 15$ terms). As τ gets larger, more of the λs become closer and closer to 1. This turns a problem of finding the corresponding eigenvectors - the one we are so much interested in! - into an ill-posed problem. Since there are so many eigenvalues very close to 1, there are lots of eigenvectors that approximately solve the equation. One could

τ	λ_1	λ_2	λ_3	λ_4	λ_5
0.5	0.7833634340	0.1915716491	0.0113687012	0.0001724833	$0.2130088e^{-5}$
1	0.9810223036	0.7270375316	0.2435807274	0.0201691014	0.0010571060
1.5	0.9988792377	0.9628356488	0.7325947429	0.2282438174	0.0347896482
2	0.9999397739	0.9969943631	0.9591497728	0.6795628985	0.2745850110
3	0.9999998317	0.9999890806	0.9996799408	0.9934916044	0.9452729461
4	0.9999999994	0.9999999648	0.9999984114	0.9999614581	0.9992757254
5	0.9999999999	0.9999999997	0.9999999943	0.9999997936	0.9999951708
6	1.0000000000	0.9999999999	0.9999999997	0.9999999990	0.9999999735

Table 1: The five largest eigenvalues of the operator A_τ.

get around this by using greater precision, but then we no longer have such a friendly method.

Since even for $\tau = 5$, the energy outside of the concentration interval $[-\tau, \tau]$ is negligible (≈ 0.0000000001), the best way to calculate the PSWFs for small values of n is to use the Legendre approximation in this interval and then use 0 outside the interval. However for many uses, in particular for wavelet construction, one is interested in smaller concentration intervals with τ close to 1.

Similarly, when $\tau < 0.5$, the $\phi_{0,\pi,\tau}$ becomes pretty much indistinguishable from the sinc function. This is not bad news either since this function does not have particularly good concentration in the interval $[-\tau, \tau]$, and hence the approximation by Lengendre polynomials would not be very good. When τ is in the range specified above, the advantages of using (10) are clear. Besides its use for the direct evaluation of the PS function, it makes differentiation and integration of the PSWFs very easy since these operations are to be performed only on the sinc function and its integer translates. Using additivity, all of the above can be applied to any bandlimited function.

5 Quadrature

We can also utilize other properties of the PSWFs to obtain simple solutions to other problems which involve using only discrete values of these functions that can be obained directly from the solution to (8). For example, quadrature problems of bandlimited functions can be simplified. We use the discrete orthogonality of the PSWFs [12]

$$\sum_{k=-\infty}^{\infty} \phi_n(k)\,\phi_m(k) = \delta_{nm}.$$

Then the PSWF expansion for any $f \in V_0$,

$$f(t) = \sum_{n=0}^{\infty} a_n \phi_n(t)$$

has coefficients given by

$$a_n = \sum_{k=-\infty}^{\infty} \phi_n(k) f(k).$$

This enables us to obtain a simple formula involving only integer values of the PSWF for the integral

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)dt &= \sum_n \sum_{k=-\infty}^{\infty} \phi_n(k) f(k) \int_{-\infty}^{\infty} \phi_n(t)dt \\ &= \sum_n \sum_{k=-\infty}^{\infty} \phi_n(k) f(k) (-1)^n \sqrt{\frac{2\tau}{\lambda_n}} \phi_n(0). \qquad (13)\end{aligned}$$

We can similarly find an expresson for the integral over the concentration interval $[-\tau, \tau]$. It comes from the calculations

$$\begin{aligned}\int_{-\tau}^{\tau} f(t)dt &= \sum_n \sum_{k=-\infty}^{\infty} \phi_n(k) f(k) \int_{-\tau}^{\tau} \phi_n(t)dt \\ &= \sum_n \sum_{k=-\infty}^{\infty} \phi_n(k) f(k) \int_{-\infty}^{\infty} \chi_\tau(t) \phi_n(t)dt \\ &= \sum_n \sum_{k=-\infty}^{\infty} \phi_n(k) f(k) \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\chi_\tau}(t) \widehat{\phi}_n(t)dt.\end{aligned}$$

But this last integral can be expressed as

$$\begin{aligned}\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\chi_\tau}(t) \widehat{\phi}_n(t)dt &= \frac{2\tau(-1)^n}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \omega\tau}{\omega\tau} \sqrt{\frac{2\tau}{\lambda_n}} \phi_n\left(\frac{\omega\tau}{\pi}\right) d\omega \\ &= (-1)^n \sqrt{\frac{2\tau}{\lambda_n}} \int_{-\tau}^{\tau} \frac{\sin \pi\xi}{\pi\xi} \phi_n(\xi)d\xi \\ &= (-1)^n \sqrt{2\tau\lambda_n}\, \phi_n(0).\end{aligned}$$

Hence we get a formula analogous to (13)

$$\int_{-\tau}^{\tau} f(t)dt = \sum_n \sum_{k=-\infty}^{\infty} \phi_n(k) f(k) (-1)^n \sqrt{2\tau\lambda_n}\, \phi_n(0). \qquad (14)$$

The series (14) may be truncated to a small number of terms in n since $|\phi_n(t)| \leq 1$ for all integers n and $t \in \mathbf{R}$ and the eigenvalues are close to 0 for $n > 2\tau$. In both series, all the values can be calculated from the discrete eigenvalue problem.

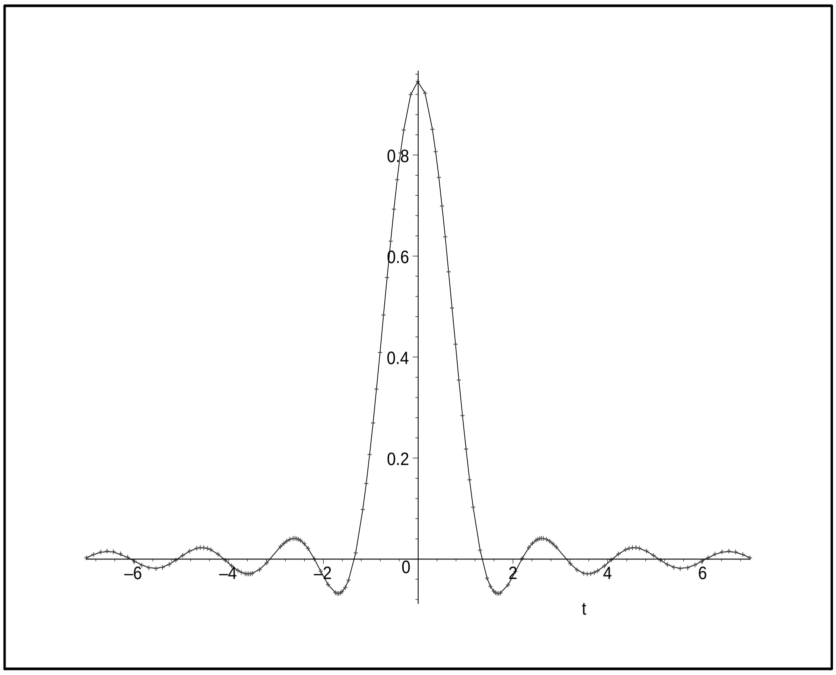

Figure 1: The graphs of the first PS function $\phi_{0,\pi,\tau}(t)$, $\tau = 1$ obtained by using the sinc function (solid line) and Legendre-Bessel approximation method (dotted line).

6 Examples

The first three figures contain the graphs of $\phi_{0,\pi,\tau}$ corresponding to $\tau = 1$, 2 and 3 calculated by our method. For comparison purposes, each picture also includes the graph of the same function obtained by using Legendre polynomials in the interval of concentration $[-\tau, \tau]$ and Bessel functions outside of it. While the two curves are extremely close when $\tau = 1$ and 2, an overshoot can be seen at $\pm\tau$ in the Legendre-Bessel - based graph with $\tau = 3$. Such artifacts are completely avoided in the curves based on our sinc-based method. Figures 4 and 5 show similar results for graphs of $\phi_{3,\pi,\tau}$ and $\phi_{4,\pi,\tau}$ approximated by both methods.

7 Conclusions

We have introduced a new method of constructing prolate spheroidal wave functions and wavelets. It consists of first finding a solution to a discrete maximization problem which is done by finding eigenvalues and eigenvectors of a certain matrix. The eigenvalues turn out to be exactly the same as those obtained in the continuous maximization problem associated with the PSWFs. The eigenvectors, in turn, are composed of the values of

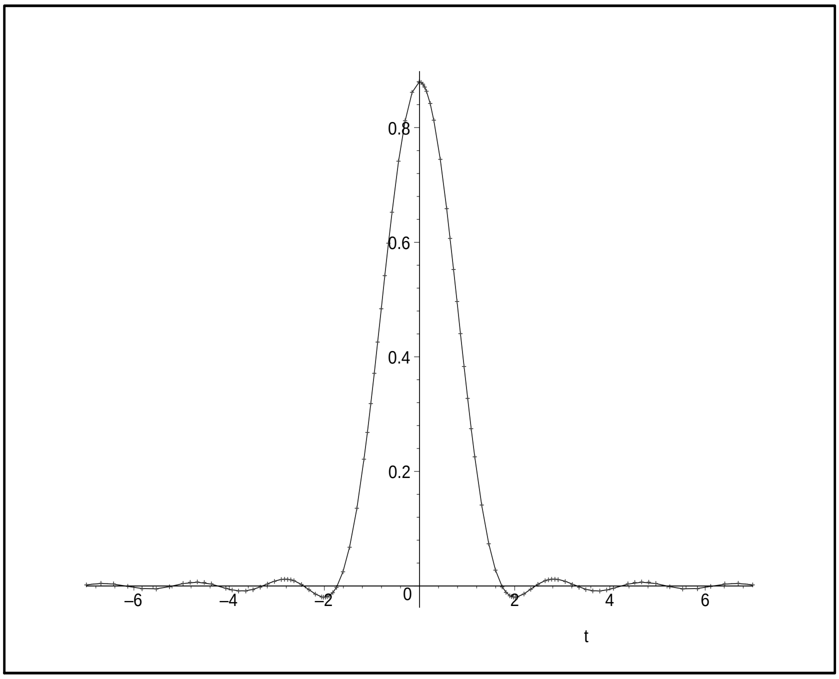

Figure 2: The graphs of $\phi_{0,\pi,\tau}(t)$, $\tau = 1.5$ obtained by using the sinc function (solid line) and Legendre-Bessel approximation method (dotted line).

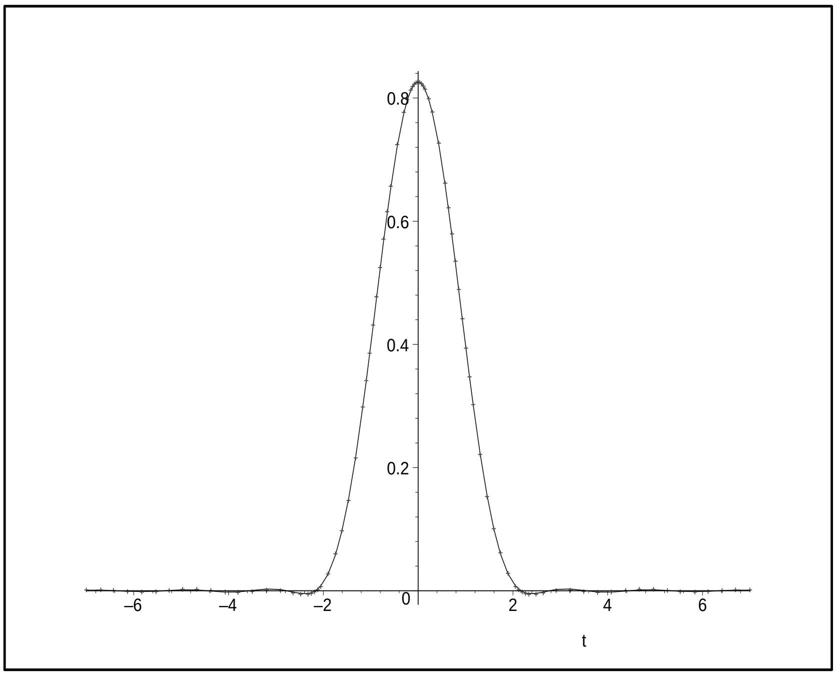

Figure 3: The graphs of $\phi_{0,\pi,\tau}(t)$, $\tau = 2$ obtained by using the sinc function (solid line) and Legendre-Bessel approximation method (dotted line).

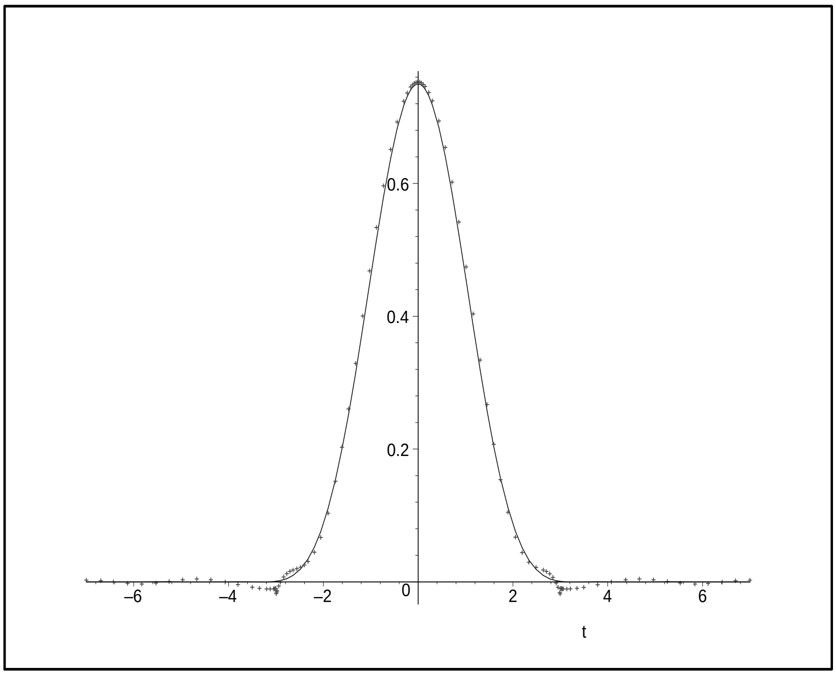

Figure 4: The graphs of $\phi_{0,\pi,\tau}(t)$, $\tau = 3$ obtained by using the sinc function (solid line) and Legendre-Bessel approximation method (dotted line).

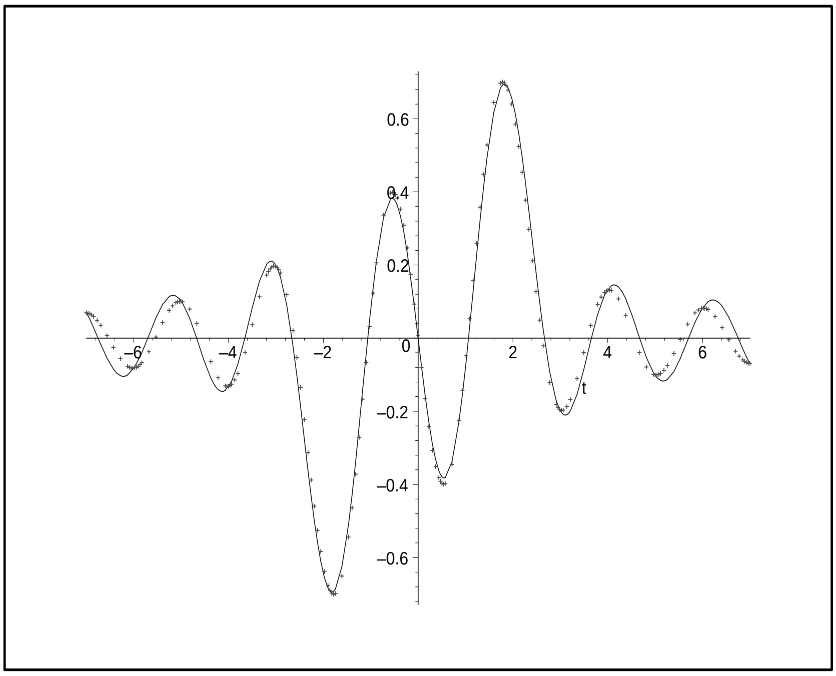

Figure 5: The graphs of $\phi_{3,\pi,\tau}(t)$, $\tau = 2$ obtained by using the sinc function (solid line) and Legendre-Bessel approximation method (dotted line).

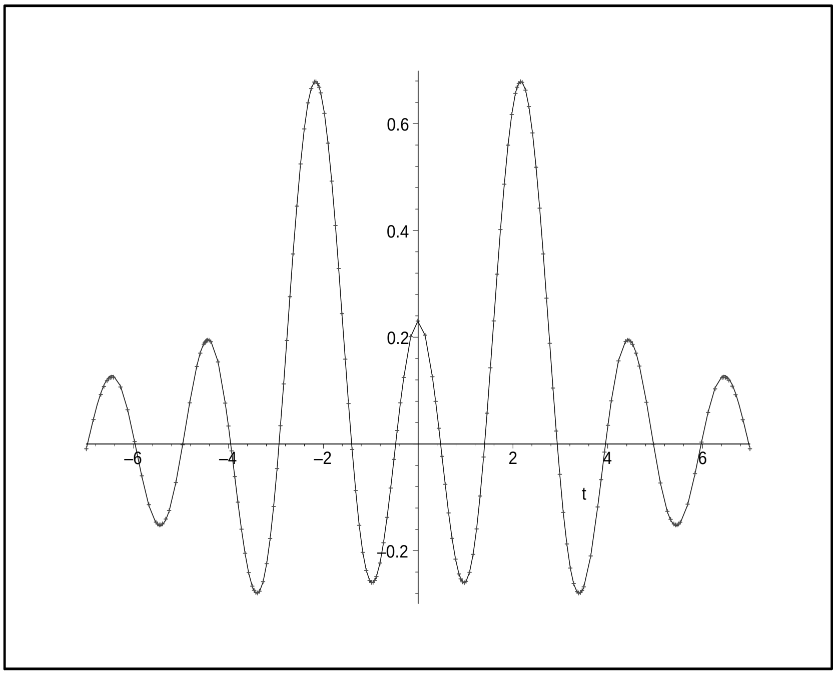

Figure 6: The graphs of $\phi_{4,\pi,\tau}(t)$, $\tau = 2$ obtained by using the sinc function (solid line) and Legendre-Bessel method (dotted line).

the PSWFs at the integers. Thus the Shannon sampling theorem can be used with these values to reconstruct the PSWFs for all real values. This method can be carried out by any program such as MATLAB or MAPLE which can find eigenvalues and eigenvectors of matrices to a reasonable precision. This method gives approximations consisting of bandlimited functions (with the same bandwidth as the PSWFs) valid on the entire real line and avoids all integration. Several explicit computations have shown it to compare favorably to the standard method.

References

[1] Flammer, C., *Spheroidal Wave Functions*, Stanford University Press, Stanford, CA (1957).

[2] Hanish, S., Baier, R.V.,Van Buren, A.L. and King, B.J., Tables of Radial Spheroidal Wave Functions, Volume 1: Prolate, $m = 0$, *Tech. Rep. NRL 7088, Naval Research Laboratory*, Washington, DC (1970) (available through NTIS, AN: AD723836XAB).

[3] Lozier, D.W. and Oliver, F.W.J., Numerical Evaluation of Special Functions, in *Mathematics of Computation 1943-1993: A Half-Century of Computational Mathematics*, W. Gautschi, Ed., AMS Proceedings of Symposia in Applied Mathematics 48 (1993) 79–125.

[4] Papoulis, A., *Signal Analysis*, McGraw–Hill, New York (1977).

[5] Shannon, C.E., Communication in the Presence of Noise, *Proc. IRE*, 37 (1949) 10–21.

[6] Slepian, D. and Pollak, H.O., Prolate spheroidal wave functions, Fourier analysis and uncertainty, I, *Bell System Tech. J.*, 40 (1961) 43–64.

[7] Slepian, D., Some comments on Fourier analysis, uncertainty and modeling, *SIAM Review*, 25 (1983) 379–393.

[8] Stakgold, I., *Green's Functions and Boundary Value Problems*, John Wiley and Sons, New York (1979).

[9] Van Buren, A.L., A Fortran Computer Program for Calculating the Linear Prolate Functions, *Tech. Rep. NRL 7994, Naval Research Laboratory*, Washington, DC (1976) (available through NTIS, AN: ADA0252106XAB).

[10] Volkmer, H., Spheroidal Wave Functions, in *Handbook of Mathematical Functions*, Nat. Bureau of Sts. Applied Math. Series (2004).

[11] Walter, G. G., Prolate Spheroidal Wavelets: Differentiation, Translation, and Convolution Made Easy, *J. of Fourier Analysis and Applications*, 11 (2005) 73–84.

[12] Walter, G. and Shen, X., Wavelets Based on Prolate Spheroidal Wave Functions, *J. of Fourier Analysis and Applications*, 10(1) (2004) 1–25.

[13] Zhang, S. and Jin, J.M., *Computation of Special Functions,* John Wiley and Sons, New York (1996).