Math 2243 Practice Final
Fall 2005

(Instructors: S. Joo / M. Kurzke)

Name: ______________________________ Discussion Session: _____________

(021: Javier Zuniga TTh 10:10, 022: Chung-I Ho TTh 10:10, 023: Javier Zuniga TTh 11:15,
024: Chung-I Ho TTh 11:15 031: Doyoon Kim TTh 2:30,
032: Javier Zuniga TTh 2:30, 033: Doyoon Kim TTh 3:35)

Exam time: Thursday, Dec 15, 1:30-4:30 (13:30-16:30)
Exam location:

- 021: Ford Hall B10
- 022: Ford Hall B15
- 023: Ford Hall B29
- 024: Ford Hall B80
- 031: Ford Hall 110
- 032: Ford Hall 115
- 033: Ford Hall 127

Exam Rules:

- One-line scientific calculators are allowed.
- No books or notes.
- All questions are partial-credit.
- Put final answers in boxes where appropriate.
- **Good Luck!**
- You will need your student ID for the real final!
- Finals without a student ID can not be accepted!

(25 pts) 1. Determine whether or not the set

$$S = \left\{5e^t,\ 3t^2e^t,\ t^2 - e^t\right\}$$

is linearly independent on $(-\infty, \infty)$. Is $2t^2 + 4t$ in the span of S? Justify your answer.
Solution: Yes (linearly independent), and No (not in the span).

Reason: Wronskian check.

$$\begin{vmatrix} 5e^t & 3t^2e^t & t^2 - e^t \\ 5e^t & (3t^2+6)e^t & 2t - e^t \\ 5e^t & (3t^2+12t+6)e^t & 2 - e^t \end{vmatrix} = 5e^{2t}\begin{vmatrix} 1 & 3t^2 & t^2 - e^t \\ 1 & 3t^2+6 & 2t - e^t \\ 1 & 3t^2+12t+6 & 2 - e^t \end{vmatrix}$$

$$= 5e^{2t}\begin{vmatrix} 1 & 3t^2 & t^2 - e^t \\ 0 & 6 & 2t - t^2 \\ 0 & 12t+6 & 2 - t^2 \end{vmatrix} = 5e^{2t}\begin{vmatrix} 6 & 2t - t^2 \\ 12t+6 & 2 - t^2 \end{vmatrix}$$

$$= 5e^{2t}(12 - 6t^2 - 24t^2 - 12t + 12t^3 + 6t^2)$$

$$= 5e^{2t}(12t^3 - 24t^2 - 12t + 12),$$

which is not the zero function.

$$\begin{vmatrix} 5e^t & 3t^2e^t & t^2 - e^t & 2t^2 + 4t \\ 5e^t & (3t^2+6)e^t & 2t - e^t & 4t + 4 \\ 5e^t & (3t^2+12t+6)e^t & 2 - e^t & 4 \\ 5e^t & (3t^2+18t+18)e^t & -e^t & 0 \end{vmatrix} = 30e^{2t}\begin{vmatrix} 1 & t^2 & t^2 - e^t & t^2 + 2t \\ 1 & t^2+2 & 2t - e^t & 2t + 2 \\ 1 & t^2+4t+2 & 2 - e^t & 2 \\ 1 & t^2+6t+6 & -e^t & 0 \end{vmatrix}$$

$$= 30e^{2t}\begin{vmatrix} 1 & t^2 & t^2 - e^t & t^2 + 2t \\ 0 & 2 & 2t - t^2 & 2 - t^2 \\ 0 & 4t & 2 - 2t & -2t \\ 0 & 2t+4 & -2 & -2 \end{vmatrix} = 120e^{2t}\begin{vmatrix} 2 & -t^2 + 2t & -t^2 + 2 \\ 2t & 1 - t & -t \\ t + 2 & -1 & -1 \end{vmatrix}$$

$$= 120e^{2t}(-t^3 + 4t^2 - 2t - 6),$$

and this is not the zero function, so $2t^2 + 4t$ is linearly independent from the other functions, hence not in the span. (Sorry for the ugly calculation).

(25 pts) 2. Find the general solution of $y'' - 3y' + 2y = e^t$.

Solution:

Characteristic equation: $r^2 - 3r + 2 = 0$, solutions: $1, 2$.

Homogeneous solution: $c_1 e^t + c_2 e^{2t}$.

Right-hand side is a solution of the homogeneous problem, so we try multiplying by t to find a particular solution:

try $y_p = ate^t$:

$y_p'' - 3y_p' + 2y_p = e^t(at + 2a - 3(at + a) + 2at)$, and this should be e^t, so $a = -1$, and the solution is $c_1 e^t + c_2 e^{2t} - te^t$.

(25 pts) 3. The number of bacteria in a colony increases at a rate proportional to the number present. If the number of bacteria doubles in 5 hours, how long will it take this colony to grow to seven times its original size?

Solution:

DE $y' = ky$ so $y(t) = y_0 e^{kt}$. (for time t in hours). We know $2y_0 = y_0 e^{5k}$, so $k = \frac{\ln 2}{5}$. We need to solve $7y_0 = y_0 e^{kt}$ so $t = \frac{\ln 7}{k} = 5\frac{\ln 7}{\ln 2}$, and the colony takes $5\frac{\ln 7}{\ln 2}$ hours to grow to seven times its original size.

(20 pts) 4. Solve the initial value problem $y' = t^2(y^2 + 1), \quad y(0) = 0.$

Solution: $dy/(y^2 + 1) = t^2 dt$, so $\arctan y = t^3/3 + C$. Since $y(0) = 0$, $C = 0$, and so $y(t) = \tan \frac{t^3}{3}$.

(30 pts) 5. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

Solution:

Characterisitc equation $\lambda^2 - 8\lambda + 25 = 0$, so $\lambda_{1,2} = 4 \pm 3i$.

Eigenvector for $\lambda_1 = 4 + 3i$: $\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

General solution of homogeneous problem: $c_1 e^{4t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$ Particular solution: try $\begin{pmatrix} a \\ b \end{pmatrix}$, we obtain $4a + 3b = 2$, $-3a + 4b = 7$. Cramer's rule gives $a = -13/25$, $b = 34/25$. We obtain by adding:

$c_1 e^{4t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} + \frac{1}{25} \begin{pmatrix} -13 \\ 34 \end{pmatrix}$

(25 pts) 6. Are these sets

$$V = \left\{A \in \mathbb{M}_{22} : A = -A^T\right\}$$

and

$$W = \{A \in \mathbb{M}_{22} : \det A = 0\}$$

subspaces of $\mathbb{M}_{22}$? Justify your answer.

Solution: Yes and no:

$V \neq \emptyset$ since the zero matrix is in V. Assume $A, B \in V$, then $A = -A^T$ and $B = -B^T$ implies $A + B = -A^t - B^T = -(A + B)^T$, and $kA = -kA^T = (-kA)^T$, so $A + B \in V$, $kA \in V$, so V is a subspace.

W is not a subspace: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in W$, but their sum is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin W$.

(25 pts) 7. Solve the initial value problem

$$\frac{1}{t}y' + y = t^2, \qquad y(0) = 5.$$

Solution:

Rewrite the equation and use the integrating factor method.
Since $y' + ty = t^3$, $\mu = e^{\int t\,dt} = e^{\frac{1}{2}t^2}$. Then

$$ye^{\frac{1}{2}t^2} = \int t^3 e^{\frac{1}{3}t^2}\,dt.$$

Use substitution, $u = \frac{1}{2}t^2$,

$$ye^{\frac{1}{2}t^2} = \int 2ue^u\,du = 2ue^u - \int 2e^u\,du = t^2e^{\frac{1}{2}}t^2 - 2e^{\frac{1}{2}t^2} + C$$

We get $y = t^2 - 2 + Ce^{-\frac{1}{2}t^2}$.
Use initial condition, $5 = -2 + C$, and hence $y(t) = t^2 - 2 + 7e^{-\frac{1}{2}t^2}$

(30 pts) 8. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 & 0 \\ 6 & 5 & 0 \\ 2 & 3 & 5 \end{pmatrix} \mathbf{x}$$

Solution:

Characteristic equation is $-(\lambda - 5)(\lambda - 7)(\lambda + 1)$.

The corresponding eigenvectors can be

$$\begin{pmatrix} 6 \\ -6 \\ 1 \end{pmatrix} \text{ for } \lambda = -1, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ for } \lambda = 5, \quad \begin{pmatrix} 2 \\ 6 \\ 11 \end{pmatrix} \text{ for } \lambda = 7.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 6 \\ -6 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{7t} \begin{pmatrix} 2 \\ 6 \\ 11 \end{pmatrix}.$$

(25 pts) 9. Compute the determinant and the inverse of the matrix

$$A = \begin{pmatrix} 2 & -3 & 5 \\ 0 & 1 & 3 \\ 1 & -1 & 2 \end{pmatrix}$$

Solution:

$$\det A = 2(2+3) + (-9-5) = -4.$$

$$\begin{pmatrix} 2 & -3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 5 & 0 & 1 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -\frac{5}{4} & -\frac{1}{4} & \frac{7}{2} \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}.$$

The inverse matrix,

$$A^{-1} = \frac{1}{4} \begin{pmatrix} -5 & -1 & 14 \\ -3 & 1 & 6 \\ 1 & 1 & -2 \end{pmatrix}.$$

(15 pts) 10. Find the general solution for the differential equation

$$y' - 2ty = 3t.$$

Solution:

Use integraing factor method,

$$\mu = e^{-\int 2t\,dt} = e^{-t^2}$$

$$e^{-t^2} y = \int 3te^{-t^2}\,dt = -\frac{3}{2}e^{-t^2} + C$$

$$y(t) = -\frac{3}{2} + Ce^{t^2}.$$

(30 pts) 11. Convert the equation

$$y'' + 5\sin(y') - y^2 + y = 0$$

to a corresponding system, and determine the location, type and stability of all equilibrium points via the linearization of that system at its equilibrium points.

Solution:

Convert to the system,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -5\sin x_2 + x_1^2 - x^2 \end{aligned}$$

There are two equilibrium points,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ 2x_1 - 1 & -5\cos x_2 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix}$$

has the trace -5, determinant 1, and discriminant is positive, thus the origin is stable attracting node.

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

has the trace -5, determinant -1 thus the equilibrium point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an unstable saddle point.

(25 pts) 12. Determine the kernel, the nullity and the rank of the transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$T(v) = \begin{pmatrix} 5 & 3 & 0 & 1 \\ 6 & 4 & 0 & 2 \\ 7 & 7 & 0 & 7 \end{pmatrix} v.$$

Solution:

$$\begin{pmatrix} 5 & 3 & 0 & 1 \\ 6 & 4 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & -4 \\ 0 & -2 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The kernel is the subspace of $\mathbb{R}^4$ spanned by $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$.

The nullity of T is 2 and the rank of T is 2.