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On strong ellipticity for isotropic hyperelastic materials based upon logarithmic strain ☆

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Abstract

This paper discusses various constitutive restrictions on the strain energy function for an isotropic hyperelastic material derived from the condition of strong ellipticity. The strain energy function is assumed to be a function of a novel set of invariants of the Hencky (logarithmic or natural) strain tensor introduced by Criscione et al. (J. Mech. Phys. Solids 48 (2000) 2445). A key step in the analysis is the derivation of an expression for the Fréchet derivative of the Hencky strain with respect to the deformation gradient that is convenient for analyzing the quadratic form over the space of second order tensors central to establishing strong ellipticity. The theory is illustrated by applying the restrictions to a model for rubber proposed by Criscione et al. (J. Mech. Phys. Solids 48 (2000) 2445) It is shown that while that model can be made to violate strong ellipticity, it does so only for very large strains.

Keywords: Finite elasticity; Strong ellipticity; Constitutive restrictions; Logarithmic strain

1. Introduction

The condition of strong ellipticity for a hyperelastic material is necessary or sufficient or both for many desirable statical and dynamical properties to hold. (See [1] for a discussion of such properties.) Many important results characterizing strong ellipticity for isotropic hyperelastic material have been derived over the past 30 years. Among the most important contributions to this subject are the works by Knowles and Sternberg [2,3], Zee and Sternberg [4], Horgan [5], Rosakis [6] and Wang and Aron [7]. The picture for anisotropic hyperelastic material is far less complete, though the work by Walton and Wilber [8,9] offers a step towards clarifying the issue.

All of the works cited above express the strain energy function for a hyperelastic material in terms of the standard list of invariants of the left Cauchy–Green strain. While convenient for checking conditions such as strong ellipticity, that standard formulation of hyperelasticity, as pointed out by Criscione [10] and Criscione et al. [11], is ill suited for the experimental characterization of real material because the standard list of invariants are highly

☆ Dedicated to C. O. Horgan on the occasion of his sixtieth birthday.

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correlated. As a consequence, Criscione et al. [11] have proposed a model for the strain energy function of an isotropic hyperelastic material that is formulated in terms of a novel set of invariants of the Hencky strain (also called the logarithmic or "natural" strain). These invariants have been shown to have a certain orthogonality property that makes models based upon them ideal for fitting experimental data. Moreover, these novel invariants have direct physical interpretation as volumetric expansion, the magnitude of distortion and the mode of distortion, which the standard list of invariants fail to have.

On the negative side, for a hyperelastic material model based upon Criscione's novel set of invariants of the Hencky strain, determining conditions characterizing strong ellipticity seems very much more complicated than for models based upon the standard list of invariants of the left Cauchy–Green strain. Criscione and Wilber [12] have recently derived conditions guaranteeing the satisfaction of the Baker–Ericksen inequalities for Criscione's formulation of isotropic hyperelastic material, but this lends no help to characterizing strong ellipticity. The primary impediments to deriving constitutive restrictions for Criscione's model based upon strong ellipticity are having a convenient expression for the Fréchet derivative of the logarithmic strain with respect to the deformation gradient and the complicated functional form of gradients of Criscione's invariants. It is shown herein how to overcome the first impediment and progress is made on the latter. Specifically, the desired convenient expression for the Fréchet derivative of the logarithmic strain with respect to the deformation gradient is derived and partial results on strong ellipticity are developed using that Fréchet derivative.

Other researchers have investigated constitutive restrictions for elastic material based upon logarithmic strain and strong ellipticity. Most notably among these efforts is the work of Hill [13,14] with later applications by Hill and Hutchinson [15] and Hutchinson and Neale [16]. Hill [14] uses clever if informal and cumbersome arguments concerning the variation of the principal axes of the "Eulerian strain" (Hill's terminology) with respect to a superimposed infinitesimal deformation to derive constitutive inequalities for a hyperelastic material based upon logarithmic strain. However, because he lacked a convenient expression for the Fréchet derivative of logarithmic strain with respect to the deformation gradient, Hill's constitutive restrictions are complicated and involve inequalities coupling both stress and strain tensoral components. In contrast, armed with the Fréchet derivative of logarithmic strain, we demonstrate herein how one can derive constitutive restrictions in terms only of the strain energy function and its partial derivatives with respect to the chosen set of isotropic invariants. The resulting restrictions are of a character fundamentally different from those of Hill [14], Hill and Hutchinson [15] and Hutchinson and Neale [16].

It should be noted that the J_2 deformation plasticity theory studied by Hutchinson and Neale [16] corresponds to an incompressible elastic material with strain energy function dependent upon only the second of the Criscione isotropic strain invariants. Such a special case is also examined herein. However, Hutchinson and Neale utilize the results of Hill [13,14] which, as described above, yield constitutive restrictions in terms of inequalities coupling both stress and strain tensoral components rather than upon the strain energy function and its partial derivatives with respect to its strain invariant arguments as derived herein. Consequently, both the character of the constitutive restrictions of Hutchinson and Neale as well as their method of derivation are fundamentally different from those provided below.

It should be noted additionally that Bruhns et al. [17] also derive constitutive restrictions for an isotropic, hyperelastic material with strain energy function written as a function of logarithmic strain. However, their set of isotropic invariants is fundamentally different from Criscione's and, in particular, lacks the latter's attractive physical interpretation and data fitting utility. Moreover, they do not compute the Fréchet derivative of the logarithmic strain as needed to develop a complete and rigorous treatment of strong ellipticity.

2. Preliminaries

Let Lin denote the set of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$, and Lin^+ be the set of linear maps with positive determinant. Second order tensors (the elements of Lin) will be denoted by uppercase boldface letters (except for the Cauchy stress tensor, which is denoted by $\mathbf{t}$), vectors (the elements of $\mathbb{R}^3$) will be denoted by lowercase boldface letters. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ a *diad* $\mathbf{a} \otimes \mathbf{b} \in$ Lin is defined by its action on $\mathbb{R}^3$ through $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} := (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$ for every $\mathbf{v} \in \mathbb{R}^3$. If $\{\mathbf{e}_i\}_{i=1}^3$ is a basis for $\mathbb{R}^3$, then $\{\mathbf{e}_i \otimes \mathbf{e}_j\}_{i,j=1}^3$ is a basis for Lin. The inner product on Lin is defined in the following way: $\mathbf{A}\colon \mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B})$. If $\hat{\mathbf{L}}:$ Lin $\to$ Lin (i.e. $\forall \mathbf{A} \in \mathrm{Lin}, \hat{\mathbf{L}}[\mathbf{A}] \in \mathrm{Lin}$), then $\frac{\partial}{\partial \mathbf{A}}\hat{\mathbf{L}}(\mathbf{A})$ is a fourth order tensor (i.e. a linear transformation) from Lin to Lin. The action of a fourth order tensor $\mathscr{L} \in \mathfrak{T}^4$ on a second order tensor $\mathbf{A} \in$ Lin is denoted by $\mathscr{L}[\mathbf{A}]$ or $\mathscr{L}\colon \mathbf{A} \in$ Lin.

Let $\mathfrak{B}$ be the reference configuration of a given body and let $\mathbf{f}$ be a deformation of $\mathfrak{B}$ (i.e. a smooth one-to-one mapping, defined on $\mathfrak{B}$). For $\mathbf{p} \in \mathfrak{B}$, $\mathbf{F}(\mathbf{p}) = \nabla \mathbf{f}(\mathbf{p}) \in \mathrm{Lin}^+$ is the local deformation gradient. Let $\mathbf{T}$ defined on $\mathfrak{B}$ denote the *First Piola–Kirchhoff tensor*. The body is said to exhibit *elastic* material behavior, if there is a function $\hat{\mathbf{T}}$, defined on Lin^+ and taking values in Lin, such that: $\mathbf{T}(\mathbf{x}) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{p}))$, where $\mathbf{x} = \mathbf{f}(\mathbf{p})$. The constitutive function $\hat{\mathbf{T}}$ is said to be *strongly elliptic* at $\mathbf{F}_0$ when the elasticity tensor $\frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}(\mathbf{F}_0)$ satisfies:

$$\mathbf{H} : \frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}(\mathbf{F}_0) : \mathbf{H} > 0, \quad \forall \mathbf{H} = \mathbf{a} \otimes \mathbf{b} \quad \text{such that } |\mathbf{a}| = |\mathbf{b}| = 1. \tag{1}$$

The strong ellipticity condition ensures important stability properties of the constitutive equation. For example, it implies that an increase in a component of strain is accompanied by an increase in the corresponding component of the First Piola–Kirchhoff stress (cf. [1]). Let $\mathbf{F} = \mathbf{VR} = \mathbf{RU}$ be the pointwise polar decomposition of $\mathbf{F}$, where $\mathbf{R} \in \mathrm{Orth}^+$, $\mathbf{U}, \mathbf{V} \in$ Sym. The right and left Cauchy–Green Strain tensors are defined respectively by $\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathrm{T}}$. Let $\mathbf{N}$ be the *natural strain* (or *Hencky strain*), defined by $\mathbf{N} = \ln(\mathbf{V})$.

A material is said to be *hyperelastic* if its material behavior can be characterized by a scalar-valued function W (called *strain energy density* function or *stored energy* function), which depends on the local deformation gradient $\mathbf{F}$. Usually, in the case of materials with isotropic material behavior, W is given as a function of the principal invariants $\{\mathscr{I}_1, \mathscr{I}_2, \mathscr{I}_3\}$ of the left Cauchy–Green strain tensor $\mathbf{B}$. In the special case of incompressible material, the Cauchy stress is then given by

$$\mathbf{t} = -p\mathbf{I} + 2W_1\mathbf{B} - 2W_2\mathbf{B}^{-1}.$$

As shown by Criscione et al. (cf. [11]), the tensors $\mathbf{B}$ and $\mathbf{B}^{-1}$ may be highly covariant, which leads to large numerical errors in fitting experimental data. Criscione et al. (cf. [11]) proposed a new invariant basis for natural strain, which leads to a representation for the Cauchy stress in terms of an orthogonal basis of second order tensors. Moreover, the invariants have the added benefit of useful physical interpretations. The three natural strain invariants are defined in the following way:

Amount of dilatation:

$$K_1 = \mathrm{tr}\,\mathbf{N} = \ln J \quad \text{where} \quad J = \det \mathbf{F}, \tag{2}$$

Magnitude of distortion:

$$K_2 = |\mathrm{dev}(\mathbf{N})| = \sqrt{\mathrm{dev}(\mathbf{N})\colon\ \mathrm{dev}(\mathbf{N})}, \tag{3}$$

where the deviatoric part of a second order tensor is defined as usual by

$$\mathrm{dev}(\mathbf{A}) := \mathbf{A} - \tfrac{1}{3}\,\mathrm{tr}(\mathbf{A})\mathbf{I},$$

and *Mode of distortion*:

$$K_3 = 3\sqrt{6}\,\det(\boldsymbol{\Phi}) \quad \text{where} \quad \boldsymbol{\Phi} = \frac{\mathrm{dev}(\mathbf{N})}{K_2}. \tag{4}$$

Let μ_1, μ_2, μ_3 denote the eigenvalues of $\mathbf{N}$ and define

$$\lambda_i := \frac{2}{3}\mu_i - \frac{1}{3}\,\mu_j - \frac{1}{3}\,\mu_k = -\frac{1}{3}\sum_s (\mu_s + \delta_{si}\mu_i) \tag{5}$$

in which i, j, k are all distinct. It is clear that $\sum \lambda_i = 0$ and hence $\lambda_k = -\lambda_j - \lambda_i$. It also proves useful to introduce Λ_i defined through

$$\Lambda_i := \frac{\lambda_i}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}. \tag{6}$$

Let $\mathbf{N} = \sum \mu_k \mathbf{e}_k \otimes \mathbf{e}_k$ be the spectral expansion of $\mathbf{N}$. Then it follows that

$$\begin{aligned}\mathrm{dev}(\mathbf{N}) &= \mathbf{N} - \tfrac{1}{3}\,(\mu_1 + \mu_2 + \mu_3)\mathbf{I} \\ &= \lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\mathbf{e}_3 \otimes \mathbf{e}_3.\end{aligned} \tag{7}$$

3. The strong ellipticity condition in terms of the natural strain invariants

In this section the form of the strong ellipticity condition will be derived for the case when the strain energy function is expressed as a function of the natural strain invariants proposed by Criscione et al.

(cf. [11]). As shown in [11], the gradients of the natural strain invariants with respect to the Hencky strain **N** are given by:

$$\frac{\partial K_1}{\partial \mathbf{N}} = \mathbf{I}, \tag{8}$$

$$\frac{\partial K_2}{\partial \mathbf{N}} = \mathbf{\Phi}, \tag{9}$$

$$\frac{\partial K_3}{\partial \mathbf{N}} = \frac{1}{K_2}\mathbf{Y}, \tag{10}$$

where **Y** is defined by

$$\mathbf{Y} := 3\sqrt{6}\mathbf{\Phi}^2 - \sqrt{6}\mathbf{I} - 3K_3\mathbf{\Phi}. \tag{11}$$

The action of the second derivatives of K_{1-3} with respect to **N** (which are fourth order tensors) on a second order tensor $\mathbf{A} \in \mathrm{Lin}$ is given by

$$\frac{\partial^2 K_1}{\partial \mathbf{N}^2}[\mathbf{A}] = \mathbf{0}, \tag{12}$$

$$\frac{\partial^2 K_2}{\partial \mathbf{N}^2}[\mathbf{A}] = \frac{\partial}{\partial \mathbf{N}}\left(\frac{\mathrm{dev}(\mathbf{N})}{K_2}\right)[\mathbf{A}] = -\frac{1}{K_2^2}(\mathbf{\Phi}\colon \mathbf{A})\mathrm{dev}(\mathbf{N}) + \frac{\mathrm{dev}(\mathbf{A})}{K_2}, \tag{13}$$

$$\frac{\partial^2 K_3}{\partial \mathbf{N}^2}[\mathbf{A}] = \frac{\partial}{\partial \mathbf{N}}\left(\frac{1}{K_2}\mathbf{Y}\right)[\mathbf{A}] = -\frac{1}{K_2^2}(\mathbf{\Phi} : \mathbf{A})\mathbf{Y} + \frac{1}{K_2}\frac{\partial \mathbf{Y}}{\partial \mathbf{N}}[\mathbf{A}], \tag{14}$$

where for $\frac{\partial \mathbf{Y}}{\partial \mathbf{N}}[\mathbf{A}]$ one obtains the following expression

$$\begin{aligned}\frac{\partial \mathbf{Y}}{\partial \mathbf{N}}[\mathbf{A}] &= \frac{\partial}{\partial \mathbf{N}}(3\sqrt{6}\mathbf{\Phi}^2 - \sqrt{6}\mathbf{I} - 3K_3\mathbf{\Phi})[\mathbf{A}] \\ &= 3\sqrt{6}\mathbf{\Phi}\left(-\frac{1}{K_2^2}(\mathbf{\Phi}\colon \mathbf{A})\mathrm{dev}(\mathbf{N}) + \frac{\mathrm{dev}(\mathbf{A})}{K_2}\right) \\ &\quad + 3\sqrt{6}\left(-\frac{1}{K_2^2}(\mathbf{\Phi}\colon \mathbf{A})\mathrm{dev}(\mathbf{N}) + \frac{\mathrm{dev}(\mathbf{A})}{K_2}\right)\mathbf{\Phi} \\ &\quad -3\frac{1}{K_2}(\mathbf{Y}\colon \mathbf{A})\mathbf{\Phi} - 3K_3 \\ &\quad \times\left(-\frac{1}{K_2^2}(\mathbf{\Phi}\colon\mathbf{A})\mathrm{dev}(\mathbf{N}) + \frac{\mathrm{dev}(\mathbf{A})}{K_2}\right).\end{aligned} \tag{15}$$

Let **t** denote the Cauchy stress tensor. The behavior of hyperelastic materials exhibiting isotropic response with respect to the reference configuration can be described by a constitutive equation of the form:

$$J\mathbf{t}(\mathbf{F}) = \frac{\partial W}{\partial \mathbf{N}} = W_{,i}\frac{\partial K_i}{\mathbf{N}} = W_{,1}\mathbf{I} + W_{,2}\mathbf{\Phi} + W_{,3}\frac{1}{K_2}\mathbf{Y}, \tag{16}$$

where $W = W(K_1, K_2, K_3)$ and $J = \det(\mathbf{F})$.

The well-known relation $\mathbf{T} = J\mathbf{t}\mathbf{F}^{-\mathrm{T}} = \frac{\partial W}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}$ between the First Piola–Kirchhoff stress **T** and the Cauchy stress **t**, implies

$$\begin{aligned}\frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}[\mathbf{H}] &= W_{,ij}\left(\frac{\partial K_j}{\partial \mathbf{N}}\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\frac{\partial K_i}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}} \\ &\quad + W_{,i}\frac{\partial^2 K_i}{\partial \mathbf{N}^2}\left[\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right]\mathbf{F}^{-\mathrm{T}} \\ &\quad - W_{,i}\frac{\partial K_i}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}\colon \mathbf{H} &= W_{,ij}\left(\frac{\partial K_j}{\partial \mathbf{N}}\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\left(\frac{\partial K_i}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}} : \mathbf{H}\right) \\ &\quad + W_{,i}\mathbf{H}\colon \left(\frac{\partial^2 K_i}{\partial \mathbf{N}^2}\left[\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right]\mathbf{F}^{-\mathrm{T}}\right). \\ &\quad - W_{,i}\mathbf{H}\colon \left(\frac{\partial K_i}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right).\end{aligned}$$

Since

$$\mathbf{H}\colon \left(\frac{\partial^2 K_i}{\partial \mathbf{N}^2}\left[\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right]\mathbf{F}^{-\mathrm{T}}\right) = \mathbf{H}\mathbf{F}^{-1} : \frac{\partial^2 K_i}{\partial \mathbf{N}^2} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}],$$

the strong ellipticity condition takes the form

$$\begin{aligned}\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}\colon \mathbf{H} &= W_{,ij}\left(\frac{\partial K_j}{\partial \mathbf{N}}\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\left(\frac{\partial K_i}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}\colon \mathbf{H}\right) \\ &\quad + W_{,i}\mathbf{H}\mathbf{F}^{-1}\colon \frac{\partial^2 K_i}{\partial \mathbf{N}^2}\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}] \\ &\quad - W_{,i}\mathbf{H}\colon \left(\frac{\partial K_i}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right) > 0.\end{aligned} \tag{17}$$

In what follows the terms involving the Fréchet derivative of the Hencky strain $\mathbf{N}$ with respect to the deformation gradient $\mathbf{F}$ will be calculated using the formula derived in Appendix A.

4. Special cases of the strong ellipticity condition

In this section, necessary conditions for the strong ellipticity condition are derived for the cases when the strain energy function depends only on one of the invariants and for the case when it is only a function of K_2 and K_3. The latter is actually the case of incompressible material behavior. Initially, consideration will be restricted to the special case of deformations in the class

$$\mathbf{F} = f_1\mathbf{e}_1 \otimes \mathbf{e}_1 + f_2\mathbf{e}_2 \otimes \mathbf{e}_2 + f_3\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{18}$$

where $\{\mathbf{e}_i\}_{i=1}^3$ are the eigenvectors of the left stretch tensor $\mathbf{V}$, (consequently of the Hencky strain $\mathbf{N}$ as well). The first three cases that are considered in the propositions which follow, are intended mostly to provide insight for the form of the constitutive equation in the general case than for their physical applicability.

Proposition 1. *If the deformation is of the form* (18) *and if* $W = W(K_1)$, *then*

1. *when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_j$, $i \neq j$

$$\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}} \hat{\mathbf{T}} : \mathbf{H} = 0$$

2. *and when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_i$

$$\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}} \hat{\mathbf{T}}\colon \mathbf{H} = W_{,11}\frac{1}{f_i^2} - W_{,1}\frac{1}{f_i^2}.$$

Thus a necessary condition for strong ellipticity at the given type of deformation for this particular kind of constitutive equation is

$$W_{,11} > W_{,1}. \tag{19}$$

The proof of Proposition 1 is given in Appendix B. It follows from (19) that $W(K_1)$ must grow exponentially as a function of K_1 for strong ellipticity to hold.

Proposition 2. *If the deformation is of the form* (18) *and if* $W = W(K_2)$, *then*

1. *when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_j$, $i \neq j$

$$\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}} \hat{\mathbf{T}}\colon \mathbf{H} = \frac{W_{,2}}{K_2}\left(\frac{\mu_i - \mu_j}{\mathrm{e}^{2\mu_i} - \mathrm{e}^{2\mu_j}}\right).$$

Thus a necessary condition for strong ellipticity of the constitutive equation is:

$$W_{,2} > 0. \tag{20}$$

2. *In the case when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_i$

$$\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}} \hat{\mathbf{T}}\colon \mathbf{H} = \lambda_i^2 \frac{1}{K_2^2 \mathrm{e}^{2\mu_i}}\left(W_{,22} - \frac{W_{,2}}{K_2}\right) - \lambda_i \frac{W_{,2}}{f_i^2 K_2} + \frac{2W_{,2}}{3K_2\mathrm{e}^{2\mu_i}}.$$

Thus a necessary condition for strong ellipticity at the given type of deformation for this particular kind of constitutive equation is

$$W_{,22} > \left(\frac{3K_2}{8} + \frac{1}{K_2}\right) W_{,2}. \tag{21}$$

The proof of Proposition 2 is given in Appendix B. It follows from (20) that $W(K_2)$ must increase with K_2 and from (21) that is must grow exponentially with K_2.

Proposition 3. *If the deformation is of the form* (18) *and if* $W = W(K_3)$, *then*

1. *when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_j$, $i \neq j$

$$\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}} \hat{\mathbf{T}}\colon \mathbf{H} = W_{,3}\frac{1}{K_2^2}\left\{\frac{-3\sqrt{6}}{K_2}\lambda_k - 3K_3\right\} \times \frac{\mu_i - \mu_j}{\mathrm{e}^{2\mu_i} - \mathrm{e}^{2\mu_j}}. \tag{22}$$

2. *In the case when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_i$

$$\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}} \hat{\mathbf{T}}\colon \mathbf{H} = W_{,33}\frac{\Theta^2}{K_2^2 f_i^2} - W_{,3}\frac{\Theta}{K_2 f_i^2}$$

$$+ W_{,3}\left(-\frac{\Theta\lambda_i}{f_i^2}\left(\frac{1}{K_2^3}+\frac{3}{K_2^4}\right)-\frac{6\sqrt{6}\lambda_i^3}{K_2^5 f_i^2}\right.$$
$$\left.+\frac{3K_3\lambda_i^2}{K_2^3 f_i^2}+\frac{4\sqrt{6}\lambda_i}{K_2^3 f_i^2}-\frac{2K_3}{K_2^2 f_i^2}\right) \tag{23}$$

where

$$\Theta = \mathbf{Y}\colon \mathbf{H} = \frac{3\sqrt{6}}{K_2^2}\,\lambda_i^2 - \frac{3K_3}{K_2}\,\lambda_i - \sqrt{6}.$$

The proof of Proposition 3 is given in Appendix B.

Note: The expression multiplying $W_{,3}$ might change sign making it difficult to deduce any general constitutive restrictions assuring strong ellipticity. Moreover, both (22) and (23) contain K_2 even though W was assumed to depend only upon K_3. The proposition given above is intended primarily as an auxiliary calculation for the proposition that is to follow.

Proposition 4. *If the deformation is of the form* (18) *and if* $W = W(K_2, K_3)$, *then*

1. *when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_j$, $i \neq j$

$$\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}}\,\hat{\mathbf{T}}\colon \mathbf{H} = \frac{\mu_i - \mu_j}{K_2(\mathrm{e}^{2\mu_i} - \mathrm{e}^{2\mu_j})} \times \left(W_{,2} + W_{,3}\left(-\frac{3\sqrt{6}\lambda_k}{K_2^2} - \frac{3K_3}{K_2}\right)\right). \tag{24}$$

2. *In the case when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_i$

$$\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}}\,\hat{\mathbf{T}}\colon \mathbf{H} = W_{,22}\,\frac{\lambda_i^2}{K_2^2 f_i^2} + W_{,23}\,\frac{\Theta\lambda_i}{K_2^2 f_i^2} + W_{,32}\,\frac{\Theta\lambda_i}{K_2^2 f_i^2} + W_{,33}\,\frac{\Theta^2}{K_2^2 f_i^2}$$
$$+ W_{,2}\left(-\frac{\lambda_i^2}{K_2^3 f_i^2} + \frac{1}{K_2}.\frac{2}{3 f_i^2}\right)$$
$$+ W_{,3}\left(-\frac{\Theta\lambda_i}{f_i^2}\left(\frac{1}{K_2^3}+\frac{3}{K_2^4}\right)-\frac{6\sqrt{6}\lambda_i^3}{K_2^5 f_i^2}\right.$$
$$\left.+\frac{3K_3\lambda_i^2}{K_2^3 f_i^2}+\frac{4\sqrt{6}\lambda_i}{K_2^3 f_i^2}-\frac{2K_3}{K_2^2 f_i^2}\right)$$
$$- W_{,2}\,\frac{\lambda_i}{K_2 f_i^2} - W_{,3}\,\frac{\Theta}{K_2 f_i^2}.$$

The proof of Proposition 4 is given in Appendix B.

The first term in (24) is always positive, so one only needs to find conditions for which

$$W_{,2} + W_{,3}\left(-\frac{3\sqrt{6}\lambda_k}{K_2^2} - \frac{3K_3}{K_2}\right) > 0. \tag{25}$$

Using

$$K_2 = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$$

and

$$K_3 = 3\sqrt{6}\det(\boldsymbol{\Phi}) = 3\sqrt{6}\det\left(\frac{\operatorname{dev}(\mathbf{N})}{K_2}\right) = \frac{3\sqrt{6}}{K_2^3}\,\lambda_1\lambda_2\lambda_3$$

and the notations given in Eqs. (5) and (6) the condition (25) takes the form

$$W_{,2} + \frac{3\sqrt{6}}{K_2}\,W_{,3}(\Lambda_k - 3\Lambda_1\Lambda_2\Lambda_3) > 0.$$

Now, the fact that $-4 \leqslant \Lambda_k - 3\Lambda_1\Lambda_2\Lambda_3 \leqslant 4$ allows for an estimate of (25) from below and above, which is useful for the derivation of both necessary conditions and sufficient conditions. For example, a sufficient condition for the positivity of the quadratic form (17) in the case described in Proposition 4 ($W = W(K_2, K_3)$, and $\mathbf{F}$ of the form (18)), when the deformation gradient is perturbed by $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_j$, $i \neq j$ is

1. $W_{,3} > 0$, and
2. $W_{,2} > \frac{12\sqrt{6}}{K_2} W_{,3}$.

Next, strong ellipticity at simple shear is considered for the case when the constitutive equation is of the form $W = W(K_2)$. The deformation gradient $\mathbf{F}$ is taken to be

$$\mathbf{F} = \mathbf{I} + \gamma\mathbf{f}_1 \otimes \mathbf{f}_2, \tag{26}$$

where $\{\mathbf{f}_k\}$ is some initial basis of $\mathbb{R}^3$. Let $\{\mathbf{e}_k\}$ be the eigenvectors of the left Cauchy–Green tensor $\mathbf{B}$. In this basis, $\mathbf{F}$ has the following form:

$$\begin{aligned}\mathbf{F} &= \left(1+\frac{\gamma\beta}{1+\beta^2}\right)\mathbf{e}_1\otimes\mathbf{e}_1+\left(1-\frac{\gamma\beta}{1+\beta^2}\right)\\ &\quad\times\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3\\ &\quad+\frac{\gamma}{1+\beta^2}\mathbf{e}_1\otimes\mathbf{e}_2-\frac{\gamma\beta^2}{1+\beta^2}\mathbf{e}_2\otimes\mathbf{e}_1\\ &= f_1\mathbf{e}_1\otimes\mathbf{e}_1+f_2\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3\\ &\quad+f_{12}\mathbf{e}_1\otimes\mathbf{e}_2+f_{21}\mathbf{e}_2\otimes\mathbf{e}_1. \end{aligned} \tag{27}$$

In the proposition that follows, the notations $\bar{f}_1 = f_2$, $\bar{f}_2 = f_1$ and $\bar{f}_3 = f_3 = 1$ are used.

Proposition 5. *If the deformation is of the form* (26) *and if* $W = W(K_2)$, *then*

1. *when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_j$, $i \neq j$

$$\begin{aligned}\mathbf{H}\colon \frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}\colon\mathbf{H} &= W_{,22}\frac{\lambda_i}{K_2^2 e^{2\mu_i}}(f_{12}\delta^{1i}\delta^{2j}+f_{21}\delta^{1j}\delta^{2i})\\ &\quad\times(-f_{12}\lambda_2\delta^{2i}\delta^{1j}-f_{21}\lambda_1\delta^{1i}\delta^{2j})\\ &\quad+W_{,2}\frac{-\lambda_i}{K_2^3 e^{2\mu_i}}(f_{12}\delta^{1i}\delta^{2j}+f_{21}\delta^{1j}\delta^{2i})\\ &\quad\times(-f_{12}\lambda_2\delta^{2i}\delta^{1j}-f_{21}\lambda_1\delta^{1i}\delta^{2j})\\ &\quad+W_{,2}\frac{1}{K_2}\left(\frac{f_j\bar{f}_j}{E(\mu_i,\mu_j)}-f_{12}f_{21}\right.\\ &\quad\times\left(\frac{\delta^{1j}(1+\delta^{2j})}{E(\mu_i,\mu_2)}+\frac{\delta^{2j}(1+\delta^{1i})}{E(\mu_i,\mu_1)}\right)\\ &\quad\left.+\frac{1}{3}\frac{f_{12}f_{21}}{e^{2\mu_i}}(\delta^{1i}\delta^{2j}+\delta^{1j}\delta^{2i})\right)\\ &\quad-W_{,2}\frac{\lambda_i}{K_2}(f_{12}^2\delta^{1j}\delta^{2i}+f_{21}^2\delta^{2j}\delta^{1i})\end{aligned}$$

2. *In the case when* $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_i$

$$\begin{aligned}\mathbf{H}\colon \frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}\colon\mathbf{H} &= W_{,22}\frac{\lambda_i^2 f_i\bar{f}_i}{K_2^2 e^{2\mu_i}}-W_{,2}\frac{\lambda_i^2 f_i\bar{f}_i}{K_2^3 e^{2\mu_i}}\\ &\quad+W_{,2}\frac{1}{K_2}\left(\frac{2f_i\bar{f}_i}{3e^{2\mu_i}}-f_{12}f_{21}\right.\\ &\quad\left.\times\left(\frac{\delta^{1i}}{E(\mu_i,\mu_2)}+\frac{\delta^{2i}}{E(\mu_i,\mu_1)}+\right)\right)\\ &\quad-\frac{\lambda_i\bar{f}_i^2}{K_2}.\end{aligned}$$

The proof of Proposition 5 is given in Appendix B. One now easily proves the following corollary.

Corollary 1. *Necessary conditions for strong ellipticity of a constitutive equation of the form* $W = W(K_2)$, *at a simple shearing deformation* $\mathbf{F}$, *given by* (26), *are*

1. *W is a strictly increasing function of* K_2 (*i.e.* $W_{,2} > 0$), *and*
2. *W is convex for large strains, i.e.* $W_{,22} > 0$ *for sufficiently large* γ.

Consequently, necessary conditions for strong ellipticity at this class of deformations are much milder; only convexity of the constitutive function W is needed, in contrast to the case of $\mathbf{F}$ being diagonal in the basis of eigenvectors of the Hencky strain $\mathbf{N}$, for which exponential growth of W with respect to the second invariant K_2 is required.

5. Application to Criscione's model for rubber data

In this section, the general considerations described above are applied to analyze the stability of an incompressible polynomial model used by Criscione et al. [11] to fit the rubber data of Jones and Treloar [18]. The model has W a polynomial in the invariants K_2 and K_3 of the form

$$\begin{aligned}W &= 0.48K_2^2-0.053K_2^3+0.088K_2^4\\ &\quad+(0.065+0.039K_2)K_2^3K_3.\end{aligned}$$

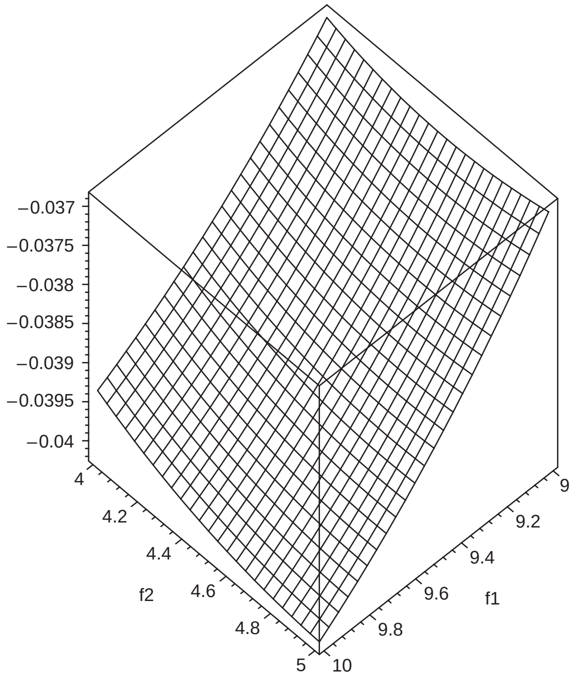

Fig. 1. Perturbation in the $\mathbf{e}_1$ direction for large strains.

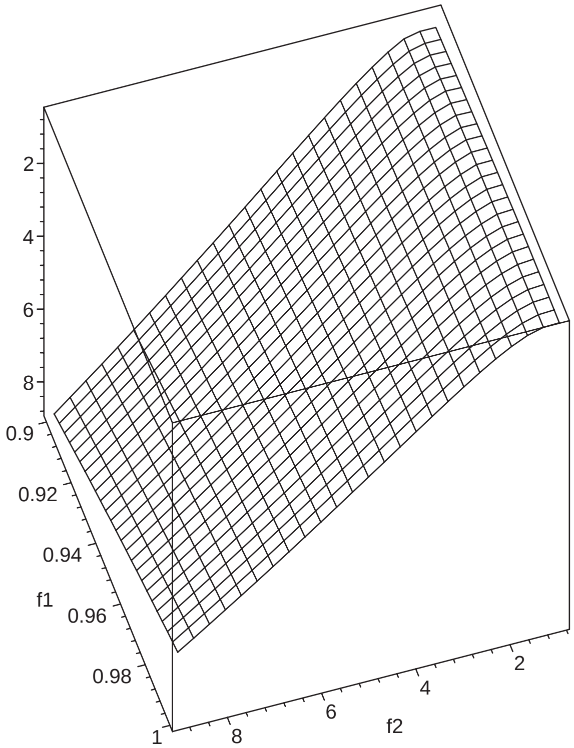

Fig. 2. Perturbation in the $\mathbf{e}_1$ direction for moderate strains.

The stability of this model was tested on a bi-axial stretch deformation of the form (18) with $f_1 f_2 f_3 = 1$ due to incompressibility. In Figs. 1 and 2, using the result from Proposition 4, the quadratic form (17) characterizing strong ellipticity is plotted as a function of f_1 and f_2 for $\mathbf{H} = \mathbf{e}_1 \otimes \mathbf{e}_1$ for large and moderate strain levels, respectively. One can see that strong ellipticity fails only for strain levels exceeding 800%. In Fig. 3, the bi-axial deformation is perturbed by a shear of the form $\mathbf{H} = \mathbf{e}_1 \otimes \mathbf{e}_2$. In this case, no strain level causes a loss of strong ellipticity. Thus, the Criscione model can lose strong ellipticity under bi-axial stretch, but only for excessively large strains not expected to be encountered in experiment.

6. Conclusions

The model for isotropic, hyperelastic material developed by Criscione et al. [11] offers many attractive features as a tool for characterizing the mechanical behavior of real material such as rubber and soft tissue. The use of the Hencky (logarithmic) strain with the novel set of isotropic invariants introduced in [11] in the strain energy function provides a framework that overcomes the correlations inherent in models utilizing the standard list of invariants of the left Cauchy–Green strain that pollute attempts to fit experimental data without excessive error. However, the modeling approach of Criscione et al. produces very complicated expressions for the Cauchy stress making the analysis of such models a difficult task. In this paper, a major impediment to analyzing the stability properties of Criscione's class of models is overcome through the derivation of an expression for the Fréchet derivative of the Hencky strain with respect to the deformation gradient. This result makes possible the derivation of various results concerning the strong ellipticity of Criscione's class of models. While constructing conditions that are both necessary and sufficient for strong ellipticity to hold for all deformations still seems a daunting task, one can utilize the results contained herein to test specific models for strong ellipticity numerically for classes and

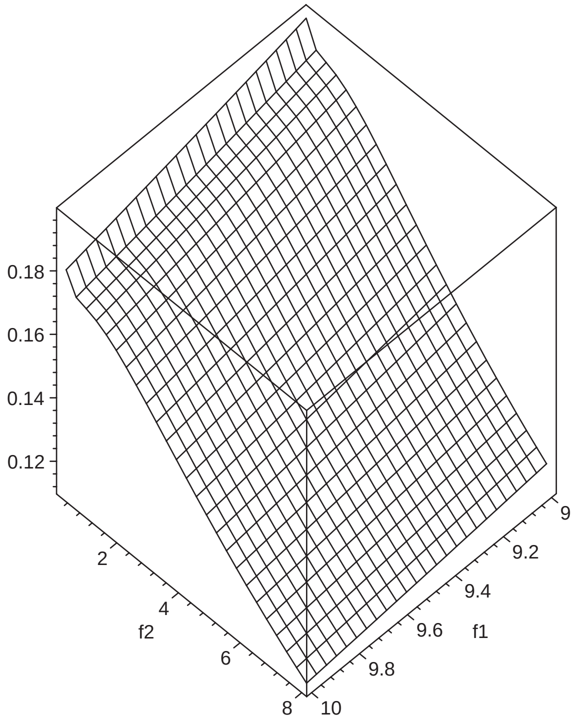

Fig. 3. Shearing perturbation in the $\mathbf{e}_1 \otimes \mathbf{e}_2$ direction for large strains.

magnitudes of deformations encountered in experimental characterizations of real material. Moreover, results of a more general nature are provided herein for the special cases of bi-axial stretch and simple shear deformations that characterize the model's stability to both uniaxial stretch and simple shear perturbations. However, it must be emphasized that these results are only a first step towards understanding the stability properties of Criscione's class of models. A more thorough investigation of the case when the strain energy function depends upon both of the invariants K_2 and K_3 needs to be carried out. These issues will be addressed by the authors in future investigations.

Appendix A. The Fréchet derivative of the Hencky strain

In this appendix, a formula is derived for the Fréchet of the Hencky strain with respect to the deformation gradient that is convenient for characterizing strong ellipticity of the Criscione model. To that end, recall that by definition the left stretch, left Cauchy–Green strain and the Hencky strain are defined by

$$\mathbf{V}^2 = \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \tag{28}$$

$$\mathbf{N} = \ln(\mathbf{V}). \tag{29}$$

The Fréchet derivative $D_{\mathbf{F}}\mathbf{N}$ will be computed implicitly.

Writing

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathrm{e}^{2\mathbf{N}} = \sum_{n=0}^{\infty} \frac{2^n}{n!} \mathbf{N}^n,$$

one has on the one hand that

$$D_{\mathbf{F}}\mathbf{B}[\mathbf{H}] = \mathbf{H}\mathbf{F}^{\mathrm{T}} + \mathbf{F}\mathbf{H}^{\mathrm{T}}. \tag{30}$$

On the other hand,

$$D_{\mathbf{F}}\mathbf{B}[\mathbf{H}] = \sum_{n=0}^{\infty} \frac{2^n}{n!} \left(\sum_{k=0}^{n-1} \mathbf{N}^k D_{\mathbf{F}}\mathbf{N}[\mathbf{H}]\mathbf{N}^{n-1-k} \right), \tag{31}$$

where the inner summation is necessitated by the non-commutativity of matrix multiplication. Now, if $\mathbf{H} = \mathbf{a} \otimes \mathbf{b}$, where $|\mathbf{a}| = |\mathbf{b}| = 1$, using Eq. (30), one sees that

$$\mathbf{a} \otimes \mathbf{b} \colon D_{\mathbf{F}}\mathbf{B}[\mathbf{a} \otimes \mathbf{b}] = \mathbf{b} \cdot (\mathbf{F}\mathbf{b}) + \mathbf{a} \cdot (\mathbf{F}\mathbf{b})(\mathbf{a} \cdot \mathbf{b}). \tag{32}$$

On the other hand, using Eq. (31), one obtains

$$\begin{aligned} &\mathbf{a} \otimes \mathbf{b} \colon D_{\mathbf{F}}\mathbf{B}[\mathbf{a} \otimes \mathbf{b}] \\ &\quad = \sum_{n=1}^{\infty} \frac{2^n}{n!} \left(\sum_{k=0}^{n-1} \mathbf{a} \otimes \mathbf{b} \colon (\mathbf{N}^k D_{\mathbf{F}}\mathbf{N}[\mathbf{a} \otimes \mathbf{b}]\mathbf{N}^{n-1-k}) \right) \\ &\quad = \sum_{n=1}^{\infty} \frac{2^n}{n!} \left(\sum_{k=0}^{n-1} (\mathbf{N}^k \mathbf{a}) \cdot (D_{\mathbf{F}}\mathbf{N}[\mathbf{a} \otimes \mathbf{b}]\mathbf{N}^{n-1-k}\mathbf{b}) \right). \end{aligned} \tag{33}$$

Letting $\mathbf{a} = \mathbf{e}_i$, $\mathbf{b} = \mathbf{e}_j$, where $\mathbf{N}\mathbf{e}_i = \mu_i \mathbf{e}_i$, $\mathbf{N}\mathbf{e}_j = \mu_j \mathbf{e}_j$, it follows from Eqs. (32) and (33) that

$$\begin{aligned} &\mathbf{e}_j \cdot (\mathbf{F}\mathbf{e}_j) + \mathbf{e}_i \cdot (\mathbf{F}\mathbf{e}_j)(\mathbf{e}_i \cdot \mathbf{e}_j) \\ &\quad = \mathbf{e}_i \cdot (D_{\mathbf{F}}\mathbf{N}[\mathbf{e}_i \otimes \mathbf{e}_j]\mathbf{e}_j) \sum_{n=1}^{\infty} \frac{2^n}{n!} \\ &\qquad \times \left(\sum_{k=0}^{n-1} \mu_i^k \mu_j^{n-1-k} \right). \end{aligned} \tag{34}$$

From the above equation, one concludes that

$$\mathbf{e}_i\otimes\mathbf{e}_j\colon (D_{\mathbf{F}}\mathbf{N}[\mathbf{e}_i\otimes\mathbf{e}_j])$$
$$=\frac{\mathbf{e}_j\cdot(\mathbf{F}\mathbf{e}_j)+\mathbf{e}_i\cdot(\mathbf{F}\mathbf{e}_j)(\mathbf{e}_i\cdot\mathbf{e}_j)}{2\sum_{n=0}^{\infty}\frac{2^n}{(n+1)!}(\sum_{k=0}^{n}\mu_i^k\mu_j^{n-k})}$$
$$=\frac{\mathbf{e}_j\cdot(\mathbf{F}\mathbf{e}_j)+\mathbf{e}_i\cdot(\mathbf{F}\mathbf{e}_j)(\mathbf{e}_i\cdot\mathbf{e}_j)}{\frac{1}{\mu_i-\mu_j}(\mathrm{e}^{2\mu_i}-\mathrm{e}^{2\mu_j})}$$
$$=\begin{cases}\dfrac{\mathbf{e}_j\cdot(\mathbf{F}\mathbf{e}_j)}{\mathrm{e}^{2\mu_j}} & \text{if } \mu_i=\mu_j\\ \dfrac{\mathbf{e}_j\cdot(\mathbf{F}\mathbf{e}_j)(\mu_i-\mu_j)}{\mathrm{e}^{2\mu_i}-\mathrm{e}^{2\mu_j}} & \text{if } \mu_i\neq\mu_j\end{cases}. \tag{35}$$

More generally one needs $\mathbf{e}_i\otimes\mathbf{e}_j\colon (D_{\mathbf{F}}\mathbf{N}[\mathbf{e}_k\otimes\mathbf{e}_l])$. It proves convenient to start with the even more general expression

$$\mathbf{a}\otimes\mathbf{b}\colon D_{\mathbf{F}}\mathbf{B}[\mathbf{c}\otimes\mathbf{d}]$$
$$=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}\cdot(\mathbf{F}\mathbf{d})+\mathbf{a}\cdot(\mathbf{F}\mathbf{d})(\mathbf{b}\cdot\mathbf{c}). \tag{36}$$

On the other hand, similarly to Eq. (33), one obtains

$$\mathbf{a}\otimes\mathbf{b}\colon D_{\mathbf{F}}\mathbf{B}[\mathbf{c}\otimes\mathbf{d}]$$
$$=\sum_{n=1}^{\infty}\frac{2^n}{n!}\left(\sum_{k=0}^{n-1}\mathbf{a}\otimes\mathbf{b}\colon (\mathbf{N}^k D_{\mathbf{F}}\mathbf{N}[\mathbf{c}\otimes\mathbf{d}]\mathbf{N}^{n-1-k})\right)$$
$$=\sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}$$
$$\times\left(\sum_{q=0}^{n}(\mathbf{N}^q\mathbf{a})\cdot(D_{\mathbf{F}}\mathbf{N}[\mathbf{c}\otimes\mathbf{d}]\mathbf{N}^{n-q}\mathbf{b})\right). \tag{37}$$

Letting $\mathbf{a}=\mathbf{e}_i$, $\mathbf{b}=\mathbf{e}_j$, $\mathbf{c}=\mathbf{e}_k$, $\mathbf{d}=\mathbf{e}_l$, it follows from Eqs. (36) and (37) that

$$(\mathbf{e}_i\cdot\mathbf{e}_k)(\mathbf{e}_j\cdot(\mathbf{F}\mathbf{e}_l))+(\mathbf{e}_i\cdot(\mathbf{F}\mathbf{e}_l))(\mathbf{e}_j\cdot\mathbf{e}_k)$$
$$=\delta^{ik}(\mathbf{e}_j\cdot(\mathbf{F}\mathbf{e}_l))+\delta^{jk}(\mathbf{e}_i\cdot(\mathbf{F}\mathbf{e}_l))$$
$$=\sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}$$
$$\times\left(\sum_{q=0}^{n}\mu_i^q\mu_j^{n-q}\mathbf{e}_i\cdot(D_{\mathbf{F}}\mathbf{N}[\mathbf{e}_k\otimes\mathbf{e}_l]\mathbf{e}_j)\right)$$
$$=\mathbf{e}_i\otimes\mathbf{e}_j\colon D_{\mathbf{F}}\mathbf{N}[\mathbf{e}_k\otimes\mathbf{e}_l]$$
$$\times\sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}\left(\sum_{q=0}^{n}\mu_i^q\mu_j^{n-q}\right)$$
$$=\mathbf{e}_i\otimes\mathbf{e}_j\colon D_{\mathbf{F}}\mathbf{N}[\mathbf{e}_k\otimes\mathbf{e}_l]\mathrm{E}(\mu_i,\mu_j), \tag{38}$$

where

$$\mathrm{E}(\mu_i,\mu_j)=\begin{cases}2\mathrm{e}^{2\mu_i} & \text{if } \mu_i=\mu_j\\ \dfrac{\mathrm{e}^{2\mu_i}-\mathrm{e}^{2\mu_j}}{\mu_i-\mu_j} & \text{if } \mu_i\neq\mu_j\end{cases}. \tag{39}$$

In this way there results the following formula used to express the Fréchet derivative of the Hencky strain with respect to the deformation gradient

$$\mathbf{e}_i\otimes\mathbf{e}_j\colon D_{\mathbf{F}}\mathbf{N}[\mathbf{e}_k\otimes\mathbf{e}_l]$$
$$=\frac{\delta^{ik}(\mathbf{e}_j\cdot(\mathbf{F}\mathbf{e}_l))+\delta^{jk}(\mathbf{e}_i\cdot(\mathbf{F}\mathbf{e}_l))}{\mathrm{E}(\mu_i,\mu_j)}. \tag{40}$$

Note, that this formula gives the components of the desired Fréchet derivative with respect to the orthonormal basis of eigenvectors of the left Cauchy–Green strain.

Appendix B. Proofs of the propositions in Section 4

This appendix contains sketches of the proofs of the various propositions in Section 4. Throughout most of this appendix, $\mathbf{F}$ is assumed to be diagonal in the basis of the eigenvectors of the natural strain $\mathbf{N}$, i.e.

$$\mathbf{F}=f_1\mathbf{e}_1\otimes\mathbf{e}_1+f_2\mathbf{e}_2\otimes\mathbf{e}_2+f_3\mathbf{e}_3\otimes\mathbf{e}_3. \tag{41}$$

Proof of Proposition 1. Using Eqs. (8) and (12), Eq. (17) takes the following form:

$$\mathbf{H}\colon \frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}\colon\mathbf{H}$$
$$=W_{,11}\left(\mathbf{I}\colon\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right)(\mathbf{F}^{-\mathrm{T}}\colon\mathbf{H})$$
$$-W_{,1}\mathbf{H}\colon\left(\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right). \tag{42}$$

Since

$$\mathbf{F}^{-\mathrm{T}}=\sum\frac{1}{f_k}\mathbf{e}_k\otimes\mathbf{e}_k,$$

it follows in the case when $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j$, $i\neq j$ that

$$\mathbf{F}^{-\mathrm{T}}\colon \mathbf{H}=0.$$

Also one easily shows that

$$\mathbf{F}^{-1}\mathbf{H}=\frac{1}{f_i}\,\mathbf{e}_i\otimes\mathbf{e}_j \quad\Rightarrow\quad (\mathbf{F}^{-1}\mathbf{H})^2=\mathbf{0}$$

$$\mathbf{H}\colon (\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}})=\mathrm{tr}((\mathbf{F}^{-1}\mathbf{H})^2)=0.$$

From Eq. (42), it now easily follows that in this particular case

$$\mathbf{H}\colon \frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}\colon \mathbf{H}=0.$$

This proves the first part of Proposition 1.

In the case when $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i$, one has

$$\mathbf{F}^{-\mathrm{T}}\colon \mathbf{H}=\frac{1}{f_i},$$

$$\mathbf{I}\colon \frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]=\frac{2\mathbf{e}_i.\mathbf{F}\mathbf{e}_i}{E(\mu_i,\mu_i)}=\frac{1}{f_i},$$

$$\mathbf{H}\colon \left(\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right)$$

$$=\mathrm{tr}((\mathbf{F}^{-1}\mathbf{H})^2)=\mathrm{tr}\left(\frac{1}{f_i^2}\,\mathbf{e}_i\otimes\mathbf{e}_i\right)=\frac{1}{f_i^2}.$$

From Eq. (42), it now follows that in this case,

$$\mathbf{H}\colon \frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}\colon \mathbf{H}=W_{,11}\frac{1}{f_i^2}-W_{,1}\frac{1}{f_i^2}.$$

Thus, a necessary condition for strong ellipticity of the constitutive equation is

$$W_{,11}>W_{,1}, \tag{43}$$

which concludes the second part of the proposition.

Proof of Proposition 2. Using Eqs. (9) and (13), Eq. (17) takes the following form

$$\begin{aligned}\mathbf{H}\colon \frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}\colon \mathbf{H} &= W_{,22}\left(\mathbf{\Phi}:\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right)\left(\mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}\colon \mathbf{H}\right)\\ &\quad+W_{,2}\left(-\frac{1}{K_2^2}\left(\mathbf{\Phi}:\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right)(\mathbf{H}\mathbf{F}^{-1}\colon\ \mathrm{dev}(\mathbf{N}))\right.\\ &\quad\left.+\frac{1}{K_2}\mathrm{dev}\left(\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right):\mathbf{H}\mathbf{F}^{-1}\right)\\ &\quad-W_{,2}\mathbf{H}\colon\ \left(\mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right).\end{aligned} \tag{44}$$

Since $\mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}$ is diagonal in the basis of the eigenvectors of $\mathbf{N}$, one has in the case when $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j$, $i\neq j$, that

$$\mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}\colon \mathbf{H}=0.$$

As in the proof of Proposition 1, $\mathbf{H}\mathbf{F}^{-1}=\frac{1}{f_j}\mathbf{e}_i\otimes\mathbf{e}_j$. Also $\mathrm{dev}(\mathbf{N})$ is diagonal in the basis of the eigenvectors of $\mathbf{N}$, from which it follows that

$$\mathbf{H}\mathbf{F}^{-1}\colon\ \mathrm{dev}(\mathbf{N})=0 \tag{45}$$

$$\mathbf{H}\colon \mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}=\mathrm{tr}(\mathbf{\Phi}(\mathbf{H}\mathbf{F}^{-1})^2)=0 \tag{46}$$

with the last equality due to $(\mathbf{H}\mathbf{F}^{-1})^2=\mathbf{0}$.

Thus, only the second term of Eq. (44) remains, for which the following holds true

$$\begin{aligned}\mathbf{H}\mathbf{F}^{-1}\colon\ \mathrm{dev}\left(\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right) &= \mathbf{H}\mathbf{F}^{-1}:\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\\ &\quad-\frac{1}{3}\,\mathrm{tr}\left(\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right)\mathbf{H}\mathbf{F}^{-1}\colon \mathbf{I}.\end{aligned} \tag{47}$$

The last term is equal to zero, since $\mathbf{H}\mathbf{F}^{-1}$ is traceless. Thus,

$$\begin{aligned}\mathbf{H}\mathbf{F}^{-1}\colon\ \mathrm{dev}\left(\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right) &= \frac{1}{f_j}\,\mathbf{e}_i\otimes\mathbf{e}_j\colon\ \frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{e}_i\otimes\mathbf{e}_j]\\ &= \frac{1}{f_j}\,\frac{\mathbf{e}_j.\mathbf{F}\mathbf{e}_j}{E(\mu_i,\mu_j)}=\frac{\mu_i-\mu_j}{\mathrm{e}^{2\mu_i}-\mathrm{e}^{2\mu_j}}.\end{aligned} \tag{48}$$

From Eq. (17), it follows that in this case,

$$\mathbf{H}\colon \frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}\colon \mathbf{H}=\frac{W_{,2}}{K_2}\left(\frac{\mu_i-\mu_j}{\mathrm{e}^{2\mu_i}-\mathrm{e}^{2\mu_j}}\right)>0.$$

Thus, a necessary condition for strong ellipticity of the constitutive equation is

$$W_{,2}>0, \tag{49}$$

which concludes the first part of Proposition 2.

For the second part of Proposition 2, in the case when $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_i$,

$$\mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}\colon \mathbf{H} = \frac{1}{K_2.f_i}\,\lambda_i \tag{50}$$

$$\mathbf{e}_k \otimes \mathbf{e}_k\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}\,[\mathbf{e}_i \otimes \mathbf{e}_i] = \frac{2\delta^{ik} f_i}{E(\mu_k,\mu_k)} = \frac{\delta^{ik} f_i}{\mathrm{e}^{2\mu_k}}. \tag{51}$$

Consequently,

$$\mathbf{\Phi}\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}\,[\mathbf{H}] = \frac{2 f_i}{E(\mu_i,\mu_i)}\,\frac{\lambda_i}{K_2}, \tag{52}$$

$$\mathbf{H}\mathbf{F}^{-1}\colon\ \mathrm{dev}(\mathbf{N}) = \frac{1}{f_i}\,\mathbf{e}_i \otimes \mathbf{e}_i\colon\ \mathrm{dev}(\mathbf{N}) = \frac{\lambda_i}{f_i}, \tag{53}$$

$$\mathrm{dev}\left(\frac{\partial \mathbf{N}}{\partial \mathbf{F}}\,[\mathbf{H}]\right) = \frac{\partial \mathbf{N}}{\partial \mathbf{F}}\,[\mathbf{H}] - \frac{1}{3}\,\mathrm{tr}\left(\frac{\partial \mathbf{N}}{\partial \mathbf{F}}\,[\mathbf{H}]\right)\mathbf{I}, \tag{54}$$

$$\mathbf{H}\mathbf{F}^{-1}\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}\,[\mathbf{H}] = \frac{1}{f_i}\,\mathbf{e}_i \otimes \mathbf{e}_i\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}\,[\mathbf{e}_i \otimes \mathbf{e}_i] = \frac{1}{\mathrm{e}^{2\mu_i}}, \tag{55}$$

$$\mathbf{H}\mathbf{F}^{-1}\colon\ \left(-\frac{1}{3}\mathrm{tr}\left(\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\mathbf{I}\right) = -\frac{1}{3 f_i}\,\mathrm{tr}\left(\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right) = -\frac{1}{3\mathrm{e}^{2\mu_i}}. \tag{56}$$

Thus,

$$\mathbf{H}\mathbf{F}^{-1}\colon\ \mathrm{dev}\left(\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right) = \frac{1}{\mathrm{e}^{2\mu_i}} - \frac{1}{3\mathrm{e}^{2\mu_i}} = \frac{2}{3\mathrm{e}^{2\mu_i}}, \tag{57}$$

$$(\mathbf{H}\mathbf{F}^{-1})^2 = \frac{1}{f_i^2}\,\mathbf{e}_i \otimes \mathbf{e}_i,$$

$$\mathbf{\Phi}(\mathbf{H}\mathbf{F}^{-1})^2 = \frac{1}{K_2 f_i^2}\,\lambda_i \mathbf{e}_i \otimes \mathbf{e}_i.$$

It now follows that

$$\mathbf{H}\colon (\mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}) = \mathrm{tr}(\mathbf{\Phi}(\mathbf{H}\mathbf{F}^{-1})^2) = \frac{1}{K_2 f_i^2}\,\lambda_i. \tag{58}$$

Next one has that Eq. (17) takes the form

$$\begin{aligned}\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}}\,\hat{\mathbf{T}}\colon \mathbf{H} &= W_{,22}\,\frac{\lambda_i^2}{K_2^2.\mathrm{e}^{2\mu_i}} + W_{,2} \\ &\quad \times \left(-\frac{1}{K_2^2}\,\frac{f_i\lambda_i}{K_2\mathrm{e}^{2\mu_i}}\,\frac{\lambda_i}{f_i} + \frac{2}{3K_2\mathrm{e}^{2\mu_i}} - \frac{1}{K_2 f_i^2}\,\lambda_i\right) \\ &= \lambda_i^2\,\frac{1}{K_2^2\mathrm{e}^{2\mu_i}}\left(W_{,22} - \frac{W_{,2}}{K_2}\right) \\ &\quad -\lambda_i\,\frac{W_{,2}}{f_i^2 K_2} + \frac{2W_{,2}}{3K_2\mathrm{e}^{2\mu_i}}. \end{aligned} \tag{59}$$

The above expression (a quadratic function of λ_i) is positive for every λ_i if and only if its discriminant D is strictly negative, i.e.

$$D = \frac{W_{,2}^2}{f_i^4 K_2^2} - \frac{1}{K_2^2\mathrm{e}^{2\mu_i}}\left(W_{,22} - \frac{W_{,2}}{K_2}\right)\frac{2W_{,2}}{3K_2\mathrm{e}^{2\mu_i}} < 0.$$

Since (49) implies $W_{,2} > 0$, it follows that it can be canceled together with $\frac{1}{K_2^2}$ and $\frac{1}{f_i^4} = \frac{1}{\mathrm{e}^{4\mu_i}}$, without affecting the inequality. Thus a necessary condition for strong ellipticity is

$$W_{,2} - \frac{8W_{,22}}{3K_2} + \frac{8W_{,2}}{3K_2^2} < 0$$

$$\Leftrightarrow\quad W_{,22} > \left(\frac{3K_2}{8} + \frac{1}{K_2}\right)W_{,2}. \tag{60}$$

This completes the proof of Proposition 2.

Proof of Proposition 3. Using Eqs. (10), (14) and (15), Eq. (17) takes the form

$$\begin{aligned}\mathbf{H}\colon \frac{\partial}{\partial \mathbf{F}}\,\hat{\mathbf{T}}\colon \mathbf{H} &= W_{,33}\left(\frac{1}{K_2}\mathbf{Y}\colon \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\left(\frac{1}{K_2}\mathbf{Y}\mathbf{F}^{-\mathrm{T}}\colon \mathbf{H}\right) \\ &\quad + W_{,3}\mathbf{H}\mathbf{F}^{-1} : \frac{\partial^2 K_3}{\partial \mathbf{N}^2} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}\,[\mathbf{H}] \\ &\quad - W_{,3}\mathbf{H}\colon\ \left(\frac{1}{K_2}\,\mathbf{Y}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right). \end{aligned} \tag{61}$$

Since $\mathbf{Y}\mathbf{F}^{-\mathrm{T}}$ is diagonal in the basis of the eigenvectors of $\mathbf{N}$, it follows in the case when $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_j$,

$i \neq j$, that

$$\mathbf{Y}\mathbf{F}^{-\mathrm{T}}\colon \mathbf{H} = 0. \tag{62}$$

As in the proof of Proposition 1, $\mathbf{H}\mathbf{F}^{-1} = \frac{1}{f_j}\mathbf{e}_i \otimes \mathbf{e}_j$ from which one sees that

$$\mathbf{H}\colon \left(\frac{1}{K_2}\mathbf{Y}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right) = \mathrm{tr}\left(\mathbf{Y}(\mathbf{H}\mathbf{F}^{-1})^2\right) = 0,$$

with the last equality is due to $(\mathbf{H}\mathbf{F}^{-1})^2 = \mathbf{0}$. Thus, in this case,

$$\mathbf{H} : \frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}} : \mathbf{H} = W_{,3}\mathbf{H}\mathbf{F}^{-1} : \frac{\partial^2 K_3}{\partial \mathbf{N}^2} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]. \tag{63}$$

From Eq. (15), it follows that $\frac{\partial \mathbf{Y}}{\partial \mathbf{N}}\colon \mathbf{A} = \mathbf{A}\colon \frac{\partial \mathbf{Y}}{\partial \mathbf{N}}$. Thus, Eq. (14) implies

$$\mathbf{H}\mathbf{F}^{-1} : \frac{\partial^2 K_3}{\partial \mathbf{N}^2} = -\frac{1}{K_2^2}(\mathbf{\Phi}\colon (\mathbf{H}\mathbf{F}^{-1}))\mathbf{Y} + \frac{1}{K_2}\frac{\partial \mathbf{Y}}{\partial \mathbf{N}}[\mathbf{H}\mathbf{F}^{-1}]. \tag{64}$$

Since $\mathbf{\Phi}$ is diagonal and $\mathbf{H}\mathbf{F}^{-1} = \frac{1}{f_j}\mathbf{e}_i \otimes \mathbf{e}_j$, one concludes that

$$\mathbf{\Phi} : (\mathbf{H}\mathbf{F}^{-1}) = 0$$

$$\mathrm{dev}(\mathbf{H}\mathbf{F}^{-1}) = \mathbf{H}\mathbf{F}^{-1} = \frac{1}{f_j}\mathbf{e}_i \otimes \mathbf{e}_j.$$

From Eq. (15), it next follows that

$$\frac{\partial \mathbf{Y}}{\partial \mathbf{N}}[\mathbf{H}\mathbf{F}^{-1}] = \frac{3\sqrt{6}}{K_2^2}(\mathrm{dev}(\mathbf{N})\mathbf{H}\mathbf{F}^{-1} + \mathbf{H}\mathbf{F}^{-1}\mathrm{dev}(\mathbf{N})) - \frac{3K_3}{K_2}\mathbf{H}\mathbf{F}^{-1} = \left(\frac{3\sqrt{6}}{K_2^2}\frac{\lambda_i + \lambda_j}{f_j} - \frac{3K_3}{K_2 f_j}\right)\mathbf{e}_i \otimes \mathbf{e}_j. \tag{65}$$

Now using Eq. (61) and the above calculations, one obtains

$$\mathbf{H} : \frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}} : \mathbf{H} = W_{,3}\frac{1}{K_2}\left\{\frac{3\sqrt{6}}{K_2^2 f_j}(-\lambda_k) - \frac{3K_3}{K_2 f_j}\right\} \times \mathbf{e}_i \otimes \mathbf{e}_j : \frac{\partial \mathbf{N}}{\partial \mathbf{F}} : \mathbf{e}_i \otimes \mathbf{e}_j = W_{,3}\frac{1}{K_2^2}\left\{\frac{-3\sqrt{6}}{K_2}\lambda_k - 3K_3\right\}\frac{\mu_i - \mu_j}{\mathrm{e}^{2\mu_i} - \mathrm{e}^{2\mu_j}}. \tag{66}$$

For the second part of Proposition 3, i.e. for the case when $\mathbf{H} = \mathbf{e}_i \otimes \mathbf{e}_i$, it proves useful to introduce

$$\Theta = \mathbf{Y}\colon \mathbf{H} = \frac{3\sqrt{6}}{K_2^2}\lambda_i^2 - \frac{3K_3}{K_2}\lambda_i - \sqrt{6},$$

$$\mathbf{Y}\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = \frac{\Theta}{f_i}, \tag{67}$$

$$\mathbf{e}_k \otimes \mathbf{e}_k : \frac{\partial \mathbf{N}}{\partial \mathbf{F}} : \mathbf{e}_i \otimes \mathbf{e}_i = \frac{2\delta_{ik}(\mathbf{e}_k . \mathbf{F}\mathbf{e}_i)}{E(\mu_k, \mu_k)} = \frac{\delta_{ik}}{f_k}.$$

One now obtains

$$\mathbf{Y} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{e}_i \otimes \mathbf{e}_i] = \frac{\Theta}{f_i}, \tag{68}$$

$$\mathbf{H}\colon \mathbf{Y}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}} = \mathrm{tr}(\mathbf{Y}(\mathbf{H}\mathbf{F}^{-1})^2) = \frac{\Theta}{f_i^2}, \tag{69}$$

$$\mathbf{H}\mathbf{F}^{-1} : \frac{\partial^2 K_3}{\partial \mathbf{N}^2} = -\frac{1}{K_2^2}(\mathbf{\Phi} : \mathbf{H}\mathbf{F}^{-1})\mathbf{Y} + \frac{1}{K_2}\frac{\partial \mathbf{Y}}{\partial \mathbf{N}}[\mathbf{H}\mathbf{F}^{-1}], \tag{70}$$

$$\mathbf{\Phi}\colon (\mathbf{H}\mathbf{F}^{-1}) = \frac{\lambda_i}{K_2 f_i},$$

$$\frac{\partial \mathbf{Y}}{\partial N}[\mathbf{H}\mathbf{F}^{-1}] = 3\sqrt{6}\mathbf{\Phi}\left(-\frac{1}{K_2^2}\frac{\lambda_i}{K_2 f_i}\mathrm{dev}(\mathbf{N}) + \frac{1}{K_2}\mathrm{dev}(\mathbf{H}\mathbf{F}^{-1})\right)$$

$$+3\sqrt{6}\left(-\frac{1}{K_2^2}\frac{\lambda_i}{K_2 f_i}\operatorname{dev}(\mathbf{N}) + \frac{1}{K_2}\operatorname{dev}(\mathbf{HF}^{-1})\right)\boldsymbol{\Phi} - \frac{3\Theta}{K_2 f_i}\boldsymbol{\Phi} - 3K_3\left(-\frac{1}{K_2^2}\frac{\lambda_i}{K_2 f_i}\operatorname{dev}(\mathbf{N}) + \frac{\operatorname{dev}(\mathbf{HF}^{-1})}{K_2}\right), \quad (71)$$

$$\operatorname{dev}(\mathbf{HF}^{-1}) = \mathbf{HF}^{-1} - \frac{1}{3}\operatorname{tr}(\mathbf{HF}^{-1})\mathbf{I} = \frac{2}{3f_i}\mathbf{e}_i\otimes\mathbf{e}_i - \frac{1}{3f_j}\mathbf{e}_j\otimes\mathbf{e}_j - \frac{1}{3f_k}\mathbf{e}_k\otimes\mathbf{e}_k.$$

It follows that

$$\frac{\partial \mathbf{Y}}{\partial N}[\mathbf{HF}^{-1}] = -\frac{6\sqrt{6}}{K_2^4}\frac{\lambda_i}{f_i}(\operatorname{dev}(\mathbf{N}))^2 + \frac{6\sqrt{6}}{K_2^2}\operatorname{dev}(\mathbf{N})\operatorname{dev}(\mathbf{HF}^{-1}) - \frac{2\Theta}{K_2^2 f_i}\boldsymbol{\Phi} + \frac{3K_3\lambda_i}{K_2^3 f_i}\operatorname{dev}(\mathbf{N}) - \frac{3K_3}{K_2}\operatorname{dev}(\mathbf{HF}^{-1}). \quad (72)$$

The above second order tensor is diagonal in the basis of the eigenvectors of $\mathbf{N}$. Thus,

$$\mathbf{e}_k\otimes\mathbf{e}_k : \frac{\partial \mathbf{N}}{\partial \mathbf{F}} : \mathbf{e}_i\otimes\mathbf{e}_i = \delta^{ik}\frac{1}{f_i}$$

and

$$\mathbf{HF}^{-1} : \frac{\partial^2 K_3}{\partial \mathbf{N}^2} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}] = -\frac{\Theta\lambda_i}{K_2^3 f_i^2} + \frac{1}{K_2 f_i}\left(-\frac{6\sqrt{6}\lambda_i^3}{K_2^4 f_i} + \frac{12\sqrt{6}\lambda_i}{3K_2^2 f_i} - \frac{3\Theta\lambda_i}{K_2^3 f_i} + \frac{3K_3\lambda_i^2}{K_2^2 f_i} - \frac{2K_3}{K_2 f_i}\right)$$

$$= -\frac{\Theta\lambda_i}{f_i^2}\left(\frac{1}{K_2^3} + \frac{3}{K_2^4}\right) - \frac{6\sqrt{6}\lambda_i^3}{K_2^5 f_i^2} + \frac{3K_3\lambda_i^2}{K_2^3 f_i^2} + \frac{4\sqrt{6}\lambda_i}{K_2^3 f_i^2} - \frac{2K_3}{K_2^2 f_i^2}. \quad (73)$$

So, the strong ellipticity condition in this case is equivalent to

$$\mathbf{H} : \frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}} : \mathbf{H} = W_{,33}\frac{\Theta^2}{K_2^2 f_i^2} - W_{,3}\frac{\Theta}{K_2 f_i^2} + W_{,3}\left(-\frac{\Theta\lambda_i}{f_i^2}\left(\frac{1}{K_2^3} + \frac{3}{K_2^4}\right) - \frac{6\sqrt{6}\lambda_i^3}{K_2^5 f_i^2} + \frac{3K_3\lambda_i^2}{K_2^3 f_i^2} + \frac{4\sqrt{6}\lambda_i}{K_2^3 f_i^2} - \frac{2K_3}{K_2^2 f_i^2}\right), \quad (74)$$

which concludes the proof of Proposition 3.

Proof of Proposition 4. For the first part of the proposition, that is, $\mathbf{H} = \mathbf{e}_i\otimes\mathbf{e}_j$, $i \neq j$, similarly to the cases when $W = W(K_2)$ and $W = W(K_3)$, Eq. (17) takes the following form

$$\mathbf{H} : \frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}} : \mathbf{H} = W_{,2}\mathbf{HF}^{-1} : \frac{\partial^2 K_2}{\partial \mathbf{N}^2} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}] + W_{,3}\mathbf{HF}^{-1} : \frac{\partial^2 K_3}{\partial \mathbf{N}^2} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]. \quad (75)$$

As in the first part of Proposition 2,

$$W_{,2}\mathbf{HF}^{-1} : \frac{\partial^2 K_2}{\partial \mathbf{N}^2} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}] = \frac{W_{,2}}{K_2}\left(\frac{\mu_i - \mu_j}{\mathrm{e}^{2\mu_i} - \mathrm{e}^{2\mu_j}}\right), \quad (76)$$

and from the first part of Proposition 3 it follows that

$$W_{,3}\mathbf{HF}^{-1} : \frac{\partial^2 K_3}{\partial \mathbf{N}^2} : \frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}] = W_{,3}\frac{1}{K_2^2}\left\{\frac{-3\sqrt{6}}{K_2}\lambda_k - 3K_3\right\}\frac{\mu_i - \mu_j}{\mathrm{e}^{2\mu_i} - \mathrm{e}^{2\mu_j}}. \quad (77)$$

Consequently, one sees that

$$\mathbf{H}:\frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}:\mathbf{H}$$
$$=\frac{W_{,2}}{K_2}\frac{\mu_i-\mu_j}{e^{2\mu_i}-e^{2\mu_j}}$$
$$+W_{,3}\frac{1}{K_2^2}\left\{\frac{-3\sqrt{6}}{K_2}\lambda_k-3K_3\right\}\frac{\mu_i-\mu_j}{e^{2\mu_i}-e^{2\mu_j}}$$
$$=\frac{\mu_i-\mu_j}{K_2(e^{2\mu_i}-e^{2\mu_j})}$$
$$\times\left(W_{,2}+W_{,3}\left(-\frac{3\sqrt{6}\lambda_k}{K_2^2}-\frac{3K_3}{K_2}\right)\right), \quad (78)$$

which completes the proof of the first part of Proposition 4.

For the second part of Proposition 4, one has $W=W(K_2,K_3)$ and $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i$ from which it follows that Eq. (17) takes the form

$$\mathbf{H}:\frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}:\mathbf{H}$$
$$=W_{,22}\left(\frac{\partial K_2}{\partial \mathbf{N}}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\left(\frac{\partial K_2}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}:\mathbf{H}\right)$$
$$+W_{,23}\left(\frac{\partial K_3}{\partial \mathbf{N}}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\left(\frac{\partial K_2}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}:\mathbf{H}\right)$$
$$+W_{,32}\left(\frac{\partial K_2}{\partial \mathbf{N}}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\left(\frac{\partial K_3}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}:\mathbf{H}\right)$$
$$+W_{,33}\left(\frac{\partial K_3}{\partial \mathbf{N}}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)\left(\frac{\partial K_3}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}:\mathbf{H}\right)$$
$$+W_{,2}\mathbf{H}\mathbf{F}^{-1}:\frac{\partial^2 K_2}{\partial \mathbf{N}^2}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]$$
$$+W_{,3}\mathbf{H}\mathbf{F}^{-1}:\frac{\partial^2 K_3}{\partial \mathbf{N}^2}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]$$
$$-W_{,2}\mathbf{H}:\left(\frac{\partial K_2}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right)$$
$$-W_{,3}\mathbf{H}:\left(\frac{\partial K_3}{\partial \mathbf{N}}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right)>0. \quad (79)$$

From Eqs. (9) and (10),

$$\frac{\partial K_2}{\partial \mathbf{N}}=\mathbf{\Phi},\qquad \frac{\partial K_3}{\partial \mathbf{N}}=\frac{1}{K_2}\mathbf{Y}.$$

From Eq. (52),

$$\mathbf{\Phi}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]=\frac{2f_i}{E(\mu_i,\mu_i)}\frac{\lambda_i}{K_2}=\frac{\lambda_i}{K_2 f_i}.$$

From Eq. (50),

$$\mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}:\mathbf{H}=\frac{1}{K_2 f_i}\lambda_i.$$

From Eq. (58),

$$\mathbf{H}:(\mathbf{\Phi}\,\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}})=\frac{1}{K_2 f_i^2}\lambda_i.$$

From Eq. (13),

$$\frac{\partial^2 K_2}{\partial \mathbf{N}^2}[\mathbf{A}]=-\frac{1}{K_2^2}(\mathbf{\Phi}:\mathbf{A})\mathrm{dev}(\mathbf{N})+\frac{\mathrm{dev}(\mathbf{A})}{K_2}.$$

Combining these facts, one has

$$\mathbf{H}\mathbf{F}^{-1}:\frac{\partial^2 K_2}{\partial \mathbf{N}^2}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]$$
$$=-\frac{1}{K_2^2}\left(\mathbf{\Phi}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)(\mathbf{H}\mathbf{F}^{-1}:\ \mathrm{dev}(\mathbf{N}))$$
$$+\frac{1}{K_2}\mathbf{H}\mathbf{F}^{-1}:\ \mathrm{dev}\left(\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]\right)$$
$$=-\frac{1}{K_2^3}\frac{\lambda_i^2}{f_i^2}+\frac{1}{K_2}\frac{2}{3f_i^2}.$$

Next, from Eq. (68),

$$\frac{1}{K_2}\mathbf{Y}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{e}_i\otimes\mathbf{e}_i]=\frac{\Theta}{K_2 f_i}.$$

From Eq. (67)

$$\frac{1}{K_2}\mathbf{Y}\mathbf{F}^{-\mathrm{T}}:\mathbf{H}=\frac{\Theta}{K_2 f_i}$$

From Eq. (69)

$$\mathbf{H}:\frac{1}{K_2}\mathbf{Y}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}=\mathrm{tr}(\mathbf{Y}(\mathbf{H}\mathbf{F}^{-1})^2)=\frac{\Theta}{K_2 f_i^2}.$$

From Eq. (73)

$$\mathbf{H}\mathbf{F}^{-1}:\frac{\partial^2 K_3}{\partial \mathbf{N}^2}:\frac{\partial \mathbf{N}}{\partial \mathbf{F}}[\mathbf{H}]$$
$$=-\frac{\Theta\lambda_i}{f_i^2}\left(\frac{1}{K_2^3}+\frac{3}{K_2^4}\right)$$
$$-\frac{6\sqrt{6}\lambda_i^3}{K_2^5 f_i^2}+\frac{3K_3\lambda_i^2}{K_2^3 f_i^2}+\frac{4\sqrt{6}\lambda_i}{K_2^3 f_i^2}-\frac{2K_3}{K_2^2 f_i^2}.$$

Finally, substituting all of the above into Eq. (79), one obtains

$$\mathbf{H}:\frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}:\mathbf{H}$$

$$=W_{,22}\frac{\lambda_i^2}{K_2^2 f_i^2}+W_{,23}\frac{\Theta\lambda_i}{K_2^2 f_i^2}$$

$$+W_{,32}\frac{\Theta\lambda_i}{K_2^2 f_i^2}+W_{,33}\frac{\Theta^2}{K_2^2 f_i^2}$$

$$+W_{,2}\left(-\frac{\lambda_i^2}{K_2^3 f_i^2}+\frac{1}{K_2}.\frac{2}{3f_i^2}\right)$$

$$+W_{,3}\left(-\frac{\Theta\lambda_i}{f_i^2}\left(\frac{1}{K_2^3}+\frac{3}{K_2^4}\right)-\frac{6\sqrt{6}\lambda_i^3}{K_2^5 f_i^2}\right.$$

$$\left.+\frac{3K_3\lambda_i^2}{K_2^3 f_i^2}+\frac{4\sqrt{6}\lambda_i}{K_2^3 f_i^2}-\frac{2K_3}{K_2^2 f_i^2}\right)$$

$$-W_{,2}\frac{\lambda_i}{K_2 f_i^2}-W_{,3}\frac{\Theta}{K_2 f_i^2}, \qquad (80)$$

which completes the proof of Proposition 4.

Proof of Proposition 5. In this proposition, a shearing deformation of form (26) is considered and the constitutive equation is taken to be $W=W(K_2)$. One can easily show that $\mathbf{F}$ takes form (27) in the basis of eigenvectors of the left Cauchy–Green tensor $\mathbf{B}$ (which are also the eigenvectors of the Hencky strain $\mathbf{N}$). Let det denote the determinate of $\mathbf{F}$, i.e. $\det = f_1 f_2 - f_{12} f_{21}$. Then

$$\mathbf{F}^{-1}=\frac{1}{\det}(f_2\mathbf{e}_1\otimes\mathbf{e}_1+f_1\mathbf{e}_2\otimes\mathbf{e}_2$$
$$+\mathbf{e}_3\otimes\mathbf{e}_3-f_{12}\mathbf{e}_1\otimes\mathbf{e}_2-f_{21}\mathbf{e}_2\otimes\mathbf{e}_1), \qquad (81)$$

and

$$\mathbf{F}^{-\mathrm{T}}=\frac{1}{\det}(f_2\mathbf{e}_1\otimes\mathbf{e}_1+f_1\mathbf{e}_2\otimes\mathbf{e}_2$$
$$+\mathbf{e}_3\otimes\mathbf{e}_3-f_{12}\mathbf{e}_2\otimes\mathbf{e}_1-f_{21}\mathbf{e}_1\otimes\mathbf{e}_2). \qquad (82)$$

For the shearing deformation considered, $\det(\mathbf{F})=1$.

Using the previous notations,

$$\boldsymbol{\Phi}=\frac{1}{K_2}\sum_k \lambda_k\mathbf{e}_k\otimes\mathbf{e}_k.$$

For the perturbation $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j$, $i\neq j$,

$$\boldsymbol{\Phi}\,\mathbf{F}^{-\mathrm{T}}:\mathbf{H}=\frac{1}{K_2}(-f_{12}\lambda_2\delta^{2i}\delta^{1j}-f_{21}\lambda_1\delta^{1i}\delta^{2j}).$$

For $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i$

$$\boldsymbol{\Phi}\,\mathbf{F}^{-\mathrm{T}}:\mathbf{H}=\frac{1}{K_2}\lambda_i\bar{f}_i.$$

Following the same steps as in the previously considered cases, for the term $\mathbf{H}:\boldsymbol{\Phi}\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$, one obtains

$$\mathbf{H}:\boldsymbol{\Phi}\,\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$$
$$=\frac{\lambda_i}{K_2}(f_{12}^2\delta^{2i}\delta^{1j}+f_{21}^2\lambda_1\delta^{1i}\delta^{2j})$$
$$\text{for}\quad \mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j,\quad i\neq j.$$

Analogously,

$$\mathbf{H}:\boldsymbol{\Phi}\,\mathbf{F}^{-\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}=\frac{\lambda_i\bar{f}_i^2}{K_2}\quad \text{for } \mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i.$$

Using $\sum_k \lambda_k\mathbf{e}_k\otimes\mathbf{e}_k$,

$$\mathbf{H}\mathbf{F}^{-1}=\bar{f}_j\mathbf{e}_i\otimes\mathbf{e}_j-f_{12}\delta^{1j}\mathbf{e}_i\otimes\mathbf{e}_2-f_{21}\delta^{2j}\mathbf{e}_i\otimes\mathbf{e}_1$$
$$\text{for}\quad \mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j,\quad i\neq j \qquad (83)$$

and

$$\mathbf{H}\mathbf{F}^{-1}=\bar{f}_i\mathbf{e}_i\otimes\mathbf{e}_j-f_{12}\delta^{1i}\mathbf{e}_i\otimes\mathbf{e}_2-f_{21}\delta^{2j}\mathbf{e}_i\otimes\mathbf{e}_1$$
$$\text{for}\quad \mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i, \qquad (84)$$

one obtains

$$\mathbf{H}\mathbf{F}^{-1}:\ \mathrm{dev}(\mathbf{N})$$
$$=(-f_{12}\lambda_2\delta^{2i}\delta^{1j}-f_{21}\lambda_1\delta^{1i}\delta^{2j})$$
$$\text{for}\quad \mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j,\quad i\neq j$$

and

$$\mathbf{H}\mathbf{F}^{-1}:\ \mathrm{dev}(\mathbf{N})$$
$$=\bar{f}_i\lambda_i\quad \text{for}\quad \mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i.$$

Now, applying formula (40) for the Fréchet derivative of the Hencky strain with respect to the deformation gradient, one can see that in the case when $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j$, $i\neq j$

$$\boldsymbol{\Phi}:\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]=\frac{\lambda_i}{K_2\mathrm{e}^{2\mu_i}}(f_{12}\delta^{1i}\delta^{2j}+f_{21}\delta^{1j}\delta^{2i}),$$

and for $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i$

$$\mathbf{\Phi}:\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]=\frac{\lambda_i f_i}{K_2 e^{2\mu_i}}.$$

Now, notice that

$$\operatorname{dev}\left(\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right):\mathbf{HF}^{-1} =\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]:(\mathbf{HF}^{-1}) -\frac{1}{3}\left(\mathbf{I}:\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right)(\mathbf{I}:\mathbf{HF}^{-1}).$$

Using Eqs. (83) and (84), and formula (40) one obtains

$$\operatorname{dev}\left(\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right):\mathbf{HF}^{-1} =\frac{f_j\bar{f}_j}{E(\mu_i,\mu_j)}-\frac{f_{12}f_{21}}{E(\mu_i,\mu_2)}\delta^{1j}(1+\delta^{2i}) -\frac{f_{12}f_{21}}{E(\mu_i,\mu_1)}\delta^{2j}(1+\delta^{1i}) -\frac{1}{3e^{2\mu_i}}\left(-f_{12}f_{21}\delta^{1i}\delta^{2j}-f_{12}f_{21}\delta^{1j}\delta^{2i}\right)$$

for $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j$, $i\neq j$, and

$$\operatorname{dev}\left(\frac{\partial\mathbf{N}}{\partial\mathbf{F}}[\mathbf{H}]\right):\mathbf{HF}^{-1} =\frac{f_i\bar{f}_i}{e^2\mu_i}-\frac{f_{12}f_{21}}{E(\mu_i,\mu_2)}\delta^{1i} -\frac{f_{12}f_{21}}{E(\mu_i,\mu_1)}\delta^{2i}-\frac{f_i\bar{f}_i}{e^{2\mu_i}}3e^{2\mu_i}$$

for $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i$, where $E(\mu_i,\mu_j)$ is defined in Eq. (39). Finally, substituting all of the above into Eq. (44), one obtains

$$\begin{aligned}\mathbf{H}:\frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}:\mathbf{H} &=W_{,22}\frac{\lambda_i}{K_2^2e^{2\mu_i}}(f_{12}\delta^{1i}\delta^{2j}+f_{21}\delta^{1j}\delta^{2i})\\ &\times(-f_{12}\lambda_2\delta^{2i}\delta^{1j}-f_{21}\lambda_1\delta^{1i}\delta^{2j})\\ &+W_{,2}\frac{-\lambda_i}{K_2^3e^{2\mu_i}}(f_{12}\delta^{1i}\delta^{2j}+f_{21}\delta^{1j}\delta^{2i})\\ &\times(-f_{12}\lambda_2\delta^{2i}\delta^{1j}-f_{21}\lambda_1\delta^{1i}\delta^{2j})\\ &+W_{,2}\frac{1}{K_2}\left(\frac{f_j\bar{f}_j}{E(\mu_i,\mu_j)}-f_{12}f_{21}\left(\frac{\delta^{1j}(1+\delta^{2j})}{E(\mu_i,\mu_2)}+\frac{\delta^{2j}}{(1+\delta^{1i})}E(\mu_i,\mu_1)\right)\right)\\ &+W_{,2}\frac{1}{3K_2}\frac{f_{12}f_{21}}{e^{2\mu_i}}(\delta^{1i}\delta^{2j}+\delta^{1j}\delta^{2i})\\ &-W_{,2}\frac{\lambda_i}{K_2}(f_{12}^2\delta^{1j}\delta^{2i}+f_{21}^2\delta^{2j}\delta^{1i}),\end{aligned}$$

when $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_j$, $i\neq j$, and

$$\begin{aligned}\mathbf{H}:\frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}:\mathbf{H} &=W_{,22}\frac{\lambda_i^2f_i\bar{f}_i}{K_2^2e^{2\mu_i}}-W_{,2}\frac{\lambda_i^2f_i\bar{f}_i}{K_2^3e^{2\mu_i}}\\ &+W_{,2}\frac{1}{K_2}\left(\frac{2f_i\bar{f}_i}{3e^{2\mu_i}}-f_{12}f_{21}\left(\frac{\delta^{1i}}{E(\mu_i,\mu_2)}+\frac{\delta^{2i}}{E(\mu_i,\mu_1)}+\right)\right)-\frac{\lambda_i\bar{f}_i^2}{K_2},\end{aligned}\tag{85}$$

when $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_i$, which concludes the proof of Proposition 5.

Now, in order to obtain some necessary conditions for strong ellipticity of the given constitutive equation at pure shear, consider $\mathbf{H}=\mathbf{e}_i\otimes\mathbf{e}_3$, when $i\neq 3$. Then Eq. (85) takes the form

$$\mathbf{H}:\frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{T}}:\mathbf{H}=W_{,2}\frac{1}{K_2E(\mu_i,\mu_2)}.$$

Since K_2 and $E(\mu_i,\mu_j)$ are strictly positive, a necessary condition for strong ellipticity for this class of constitutive equations at the considered class of deformations is

$$W_{,2}>0,$$

which concludes the proof of the first part of Corollary 1.

For the second part, consider a perturbation $\mathbf{F} = \mathbf{e}_1 \otimes \mathbf{e}_1$. Then Eq. (85) takes the following form:

$$
\begin{aligned}
\mathbf{H} : \frac{\partial}{\partial \mathbf{F}} \hat{\mathbf{T}} : \mathbf{H}
&= \frac{1}{4K_2} \left(\ln \frac{2+\gamma^2+\gamma\sqrt{\gamma^2+4}}{2} \right)^2 \\
&\quad \times \left(1 - \frac{\gamma^2\beta^2}{(1+\beta^2)^2} \right) \frac{2}{2+\gamma^2+\gamma\sqrt{\gamma^2+4}} W_{,22} \\
&\quad - \frac{1}{4K_2^3} \left(\ln \frac{2+\gamma^2+\gamma\sqrt{\gamma^2+4}}{2} \right)^2 \\
&\quad \times \left(1 - \frac{\gamma^2\beta^2}{(1+\beta^2)^2} \right) \frac{2}{2+\gamma^2+\gamma\sqrt{\gamma^2+4}} W_{,2} \\
&\quad + \frac{1}{K_2} \left(\frac{2}{3} \left(1 - \frac{\gamma^2\beta^2}{(1+\beta^2)^2} \right) \frac{2}{2+\gamma^2+\gamma\sqrt{\gamma^2+4}} \right. \\
&\quad \left. + \frac{\gamma^2\beta^2}{(1+\beta^2)^2} \cdot \frac{1}{E(\mu_1,\mu_2)} \right) W_{,2} \\
&\quad - \frac{1}{2K_2} \ln \left(\frac{2+\gamma^2+\gamma\sqrt{\gamma^2+4}}{2} \right) \\
&\quad \times \left(1 - \frac{\gamma\beta}{1+\beta^2} \right)^2 W_{,2} \\
&= g_1(\gamma) W_{,22} + g_2(\gamma) W_{,2}.
\end{aligned}
$$

Now, considering the behavior of $g_1(\gamma)$ and $g_2(\gamma)$ as $\gamma \to \infty$, one has

$$
\begin{aligned}
\frac{g_2(\gamma)}{g_1(\gamma)} &= \frac{4}{3} + 2\gamma^2 \ln\gamma - \sqrt{2} \\
&\quad - \frac{1}{\sqrt{2}\ln\gamma} \to \infty, \quad \text{as } \gamma \to \infty
\end{aligned}
$$

as required.

References

[1] S.S. Antman, Nonlinear Problems of Elasticity, Springer, Berlin, 1995.

[2] J.K. Knowles, E. Sternberg, On the ellipticity of the equations of nonlinear elastostatics for a special material, J. Elasticity 5 (1975) 341–361.

[3] J.K. Knowles, E. Sternberg, On the failure of ellipticity of the equations for finite elastostatic plane strain, Arch. Ration. Mech. Anal. 63 (1977) 321–335.

[4] L. Zee, E. Sternberg, Ordinary and strong ellipticity in the equilibrium theory of incompressible hyperelastic solids, Arch. Ration. Mech. Anal. 83 (1983) 53–90.

[5] C.O. Horgan, Remarks on ellipticity for the generalized blatz-ko constitutive model for a compressible nonlinearly elastic solid, J. Elasticity 42 (1996) 165–176.

[6] P. Rosakis, Ellipticity and deformations with discontinuous gradients in finite elastostatics, Arch. Ration. Mech. Anal. 109 (1990) 1–37.

[7] Y. Wang, M. Aron, A reformulation of the strong ellipticity conditions for unconstrained hyperelastic media, J. Elasticity 44 (1996) 89–96.

[8] J.P. Wilber, J.R. Walton, The convexity properties of a class of constitutive models for biological soft tissues, Math. Mech. Solids 7 (2002) 217–235.

[9] J.R. Walton, J.P. Wilber, Sufficient conditions for strong ellipticity for a class of anisotropic materials, Int. J. Nonlinear Mech. 38 (2003) 441–455.

[10] J.C. Criscione, Rivlin's representation formula is ill-conceived for the determination of response functions via biaxial testing, J. Elast. 79 (1) (2003) 129–147.

[11] J.C. Criscione, J.D. Humphrey, A.S. Douglas, W.C. Hunter, An invariant basis for natural strain which yields orthogonal stress response terms in isotropic hyperelasticity, J. Mech. Phys. Solids 48 (2000) 2445–2465.

[12] J.C. Criscione, J.P. Wilber, The baker-ericksen inequalities for isotropic hyperelastic materials modeled using a novel set of invariants of hencky strain, preprint, 2004.

[13] R. Hill, On constitutive inequalities for simple materials–i, J. Mech. Phys. Solids 16 (1968) 229–242.

[14] R. Hill, Constitutive inequalities for isotropic elastic solids under finite strain, Proc. Roy. Soc. London, Series 314 (1970) 457–472.

[15] R. Hill, J.W. Hutchinson, Bifurcation phenomena in the plane tension test, J. Mech. Phys. Solids 23 (1975) 239–264.

[16] J.W. Hutchinson, K.W. Neale, Finite strain j_2 deformation theory, in: D.E. Carlson, R.T. Shield (Eds.), Finite Elasticity: Proceedings of the IUTAM Symposium on Finite Elasticity 1980, Martinus Nijhoff, Dordrecht, 1982, pp. 237–247.

[17] O.T. Bruhns, H. Xiao, A. Meyers, Constitutive inequalities for an isotropic elastic strain-energy function based upon hencky's logarithmic strain tensor, Proc. Roy. Soc. London, Series 457 (2001) 2207–2226.

[18] D.F. Jones, L.R.G. Treloar, The properties of rubber in pure homogeneous strain, J. Phys. D (1975) 1285–1304.